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Research Article

Weighted Composition Operators from H^∞ to the Bloch Space on the Polydisc

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Let $\mathbb{D}^n$ be the unit polydisc of $\mathbb{C}^n$, $\varphi(z) = (\varphi_1(z), \dots, \varphi_n(z))$ be a holomorphic self-map of $\mathbb{D}^n$, and $\psi(z)$ a holomorphic function on $\mathbb{D}^n$. Let $H(\mathbb{D}^n)$ denote the space of all holomorphic functions with domain $\mathbb{D}^n$, $H^\infty(\mathbb{D}^n)$ the space of all bounded holomorphic functions on $\mathbb{D}^n$, and $\mathfrak{B}(\mathbb{D}^n)$ the Bloch space, that is, $\mathfrak{B}(\mathbb{D}^n) = \{f \in H(\mathbb{D}^n) \mid \|f\|_{\mathfrak{B}} = |f(0)| + \sup_{z\in\mathbb{D}^n} \sum_{k=1}^{n} |(\partial f/\partial z_k)(z)|(1-|z_k|^2) < +\infty\}$. We give necessary and sufficient conditions for the weighted composition operator ψC_φ induced by $\varphi(z)$ and $\psi(z)$ to be bounded and compact from $H^\infty(\mathbb{D}^n)$ to the Bloch space $\mathfrak{B}(\mathbb{D}^n)$.

1. Introduction

Let $\mathbb{D}^n$ be the unit polydisc of $\mathbb{C}^n$. The class of all holomorphic functions with domain $\mathbb{D}^n$ will be denoted by $H(\mathbb{D}^n)$. Let φ be a holomorphic self-map of $\mathbb{D}^n$, the composition operator C_φ induced by φ is defined by $(C_\varphi f)(z) = f(\varphi(z))$ for $z \in \mathbb{D}^n$ and $f \in H(\mathbb{D}^n)$. If, in addition, ψ is a holomorphic function defined on $\mathbb{D}^n$, the weighted composition operator ψC_φ induced by ψ and φ is defined by $\psi C_\varphi(z) = \psi(z) f(\varphi(z))$ for z in $\mathbb{D}^n$ and $f \in H(\mathbb{D}^n)$.

A function f holomorphic in $\mathbb{D}^n$ is said to belong to the Bloch space $\mathcal{B}(\mathbb{D}^n)$ if

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}^n} \sum_{k=1}^{n} \left|\frac{\partial f}{\partial z_k}(z)\right| (1-|z_k|^2) < +\infty. \tag{1.1}$$

It is easy to show that $\mathcal{B}(\mathbb{D}^n)$ is a Banach space with the norm $\|\cdot\|_{\mathcal{B}}$ (see, e.g., [1]).

As usual, $H^\infty(\mathbb{D}^n)$ denotes the space of all bounded holomorphic functions on $\mathbb{D}^n$ with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}^n}|f(z)|$, that is,

$$H^\infty(\mathbb{D}^n) = \Big\{ f \in H(\mathbb{D}^n) \mid \|f\|_\infty = \sup_{z\in\mathbb{D}^n} |f(z)| < \infty \Big\}. \tag{1.2}$$

Weighted composition operators between $H^\infty(\mathbb{D})$ and the Bloch space $\mathcal{B}(\mathbb{D})$ were investigated in [2] which was the starting point for our investigations for the case of n-dimensional settings. The corresponding results for the unit ball were obtained in [3]. Composition operators between Bloch type spaces on the unit ball are investigated by Shi and Luo in [4]. In [5] the second author investigated the composition operators from $H^\infty(\mathbb{D}^n)$ to the Bloch space $\mathcal{B}(\mathbb{D}^n)$. Composition operators between Bloch spaces on the unit polydisc are investigated in [6, 7] where some sufficient and necessary conditions are given so that C_φ be compact on the Bloch space. The following statement was formulated in [7]: let $\varphi = (\varphi_1,\dots,\varphi_n)$ be a holomorphic self-map of $\mathbb{D}^n$, then C_φ is compact on $\mathcal{B}(\mathbb{D}^n)$ if and only if for every $\varepsilon > 0$, there is a $\delta \in (0,1)$ such that

$$\sum_{k,l=1}^{n} \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{1-|z_k|^2}{1-|\varphi_l(z)|^2} < \varepsilon, \tag{1.3}$$

whenever $\operatorname{dist}(\varphi(z),\partial\mathbb{D}^n) < \delta$. However, the proof of necessity contains a gap. More specifically, if $(z^j)_{j\in\mathbb{N}}$ is a sequence in $\mathbb{D}^n$ such that $\varphi(z^j) \to \partial\mathbb{D}^n$ as $j \to \infty$, and if inequality (3.13) in [7, page 289] holds, then one cannot omit consideration of the case when $|\varphi_1(z^j)| \nrightarrow 1$ as $j \to \infty$.

The method in [5] can be used to correct the proof of the results. For some basics on composition operators, see, for example, [9]. Closely related results devoted to some operators on the polydisc can be found, for example, in [8, 10–12].

In this paper, we study the weighted composition operator from $H^\infty(\mathbb{D}^n)$ to the Bloch space on the polydisc. The main results in the paper extend in [2, Theorems 2 and 3] (where one-dimensional case was considered) and those ones in [5]. The proofs are modifications of the corresponding ones in [2, 5].

Theorem 1.1. *Let $\varphi = (\varphi_1,\dots,\varphi_n)$ be a holomorphic self-map of $\mathbb{D}^n$ and $\psi(z)$ a holomorphic function on $\mathbb{D}^n$. Then $\psi C_\varphi : H^\infty(\mathbb{D}^n) \to \mathcal{B}(\mathbb{D}^n)$ is bounded if and only if the following conditions are satisfied:*

(i) $\psi \in \mathcal{B}(\mathbb{D}^n)$;

(ii)

$$\sup_{z\in\mathbb{D}^n} |\psi(z)| \sum_{k,l=1}^{n} \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{1-|z_k|^2}{1-|\varphi_l(z)|^2} < \infty. \tag{1.4}$$

Theorem 1.2. *Let $\varphi = (\varphi_1,\dots,\varphi_n)$ be a holomorphic self-map of $\mathbb{D}^n$ and $\psi(z)$ a holomorphic function of $\mathbb{D}^n$. Then $\psi C_\varphi : H^\infty(\mathbb{D}^n) \to \mathcal{B}(\mathbb{D}^n)$ is compact if and only if the following* *conditions are satisfied:*

(i) $\psi C_\varphi : H^\infty(\mathbb{D}^n) \to \mathcal{B}(\mathbb{D}^n)$ *is bounded;*

(ii)

$$\sum_{k=1}^{n}\left(1-|z_k|^2\right)\left|\frac{\partial\psi}{\partial z_k}(z)\right| = o(1) \quad (\textit{as } \varphi(z)\longrightarrow\partial\mathbb{D}^n); \tag{1.5}$$

(iii)

$$|\psi(z)|\sum_{k,l=1}^{n}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\frac{1-|z_k|^2}{1-|\varphi_l(z)|^2} = o(1) \quad (\textit{as } \varphi(z)\longrightarrow\partial\mathbb{D}^n). \tag{1.6}$$

Throughout the remainder of this paper C will denote a positive constant, the exact value of which may vary from one appearance to the next.

2. Auxiliary results

In this section we prove some auxiliary results which we use in the proof of the main results. The first two lemmas could be folklore. For a proof of the first lemma see, for example, [5].

Lemma 2.1. *Let* $f \in \mathcal{B}(\mathbb{D}^n)$. *Then*

$$|f(z)| \le |f(0)| + \|f\|_{\mathcal{B}}\sum_{j=1}^{n}\ln\frac{1}{1-|z_j|}. \tag{2.1}$$

A proof of the following lemma can be also found in [5]. We present here another proof for the benefit of the reader.

Lemma 2.2. *If* $f \in H^\infty(\mathbb{D}^n)$, *then*

$$\left|\frac{\partial f}{\partial z_k}(z)\right| = O\left(\frac{1}{1-|z_k|^2}\right), \tag{2.2}$$

that is, the inclusion $H^\infty(\mathbb{D}^n)\subset\mathcal{B}(\mathbb{D}^n)$ *holds. Moreover, there is a positive constant* C *independent of* f *such that*

$$\|f\|_{\mathcal{B}} \le C\|f\|_\infty. \tag{2.3}$$

Proof. For $k\in\{1,\dots,n\}$, let $u=(z_1,\dots,z_{k-1},u_k,z_{k+1},\dots,z_n)$. Assume that $f\in H^\infty(\mathbb{D}^n)$, then by a well-known result, we have

$$f(z) = \int_{|u_k|<1}\frac{f(u)}{(1-z_k\overline{u}_k)^2}\,dm(u_k), \tag{2.4}$$

where $dm(\cdot)$ is the normalized Lebesgue area measure on the unit disk.

Hence,

$$\frac{\partial f}{\partial z_k}(z) = \int_{|u_k|<1} \frac{2\overline{u}_k f(u)}{(1 - z_k\overline{u}_k)^3} dm(u_k). \tag{2.5}$$

By [13, Theorem 1.4.10], we have

$$\left|\frac{\partial f}{\partial z_k}(z)\right| \le 2\int_{|u_k|<1} \frac{|f(u)|}{|1 - z_k\overline{u}_k|^3} dm(u_k) \le \frac{C\|f\|_\infty}{1 - |z_k|^2}, \tag{2.6}$$

which implies that

$$\sup_{z\in\mathbb{D}^n} (1 - |z_k|^2) \left|\frac{\partial f}{\partial z_k}(z)\right| \le C\|f\|_\infty. \tag{2.7}$$

Therefore, we have

$$\|f\|_{\mathcal{B}} \le |f(0)| + Cn\|f\|_\infty \le (1 + Cn)\|f\|_\infty \tag{2.8}$$

as desired. □

Lemma 2.3. *Suppose that* $\psi C_\varphi : H^\infty(\mathbb{D}^n) \to \mathcal{B}(\mathbb{D}^n)$ *is bounded. Then the operator* $\psi C_\varphi : H^\infty(\mathbb{D}^n) \to \mathcal{B}(\mathbb{D}^n)$ *is compact if and only if for any bounded sequence* $(f_k)_{k\in\mathbb{N}}$ *in* $H^\infty(\mathbb{D}^n)$ *converging to zero uniformly on compact subsets of* $\mathbb{D}^n$*, one has* $\lim_{k\to\infty} \|\psi C_\varphi f_k\|_{\mathcal{B}} = 0$.

Proof. Assume that ψC_φ is compact and assume that $(f_k)_{k\in\mathbb{N}}$ is a sequence in $H^\infty(\mathbb{D}^n)$ with $\sup_{k\in\mathbb{N}} \|f_k\|_\infty < \infty$ and $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}^n$, as $k \to \infty$. By the compactness of ψC_φ we have that the sequence $(\psi C_\varphi(f_k))_{k\in\mathbb{N}}$ has a subsequence $(\psi C_\varphi(f_{k_m}))_{m\in\mathbb{N}}$ which converges in $\mathcal{B}(\mathbb{D}^n)$, say, to f. By Lemma 2.1 and $|f(0)| \le \|f\|_{\mathcal{B}}$ we have that for any compact $K \subset \mathbb{D}^n$ there is a positive constant C_K independent of f such that

$$|\psi C_\varphi(f_{k_m})(z) - f(z)| \le C_K\|\psi C_\varphi(f_{k_m}) - f\|_{\mathcal{B}}, \tag{2.9}$$

for all $z \in K$. This implies that $\psi C_\varphi(f_{k_m})(z) - f(z) \to 0$ uniformly on compact subsets of $\mathbb{D}^n$, as $m \to \infty$. Since $f_{k_m} \to 0$ on compacts, by the definition of the operator ψC_φ it is easy to see that for each $z \in \mathbb{D}^n$, $\lim_{m\to\infty} \psi C_\varphi(f_{k_m})(z) = 0$. Hence the limit function f is equal to 0. Since this is true for arbitrary subsequence of $(f_k)_{k\in\mathbb{N}}$, we obtain that $\psi C_\varphi(f_k) \to 0$ in $\mathcal{B}(\mathbb{D}^n)$, as $k \to \infty$.

Conversely, let $(h_k)_{k\in\mathbb{N}}$ be any sequence in the ball $\mathcal{K}_M = B_{H^\infty}(0,M)$ of the space $H^\infty(\mathbb{D}^n)$. Since $\sup_{k\in\mathbb{N}} \|h_k\|_\infty \le M < \infty$, the sequence $(h_k)_{k\in\mathbb{N}}$ is uniformly bounded on compact subsets of $\mathbb{D}^n$ and hence normal by Montel's theorem. Hence we may extract a subsequence $(h_{k_j})_{j\in\mathbb{N}}$ which converges uniformly on compact subsets of $\mathbb{D}^n$ to some $h \in H(\mathbb{D}^n)$, moreover $h \in H^\infty(\mathbb{D}^n)$ and $\|h\|_\infty \le M$, hence the sequence $(h_{k_j} - h)_{j\in\mathbb{N}}$ is such that $\|h_{k_j} - h\|_\infty \le 2M < \infty$, and converges to 0 on compact subsets of $\mathbb{D}^n$. By the hypothesis we have that

$$\psi h_{k_j} \circ \varphi \longrightarrow \psi h \circ \varphi \tag{2.10}$$

in $\mathcal{B}(\mathbb{D}^n)$. Thus the set $\psi C_\varphi(\mathcal{K}_M)$ is relatively compact, as desired. □

3. Proof of the main results

Let $w=\varphi(z)$ in this section. Now we prove the main results of this paper.

Proof of Theorem 1.1. Suppose that (i) and (ii) hold. For a function $f\in H^\infty(\mathbb{D}^n)$, we have

$$\begin{aligned}\sum_{k=1}^n\left|\frac{\partial(\psi C_\varphi f)}{\partial z_k}\right|(1-|z_k|^2)\\
&\le\sum_{k=1}^n(1-|z_k|^2)\left|\frac{\partial\psi}{\partial z_k}(z)\right||f(\varphi(z))|\\
&\quad+\sum_{k=1}^n\sum_{l=1}^n|\psi(z)|\left|\frac{\partial f}{\partial w_l}(\varphi(z))\right|\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|(1-|z_k|^2)\\
&\le\|\psi\|_{\mathcal{B}}\|f\|_\infty+\sum_{k=1}^n\sum_{l=1}^n|\psi(z)|\left|\frac{\partial f}{\partial w_l}(\varphi(z))\right|\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\frac{1-|z_k|^2}{1-|\varphi_l(z)|^2}(1-|\varphi_l(z)|^2)\\
&\le\|\psi\|_{\mathcal{B}}\|f\|_\infty+\|f\|_{\mathcal{B}}\sum_{k=1}^n\sum_{l=1}^n|\psi(z)|\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\frac{1-|z_k|^2}{1-|\varphi_l(z)|^2}.\end{aligned}\tag{3.1}$$

Since by Lemma 2.2 $\|f\|_{\mathcal{B}}\le C\|f\|_\infty$ for every $f\in H^\infty(\mathbb{D}^n)$ and by conditions (i) and (ii), it follows that the last quantity above is bounded by some constant multiplied by $\|f\|_\infty$. Hence, the operator $\psi C_\varphi : H^\infty(\mathbb{D}^n)\to\mathcal{B}(\mathbb{D}^n)$ is bounded.

Conversely, suppose that $\psi C_\varphi : H^\infty(\mathbb{D}^n)\to\mathcal{B}(\mathbb{D}^n)$ is bounded, that is, there is a constant C (e.g., $C=\|\psi C_\varphi\|_{H^\infty\to\mathcal{B}}$) such that

$$||\psi C_\varphi f||_{\mathcal{B}}\le C\|f\|_\infty\tag{3.2}$$

for all $f\in H^\infty(\mathbb{D}^n)$. Taking $f(z)\equiv 1$ and $f(z)=z_l$, $l\in\{1,\dots,n\}$, it follows that $\psi\in\mathcal{B}(\mathbb{D}^n)$ and

$$\sup_{z\in\mathbb{D}^n}\sum_{k=1}^n|\psi(z)|(1-|z_k|^2)\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|<\infty,\tag{3.3}$$

for every $l\in\{1,\dots,n\}$.

For fixed l $(1\le l\le n)$ and $\lambda\in\mathbb{D}^n$, if $\varphi_l(\lambda)\ne 0$, we define the following family of test functions

$$f(z)=\frac{1-|\varphi_l(\lambda)|^2}{1-\overline{\varphi_l(\lambda)}z_l}.\tag{3.4}$$

It is easy to see that $f\in H^\infty(\mathbb{D}^n)$ and $\|f\|_\infty\le 2$. Therefore we have

$$\begin{aligned}2||\psi C_\varphi||_{H^\infty\to\mathcal{B}}\ge||\psi C_\varphi f||_{\mathcal{B}}\ge&\sum_{k=1}^n\frac{|\psi(\lambda)|}{1-|\varphi_l(\lambda)|^2}\left|\frac{\partial\varphi_l}{\partial\lambda_k}(\lambda)\right||\varphi_l(\lambda)|(1-|\lambda_k|^2)\\
&-\sum_{k=1}^n\left|\frac{\partial\psi}{\partial\lambda_k}(\lambda)\right|(1-|\lambda_k|^2).\end{aligned}\tag{3.5}$$

From this and since $\psi \in \mathcal{B}(\mathbb{D}^n)$, we obtain

$$\sup_{\lambda\in\mathbb{D}^n}\sum_{k=1}^{n}\frac{|\psi(\lambda)|}{1-|\varphi_l(\lambda)|^2}\left|\frac{\partial\varphi_l}{\partial\lambda_k}(\lambda)\right||\varphi_l(\lambda)|(1-|\lambda_k|^2)<\infty. \tag{3.6}$$

Thus, for a fixed $\delta\in(0,1)$, by (3.6) we have that for each $l\in\{1,\dots,n\}$

$$\sup_{\lambda\in\mathbb{D}^n}\left\{\sum_{k=1}^{n}\frac{|\psi(\lambda)|}{1-|\varphi_l(\lambda)|^2}\left|\frac{\partial\varphi_l}{\partial\lambda_k}(\lambda)\right|(1-|\lambda_k|^2):|\varphi_l(\lambda)|>\delta\right\}<\infty. \tag{3.7}$$

For $\lambda\in\mathbb{D}^n$ such that $|\varphi_l(\lambda)|\le\delta$, we have

$$\sum_{k=1}^{n}\frac{|\psi(\lambda)|}{1-|\varphi_l(\lambda)|^2}\left|\frac{\partial\varphi_l}{\partial\lambda_k}(\lambda)\right|(1-|\lambda_k|^2)\le\frac{1}{1-\delta^2}\sum_{k=1}^{n}|\psi(\lambda)|\left|\frac{\partial\varphi_l}{\partial\lambda_k}(\lambda)\right|(1-|\lambda_k|^2). \tag{3.8}$$

Hence, by (3.3) we have

$$\sup_{\lambda\in\mathbb{D}^n}\left\{\sum_{k=1}^{n}\frac{|\psi(\lambda)|}{1-|\varphi_l(\lambda)|^2}\left|\frac{\partial\varphi_l}{\partial\lambda_k}(\lambda)\right|(1-|\lambda_k|^2):|\varphi_l(\lambda)|\le\delta\right\}<\infty. \tag{3.9}$$

Consequently, by (3.7) and (3.9), for each $l\in\{1,\dots,n\}$

$$\sup_{\lambda\in\mathbb{D}^n,\varphi_l(\lambda)\neq 0}\sum_{k=1}^{n}|\psi(\lambda)|\frac{1-|\lambda_k|^2}{1-|\varphi_l(\lambda)|^2}\left|\frac{\partial\varphi_l}{\partial\lambda_k}(\lambda)\right|<\infty. \tag{3.10}$$

If $\varphi_l(\lambda)=0$ for some $l\in\{1,\dots,n\}$, set $f(z)=z_l$. From (3.2) it follows that

$$|\psi(\lambda)|\left|\frac{\partial\varphi_l}{\partial z_k}(\lambda)\right|\frac{1-|\lambda_k|^2}{1-|\varphi_l(\lambda)|^2}=\left|\frac{\partial\psi}{\partial z_k}(\lambda)\varphi_l(\lambda)+\psi(\lambda)\frac{\partial\varphi_l}{\partial z_k}(\lambda)\right|(1-|\lambda_k|^2)\le C. \tag{3.11}$$

Hence for any $z\in\mathbb{D}^n$, we have

$$\sup_{z\in\mathbb{D}^n}|\psi(z)|\sum_{k,l=1}^{n}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\frac{1-|z_k|^2}{1-|\varphi_l(z)|^2}\le C, \tag{3.12}$$

which completes the proof of the theorem. □

Proof of Theorem 1.2. Assume that $\psi C_\varphi : H^\infty(\mathbb{D}^n)\to\mathcal{B}(\mathbb{D}^n)$ is bounded, and that (1.5) and (1.6) hold. Further, assume that a sequence $(f_j)_{j\in\mathbb{N}}$ is such that $\sup_{j\in\mathbb{N}}\|f_j\|_\infty\le C$ and f_j converges to zero uniformly on compact subsets of $\mathbb{D}^n$. We need to prove $\|\psi C_\varphi f_j\|_{\mathcal{B}}\to 0$ as $j\to\infty$.

Since $\psi C_\varphi : H^\infty(\mathbb{D}^n) \to \mathcal{B}(\mathbb{D}^n)$ is bounded, by Theorem 1.1 we have

$$\sup_{z\in\mathbb{D}^n} \sum_{k=1}^n (1-|z_k|^2)\left|\frac{\partial\psi}{\partial z_k}(z)\right| \le C, \tag{3.13}$$

$$\sup_{z\in\mathbb{D}^n} |\psi(z)| \sum_{k,l=1}^n \left|\frac{\partial\varphi_l}{\partial z_k}(z)\right| \frac{1-|z_k|^2}{1-|\varphi_l(z)|^2} \le C. \tag{3.14}$$

Conditions (ii) and (iii) imply that for every $\varepsilon > 0$ there exists an $r \in (0,1)$ such that

$$\begin{gathered} \sum_{k=1}^n (1-|z_k|^2)\left|\frac{\partial\psi}{\partial z_k}(z)\right| < \varepsilon, \\ |\psi(z)| \sum_{k,l=1}^n \left|\frac{\partial\varphi_l}{\partial z_k}(z)\right| \frac{1-|z_k|^2}{1-|\varphi_l(z)|^2} < \varepsilon \end{gathered} \tag{3.15}$$

whenever $\operatorname{dist}(\varphi(z), \partial\mathbb{D}^n) < r$. Hence we have

$$\begin{aligned} &\sum_{k=1}^n \left|\frac{\partial(\psi C_\varphi f_j)}{\partial z_k}\right| (1-|z_k|^2) \\ &\quad \le \sum_{k=1}^n (1-|z_k|^2)\left|\frac{\partial\psi}{\partial z_k}(z)\right| |f_j(\varphi(z))| \\ &\qquad + \sum_{k,l=1}^n |\psi(z)| \left|\frac{\partial f_j}{\partial w_l}(\varphi(z))\right| \left|\frac{\partial\varphi_l}{\partial z_k}(z)\right| (1-|z_k|^2) \\ &\quad \le \|f_j\|_\infty \sum_{k=1}^n (1-|z_k|^2)\left|\frac{\partial\psi}{\partial z_k}(z)\right| + \|f_j\|_{\mathcal{B}} \sum_{k,l=1}^n |\psi(z)| \left|\frac{\partial\varphi_l}{\partial z_k}(z)\right| \frac{(1-|z_k|^2)}{(1-|\varphi_l(z)|^2)} \le C\varepsilon, \end{aligned} \tag{3.16}$$

whenever $\operatorname{dist}(\varphi(z), \partial\mathbb{D}^n) < r$, where the last inequality comes from (3.15) .

On the other hand, let $E = \{w \in \mathbb{D}^n : \operatorname{dist}(w, \partial\mathbb{D}^n) \ge r\}$. Then, E is a compact subset of $\mathbb{D}^n$. Hence, $f_j(w) \to 0$ uniformly on E as $j \to \infty$, and from this and by the Cauchy estimate we have that $(\partial f_j/\partial z_k)(w) \to 0$ uniformly on E as $j \to \infty$. Hence

$$\begin{aligned} &\sum_{k=1}^n \left|\frac{\partial(\psi C_\varphi f_j)}{\partial z_k}(z)\right| (1-|z_k|^2) \\ &\quad \le \sum_{k,l=1}^n |\psi(z)| \left|\frac{\partial f_j}{\partial w_l}(\varphi(z))\right| \left|\frac{\partial\varphi_l}{\partial z_k}(z)\right| (1-|z_k|^2) \\ &\qquad + \sum_{k=1}^n (1-|z_k|^2)\left|\frac{\partial\psi}{\partial z_k}(z)\right| |f_j(\varphi(z))| \\ &\quad \le C\left(\sup_{w\in E} |f_j(w)| + \sum_{l=1}^n \sup_{w\in E}\left|\frac{\partial f_j}{\partial w_l}(w)\right|\right) \le C\varepsilon, \end{aligned} \tag{3.17}$$

where we have used inequalities (3.13) and (3.14). Since $\lim_{j\to\infty}\psi(0)f_j(\varphi(0))=0$, by using (3.16) and (3.17) and the fact that ε is an arbitrary positive number, we obtain that $\lim_{j\to\infty}\|\psi C_\varphi f_j\|_{\mathcal{B}}=0$. Hence, by Lemma 2.3 the implication follows.

Suppose that $\psi C_\varphi : H^\infty(\mathbb{D}^n)\to\mathcal{B}(\mathbb{D}^n)$ is compact. Then $\psi C_\varphi : H^\infty(\mathbb{D}^n)\to\mathcal{B}(\mathbb{D}^n)$ is bounded. We need to prove (1.5) and (1.6).

First we prove (1.5). Assume that (1.5) fails, then there is a sequence $(z^j)_{j\in\mathbb{N}}$ in $\mathbb{D}^n$ such that $w^j=\varphi(z^j)\to\partial\mathbb{D}^n$, as $j\to\infty$, and $\varepsilon_0>0$, such that

$$\sum_{k=1}^{n}\left(1-|z_k^j|^2\right)\left|\frac{\partial\psi}{\partial z_k}(z^j)\right|\geq\varepsilon_0 \tag{3.18}$$

for all $j\in\mathbb{N}$. Since ψC_φ is bounded, by Theorem 1.1, we know that $\psi\in\mathcal{B}(\mathbb{D}^n)$. Hence there is a subsequence of $(z^j)_{j\in\mathbb{N}}$ (we keep the same notation $(z^j)_{j\in\mathbb{N}}$) such that for any $k\in\{1,\dots,n\}$,

$$\left(1-|z_k^j|^2\right)\left|\frac{\partial\psi}{\partial z_k}(z^j)\right| \tag{3.19}$$

converges to a finite number as $j\to\infty$. We may assume that for every $l\in\{1,\dots,n\}$ there is finite limit $\lim_{j\to\infty}|w_l^j|$, where w_l^j denote $\varphi_l(z^j)$, and we may assume that

$$\lim_{j\to\infty}\left(1-|z_{k_0}^j|^2\right)\left|\frac{\partial\psi}{\partial z_{k_0}}(z^j)\right|=\varepsilon_1>0 \tag{3.20}$$

for some $k_0\in\{1,\dots,n\}$.

We construct a sequence of functions $(g_j)_{j\in\mathbb{N}}$ satisfying the following conditions:

(a) $(g_j)_{j\in\mathbb{N}}$ is a bounded sequence in $H^\infty(\mathbb{D}^n)$;

(b) $(g_j)_{j\in\mathbb{N}}$ tends to zero uniformly on compact subset of $\mathbb{D}^n$;

(c) $\|\psi C_\varphi g_j\|_{\mathcal{B}}\nrightarrow 0$, as $j\to\infty$,

and by Lemma 2.3 we will arrive at a contradiction.

Since $w^j\to\partial\mathbb{D}^n$ it follows that there is $s\in\{1,\dots,n\}$ such that $|w_s^j|\to 1$ as $j\to\infty$. We use the following test functions:

$$g_{w_s^j}(z)=2\frac{1-|w_s^j|^2}{1-\overline{w_s^j}z_s}-\frac{\left(1-|w_s^j|^2\right)^2}{\left(1-\overline{w_s^j}z_s\right)^2}. \tag{3.21}$$

It is easy to see that $(g_{w_s^j})_{j\in\mathbb{N}}$ is a bounded sequence in $H^\infty(\mathbb{D}^n)$ and $g_{w_s^j}(z)\to 0$ uniformly on compacts of $\mathbb{D}^n$. By Lemma 2.3, it follows that $\|\psi C_\varphi g_{w_s^j}\|_{\mathcal{B}}\to 0$ as $j\to\infty$.

On the other hand, we see that $g_{w_s^j}(\varphi(z^j))=1$ and

$$\frac{\partial g_{w_s^j}}{\partial z_s}(\varphi(z^j))=0. \tag{3.22}$$

Combining (3.18) and (3.20) with these, we have

$$\begin{aligned}
\|\psi C_\varphi g_{w_s^j}\|_{\mathcal{B}} &\geq \sum_{k=1}^{n}\left(1-|z_k^j|^2\right)\left|\frac{\partial\psi}{\partial z_k}(z^j)g_{w_s^j}(\varphi(z^j))+\psi(z^j)\sum_{l=1}^{n}\frac{\partial g_{w_s^j}}{\partial\zeta_l}(\varphi(z^j))\frac{\partial\varphi_l}{\partial z_k}(z^j)\right| \\
&= \sum_{k=1}^{n}\left(1-|z_k^j|^2\right)\left|\frac{\partial\psi}{\partial z_k}(z^j)\right| \\
&\geq \left(1-|z_{k_0}^j|^2\right)\left|\frac{\partial\psi}{\partial z_{k_0}}(z^j)\right| \longrightarrow \varepsilon_1>0, \quad \text{as } j\longrightarrow\infty,
\end{aligned} \tag{3.23}$$

which is a contradiction.

Now assume that (1.6) fails, then there is a sequence $(z^j)_{j\in\mathbb{N}}$ in $\mathbb{D}^n$ such that $w^j = \varphi(z^j)\to\partial\mathbb{D}^n$ as $j\to\infty$, and $\varepsilon_2>0$ such that

$$\sum_{k,l=1}^{n}|\psi(z^j)|\left|\frac{\partial\varphi_l}{\partial z_k}(z^j)\right|\frac{1-|z_k^j|^2}{1-|\varphi_l(z^j)|^2}\geq\varepsilon_2 \tag{3.24}$$

for all $j\in\mathbb{N}$. On the other hand, by Theorem 1.1, we know that (1.4) holds. Hence, there is a subsequence of $(z^j)_{j\in\mathbb{N}}$ (we keep the same notation (z^j)) such that

$$|\psi(z^j)|\left|\frac{\partial\varphi_l}{\partial z_k}(z^j)\right|\frac{1-|z_k^j|^2}{1-|\varphi_l(z^j)|^2} \tag{3.25}$$

for any $k,l\in\{1,\dots,n\}$, converges to a finite number as $j\to\infty$. Also we may assume that for every $l\in\{1,\dots,n\}$ there is finite limit $\lim_{j\to\infty}|w_l^j|$. From (3.24) and (3.25), without loss of generality we may assume that

$$\lim_{j\to\infty}|\psi(z^j)|\left|\frac{\partial\varphi_1}{\partial z_{k_0}}(z^j)\right|\frac{1-|z_{k_0}^j|^2}{1-|\varphi_1(z^j)|^2}=\varepsilon_3>0, \tag{3.26}$$

for some $k_0\in\{1,\dots,n\}$.

As above we construct a sequence of functions $(f_j)_{j\in\mathbb{N}}$ satisfying the following conditions:

(a) $(f_j)_{j\in\mathbb{N}}$is a bounded sequence in $H^\infty(\mathbb{D}^n)$;

(b) $(f_j)_{j\in\mathbb{N}}$ tends to zero uniformly on compact subset of $\mathbb{D}^n$;

(c) $\|\psi C_\varphi f_j\|_{\mathcal{B}}\nrightarrow 0$, as $j\to\infty$,

arriving at a contradiction.

Case 1. Assume that $|w_1^j|\to 1$ as $j\to\infty$ and

$$f_{w_1^j}(z)=\frac{1-|w_1^j|^2}{1-\overline{w_1^j}z_1}-\left(\frac{1-|w_1^j|^2}{1-\overline{w_1^j}z_1}\right)^{1/2}. \tag{3.27}$$

Then $f_{w_1^j}(z)$ is a bounded sequence in $H^\infty(\mathbb{D}^n)$ and $f_{w_1^j}(z) \to 0$ uniformly on every compact subset of $\mathbb{D}^n$. Moreover, $f_{w_1^j}(w_1^j) = 0$ and

$$\frac{\partial f_{w_1^j}(w_1^j)}{\partial z_1} = \frac{\overline{w_1^j}}{2(1-|w_1^j|^2)}, \quad \frac{\partial f_{w_1^j}}{\partial z_l}(z) = 0,\ l \neq 1. \tag{3.28}$$

We show that $\|\psi C_\varphi f_{w_1^j}\|_{\mathcal{B}} \nrightarrow 0$ as $j \to \infty$. Let

$$I_{f_{w_1^j}}(z^j) = |\psi(z^j)| \sum_{k=1}^{n} (1-|z_k^j|^2) \left| \frac{\partial f_{w_1^j}}{\partial z_1}(\varphi(z^j)) \right| \left| \frac{\partial \varphi_1(z^j)}{\partial z_k} \right|. \tag{3.29}$$

Then we have

$$\begin{aligned} \|\psi C_\varphi f_{w_1^j}\|_{\mathcal{B}} &\geq I_{f_{w_1^j}}(z^j) = \frac{1}{2}|\psi(z^j)| \sum_{k=1}^{n} \frac{1-|z_k^j|^2}{1-|\varphi_1(z^j)|^2} |w_1^j| \left| \frac{\partial \varphi_1}{\partial z_k}(z^j) \right| \\ &\geq \frac{1}{2}|\psi(z^j)| \frac{1-|z_{k_0}^j|^2}{1-|\varphi_1(z^j)|^2} |w_1^j| \left| \frac{\partial \varphi_1}{\partial z_{k_0}}(z^j) \right| \longrightarrow \frac{\varepsilon_3}{2} > 0 \end{aligned} \tag{3.30}$$

as $j \to \infty$. From which the result follows in this case.

Case 2. Assume that $|w_1^j| \to \rho < 1$ as $j \to \infty$. Since $w^j \to \partial\mathbb{D}^n$ there is an $l \in \{2,\ldots,n\}$ such that $|w_l^j| \to 1$ as $j \to \infty$. If there is a $k_1 \in \{1,\ldots,n\}$ and $\varepsilon_4 > 0$ such that

$$\lim_{j\to\infty} |\psi(z^j)| \left| \frac{\partial \varphi_l}{\partial z_{k_1}}(z^j) \right| \frac{1-|z_{k_1}^j|^2}{1-|\varphi_l(z^j)|^2} = \varepsilon_4 > 0, \tag{3.31}$$

then we obtain a contradiction using the following test function:

$$g_{w_l^j}(z) = \frac{1-|w_l^j|^2}{1-\overline{w_l^j} z_l} - \left(\frac{1-|w_l^j|^2}{1-\overline{w_l^j} z_l} \right)^{1/2} \tag{3.32}$$

similarly as in Case 1.

Hence, we may assume that

$$\lim_{j\to\infty} |\psi(z^j)| \left| \frac{\partial \varphi_l}{\partial z_k}(z^j) \right| \frac{1-|z_k^j|^2}{1-|\varphi_l(z^j)|^2} = 0, \tag{3.33}$$

for each $k \in \{1,\ldots,n\}$.

Set

$$f_{w_l^j}(z) = (z_1+2) \frac{1-|w_l^j|^2}{1-\overline{w_l^j} z_l} - (w_1^j+2) \left(\frac{1-|w_l^j|^2}{1-\overline{w_l^j} z_l} \right)^{1/2}. \tag{3.34}$$

It is easy to see that $\|f_{w_l^j}\|_\infty \le 12$ and that $f_{w_l^j}(z) \to 0$ uniformly on every compact subsets of $\mathbb{D}^n$. Moreover, $f_{w_l^j}(\varphi(z^j)) = 0$ and

$$\frac{\partial f_{w_l^j}}{\partial z_1}(\varphi(z^j)) = 1, \quad \frac{\partial f_{w_l^j}}{\partial z_l}(\varphi(z^j)) = \frac{2+w_1^j}{2}\frac{\overline{w_l^j}}{(1-|w_l^j|^2)}, \quad \frac{\partial f_{w_l^j}(z)}{\partial z_m} = 0, \quad (m \ne 1, l). \tag{3.35}$$

We show that $\|\psi C_\varphi f_{w_l^j}\|_{\mathcal{B}} \nrightarrow 0$ as $j \to \infty$. Let

$$\begin{aligned} I_j &= |\psi(z^j)| \sum_{k=1}^n (1-|z_k^j|^2) \left| \frac{\partial f_{w_l^j}}{\partial \zeta_1}(\varphi(z^j)) \frac{\partial \varphi_1}{\partial z_k}(z^j) \right| \\ &= |\psi(z^j)| \sum_{k=1}^n (1-|z_k^j|^2) \left| \frac{\partial \varphi_1}{\partial z_k}(z^j) \right|. \end{aligned} \tag{3.36}$$

We have

$$\begin{aligned} I_j &\le |\psi(z^j)| \sum_{k=1}^n (1-|z_k^j|^2) \left| \frac{\partial f_{w_l^j}}{\partial \zeta_l}(\varphi(z^j)) \frac{\partial \varphi_l(z^j)}{\partial z_k} \right| \\ &\quad + \|\psi C_\varphi f_{w_l^j}\|_{\mathcal{B}} + |f_{w_l^j}(\varphi(z^j))| \sum_{k=1}^n \left| \frac{\partial \psi}{\partial z_k}(z^j) \right| (1-|z_k^j|^2) \\ &\le |\psi(z^j)| \sum_{k=1}^n (1-|z_k^j|^2) \left| \frac{\partial \varphi_l(z^j)}{\partial z_k} \right| \left| \frac{2+w_1^j}{2} \right| \frac{|\overline{w_l^j}|}{(1-|w_l^j|^2)} \\ &\quad + \|\psi C_\varphi f_{w_l^j}\|_{\mathcal{B}} + |f_{w_l^j}(\varphi(z^j))| \sum_{k=1}^n \left| \frac{\partial \psi}{\partial z_k}(z^j) \right| (1-|z_k^j|^2). \\ &\le \frac{3}{2} |\psi(z^j)| \sum_{k=1}^n \frac{1-|z_k^j|^2}{1-|w_l^j|^2} \left| \frac{\partial \varphi_l}{\partial z_k}(z^j) \right| \\ &\quad + \|\psi C_\varphi f_{w_l^j}\|_{\mathcal{B}} + \|f_{w_l^j}\|_\infty \sum_{k=1}^n \left| \frac{\partial \psi}{\partial z_k}(z^j) \right| (1-|z_k^j|^2). \end{aligned} \tag{3.37}$$

As we have already proved

$$\lim_{\varphi(z) \to \partial \mathbb{D}^n} \sum_{k=1}^n \left| \frac{\partial \psi}{\partial z_k}(z) \right| (1-|z_k|^2) = 0. \tag{3.38}$$

Hence we have that

$$\lim_{j \to \infty} \sum_{k=1}^n \left| \frac{\partial \psi}{\partial z_k}(z^j) \right| (1-|z_k^j|^2) = 0. \tag{3.39}$$

Letting $j \to \infty$ and using (3.33) and (3.39), it follows that

$$\liminf_{j \to \infty} I_j \le \liminf_{j \to \infty} \|\psi C_\varphi f_{w_l^j}\|_{\mathcal{B}}. \tag{3.40}$$

On the other hand, $|w_1^j| \le \rho < 1$ for sufficiently large j. Thus we have that

$$\begin{aligned}(1-\rho^2)\varepsilon_3/2 &\le (1-\rho^2)\,|\psi(z^j)|\left|\frac{\partial \varphi_1}{\partial z_{k_0}}(z^j)\right|\frac{1-|z_{k_0}^j|^2}{1-|\varphi_1(z^j)|^2}\\ &\le |\psi(z^j)|\sum_{k=1}^{n}(1-|z_k^j|^2)\left|\frac{\partial \varphi_1}{\partial z_k}(z^j)\right| = I_j,\end{aligned} \tag{3.41}$$

for sufficiently large j.

Combining (3.40) with (3.41), we obtain that

$$0<(1-\rho^2)\varepsilon_3/2 \le \liminf_{j\to\infty}\|\psi C_\varphi f_{w_l^j}\|_{\mathcal{B}}, \tag{3.42}$$

and so $\|\psi C_\varphi f_{w_l^j}\|_{\mathcal{B}} \nrightarrow 0$, as $j \to \infty$. From Cases 1 and 2, we obtain

$$\lim_{j\to\infty}|\psi(z^j)|\sum_{k,l=1}^{n}\frac{1-|z_k^j|^2}{1-|w_l^j|^2}\left|\frac{\partial \varphi_l}{\partial z_k}(z^j)\right| = 0. \tag{3.43}$$

Hence, condition (1.6) holds, finishing the proof of the theorem. □

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References

[1] R. M. Timoney, "Bloch functions in several complex variables. I," *The Bulletin of the London Mathematical Society*, vol. 12, no. 4, pp. 241–267, 1980.

[2] S. Ohno, "Weighted composition operators between H^∞ and the Bloch space," *Taiwanese Journal of Mathematics*, vol. 5, no. 3, pp. 555–563, 2001.

[3] S. Li and S. Stević, "Weighted composition operator between H^∞ and α-Bloch spaces in the unit ball," to appear in *Taiwanese Journal of Mathematics*.

[4] J. Shi and L. Luo, "Composition operators on the Bloch space of several complex variables," *Acta Mathematica Sinica*, vol. 16, no. 1, pp. 85–98, 2000.

[5] S. Stević, "Composition operators between H^∞ and a-Bloch spaces on the polydisc," *Zeitschrift für Analysis und ihre Anwendungen*, vol. 25, no. 4, pp. 457–466, 2006.

[6] Z. Zhou, "Composition operators on the Lipschitz space in polydiscs," *Science in China. Series A*, vol. 46, no. 1, pp. 33–38, 2003.

[7] Z. Zhou and J. Shi, "Compact composition operators on the Bloch space in polydiscs," *Science in China. Series A*, vol. 44, no. 3, pp. 286–291, 2001.

[8] S. Stević, "The generalized Libera transform on Hardy, Bergman and Bloch spaces on the unit polydisc," *Zeitschrift für Analysis und ihre Anwendungen*, vol. 22, no. 1, pp. 179–186, 2003.

[9] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, Fla, USA, 1995.

[10] G. Benke and D.-C. Chang, "A note on weighted Bergman spaces and the Cesàro operator," *Nagoya Mathematical Journal*, vol. 159, pp. 25–43, 2000.

[11] S. Stević, "Cesàro averaging operators," *Mathematische Nachrichten*, vol. 248-249, pp. 185–189, 2003.

[12] S. Stević, “Hilbert operator on the polydisk,” *Bulletin of the Institute of Mathematics. Academia Sinica*, vol. 31, no. 2, pp. 135–142, 2003.

[13] W. Rudin, *Function Theory in the Unit Ball of* $\mathbb{C}^n$, vol. 241 of *Grundlehren der mathematischen Wissenschaften*, Springer, New York, NY, USA, 1980.

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