

Research Article

Essential Norms of Weighted Composition Operators from the α -Bloch Space to a Weighted-Type Space on the Unit Ball

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This paper finds some lower and upper bounds for the essential norm of the weighted composition operator from α -Bloch spaces to the weighted-type space H_μ^∞ on the unit ball for the case $\alpha \geq 1$.

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1. Introduction

Let $\mathbb{B} = \{z \in \mathbb{C}^n : |z| < 1\}$ be the open unit ball in \mathbb{C}^n , $H(\mathbb{B})$ be the class of all holomorphic functions on the unit ball, and let $H^\infty(\mathbb{B})$ be the class of all bounded holomorphic functions on \mathbb{B} with the norm

$$\|f\|_\infty = \sup_{z \in \mathbb{B}} |f(z)|. \quad (1.1)$$

Let $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ be points in \mathbb{C}^n and $\langle z, w \rangle = \sum_{k=1}^n z_k \bar{w}_k$. For a holomorphic function f , we denote

$$\nabla f = \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n} \right). \quad (1.2)$$

A positive continuous function ϕ on the interval $[0, 1)$ is called normal (see [1]) if there is $\delta \in [0, 1)$ and a and b , $0 < a < b$ such that

$$\frac{\phi(r)}{(1-r)^a} \text{ is decreasing on } [\delta, 1) \text{ and } \lim_{r \rightarrow 1} \frac{\phi(r)}{(1-r)^a} = 0,$$

$$\frac{\phi(r)}{(1-r)^b} \text{ is increasing on } [\delta, 1) \text{ and } \lim_{r \rightarrow 1} \frac{\phi(r)}{(1-r)^b} = \infty. \quad (1.3)$$

From now on, if we say that a function $\phi : \mathbb{B} \rightarrow [0, \infty)$ is normal, we will also assume that it is radial, that is, $\phi(z) = \phi(|z|)$, $z \in \mathbb{B}$.

The weighted space $H_\mu^\infty = H_\mu^\infty(\mathbb{B})$ consists of all $f \in H(\mathbb{B})$ such that

$$\sup_{z \in \mathbb{B}} \mu(z) |f(z)| < \infty, \quad (1.4)$$

where μ is normal on the interval $[0, 1)$. For $\mu(z) = (1 - |z|^2)^\beta$, $\beta > 0$, we obtain the weighted space $H_\beta^\infty = H_\beta^\infty(\mathbb{B})$ (see, e.g., [2, 3]).

The α -Bloch space $\mathcal{B}^\alpha = \mathcal{B}^\alpha(\mathbb{B})$, $\alpha > 0$, is the space of all $f \in H(\mathbb{B})$ such that

$$b_\alpha(f) = \sup_{z \in \mathbb{B}} (1 - |z|^2)^\alpha |\nabla f(z)| < \infty. \quad (1.5)$$

With the norm

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + b_\alpha(f), \quad (1.6)$$

the space \mathcal{B}^α is a Banach space ([4–6]).

The little α -Bloch space $\mathcal{B}_0^\alpha = \mathcal{B}_0^\alpha(\mathbb{B})$ is the subspace of \mathcal{B}^α consisting of all $f \in H(\mathbb{B})$ such that

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^\alpha |\nabla f(z)| = 0. \quad (1.7)$$

Let $u \in H(\mathbb{B})$ and φ be a holomorphic self-map of the unit ball. Weighted composition operator on $H(\mathbb{B})$, induced by u and φ is defined by

$$(uC_\varphi f)(z) = u(z)f(\varphi(z)), \quad z \in \mathbb{B}. \quad (1.8)$$

This operator can be regarded as a generalization of a multiplication operator and a composition operator. It is interesting to provide a function theoretic characterization when u and φ induce a bounded or compact weighted composition operator between some spaces of holomorphic functions on \mathbb{B} . (For some classical results in the topic see, e.g., [5]. For some recent results on this and related operators, see, e.g., [2–4, 7–25] and the references therein.)

In [18], Ohno has characterized the boundedness and compactness of weighted composition operators between H^∞ and the Bloch space \mathcal{B} on the unit disk. In the setting of the unit polydisk \mathbb{D}^n , we have given some necessary and sufficient conditions for a weighted composition operator to be bounded or compact from $H^\infty(\mathbb{D}^n)$ to the Bloch space $\mathcal{B}(\mathbb{D}^n)$ in [12] (see, also [21]). Corresponding results for the case of the unit ball are given in [14]. Among other results, in [14], we have given some necessary and sufficient conditions for the

compactness of the operator $uC_\varphi : \mathcal{B}^\alpha(\mathbb{B}) \rightarrow H^\infty(\mathbb{B})$, which we incorporate in the following theorem.

Theorem A. *Let $\varphi = (\varphi_1, \dots, \varphi_n)$ be a holomorphic self-map of \mathbb{B} and $u \in H(\mathbb{B})$.*

(a) *If $\alpha > 1$, then the following statements are equivalent:*

- (a1) $uC_\varphi : \mathcal{B}_0^\alpha \rightarrow H^\infty$ is a compact operator,
- (a2) $uC_\varphi : \mathcal{B}^\alpha \rightarrow H^\infty$ is a compact operator,
- (a3) $u \in H^\infty$, and

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{|u(z)|}{(1 - |\varphi(z)|^2)^{\alpha-1}} = 0. \quad (1.9)$$

(b) *If $\alpha = 1$, then the following statements are equivalent:*

- (b1) $uC_\varphi : \mathcal{B}_0 \rightarrow H^\infty$ is a compact operator,
- (b2) $uC_\varphi : \mathcal{B} \rightarrow H^\infty$ is a compact operator,
- (b3) $u \in H^\infty$, and

$$\lim_{|\varphi(z)| \rightarrow 1} |u(z)| \ln \frac{2}{1 - |\varphi(z)|^2} = 0. \quad (1.10)$$

We would also like to point out that if $\alpha \geq 1$, then the boundedness of $uC_\varphi : \mathcal{B}^\alpha \rightarrow H_\mu^\infty$ and $uC_\varphi : \mathcal{B}_0^\alpha \rightarrow H_\mu^\infty$ are equivalent (see [22] for the case $\alpha = 1$, and the proof of Theorem 3 in [14]).

The essential norm of an operator is its distance in the operator norm from the compact operators. More precisely, assume that X_1 and X_2 are Banach spaces and $A : X_1 \rightarrow X_2$ is a bounded linear operator, then the essential norm of A , denoted by $\|A\|_{e, X_1 \rightarrow X_2}$, is defined as follows:

$$\|A\|_{e, X_1 \rightarrow X_2} = \inf \{ \|A + L\|_{X_1 \rightarrow X_2} : L : X_1 \rightarrow X_2, L \text{ is compact} \}, \quad (1.11)$$

where $\|\cdot\|_{X_1 \rightarrow X_2}$ denotes the operator norm. If $X_1 = X_2$, it is simply denoted by $\|\cdot\|_e$ (see, e.g., [5, page 132]). If $A : X_1 \rightarrow X_2$ is an unbounded linear operator, then clearly $\|A\|_{e, X_1 \rightarrow X_2} = \infty$.

Since the set of all compact operators is a closed subset of the set of bounded operators, it follows that an operator A is compact if and only if $\|A\|_{e, X_1 \rightarrow X_2} = 0$.

Motivated by Theorem A, in this paper, we find some lower and upper bounds for the essential norm of the weighted composition operator $uC_\varphi : \mathcal{B}^\alpha(\mathbb{B})$ (or $\mathcal{B}_0^\alpha(\mathbb{B})$) $\rightarrow H_\mu^\infty(\mathbb{B})$, when $\alpha \geq 1$.

Throughout this paper, constants are denoted by C , they are positive and may differ from one occurrence to another. The notation $a \preceq b$ means that there is a positive constant C such that $a \leq Cb$. If both $a \preceq b$ and $b \preceq a$ hold, then we say that $a \asymp b$.

2. Auxiliary results

In this section, we quote several auxiliary results which we need in the proofs of the main results in this paper. The following lemma should be folklore.

Lemma 2.1. Let $f \in \mathcal{B}^\alpha(\mathbb{B})$, $0 < \alpha < \infty$. Then,

$$|f(z)| \leq \begin{cases} C\|f\|_{\mathcal{B}^\alpha}, & \alpha \in (0, 1), \\ |f(0)| + b_1(f) \frac{1}{2} \ln \frac{1+|z|}{1-|z|}, & \alpha = 1, \\ C \frac{\|f\|_{\mathcal{B}^\alpha}}{(1-|z|^2)^{\alpha-1}}, & \alpha > 1, \end{cases} \quad (2.1)$$

for some $C > 0$ independent of f .

The proof of the lemma for the case $\alpha \neq 1$ can be found, for example, in [26]. The formulation of the corresponding estimate in [26], for the case $\alpha = 1$, is slightly different. In this case, Lemma 2.1 follows from the following estimate:

$$|f(z) - f(0)| = \left| \int_0^1 \langle \nabla f(tz), \bar{z} \rangle \right| \leq b_1(f) \int_0^1 \frac{|z| dt}{1-|z|^2 t^2} = b_1(f) \frac{1}{2} \ln \frac{1+|z|}{1-|z|}. \quad (2.2)$$

The next lemma can be proved in a standard way (see, e.g., the proofs of the corresponding results in [5, 27–29]).

Lemma 2.2. Assume $\alpha > 0$, $g \in H(\mathbb{B})$, μ is normal, and φ is an analytic self-map of \mathbb{B} . Then, $uC_\varphi : \mathcal{B}^\alpha(\text{or } \mathcal{B}_0^\alpha) \rightarrow H_\mu^\infty$ is compact if and only if $uC_\varphi : \mathcal{B}^\alpha(\text{or } \mathcal{B}_0^\alpha) \rightarrow H_\mu^\infty$ is bounded and for any bounded sequence $(f_k)_{k \in \mathbb{N}}$ in $\mathcal{B}^\alpha(\text{or } \mathcal{B}_0^\alpha)$ converging to zero uniformly on compacts of \mathbb{B} as $k \rightarrow \infty$, one has $\|uC_\varphi f_k\|_{H_\mu^\infty} \rightarrow 0$ as $k \rightarrow \infty$.

Lemma 2.3. Let

$$f_w(z) = \frac{(1-|w|^2)^\varepsilon}{(1-\langle z, w \rangle)^{\alpha+\varepsilon-1}}, \quad w \in \mathbb{B}, \quad (2.3)$$

where $\alpha > 1$ and $\varepsilon \in (0, 1]$. Then,

$$\|f_w\|_{\mathcal{B}^\alpha} = (1-|w|^2)^\varepsilon + \frac{(\alpha+\varepsilon-1)(2\alpha)^\alpha |w| \left(\sqrt{\alpha^2 + |w|^2 \varepsilon^2 - |w|^2 \alpha^2 + \varepsilon} \right)^\varepsilon}{(\alpha+\varepsilon)^\varepsilon \left(\sqrt{\alpha^2 + |w|^2 \varepsilon^2 - |w|^2 \alpha^2 + \varepsilon} \right)^\alpha}. \quad (2.4)$$

Proof. We have

$$(1-|z|^2)^\alpha |\nabla f_w(z)| = (\alpha+\varepsilon-1)(1-|z|^2)^\alpha \frac{(1-|w|^2)^\varepsilon |w|}{|1-\langle z, w \rangle|^{\alpha+\varepsilon}}, \quad (2.5)$$

from which it easily follows that

$$b_\alpha(f_w) \leq (\alpha+\varepsilon-1)2^{\alpha+\varepsilon}. \quad (2.6)$$

Set

$$g_s(x) = \frac{(\alpha + \varepsilon - 1)s(1 - s^2)^\varepsilon (1 - x^2)^\alpha}{(1 - sx)^{\alpha + \varepsilon}}, \quad x \in [0, 1], \quad s \in [0, 1]. \quad (2.7)$$

Then,

$$g'_s(x) = (\alpha + \varepsilon - 1)s(1 - s^2)^\varepsilon (1 - x^2)^{\alpha - 1} \frac{s(\alpha - \varepsilon)x^2 - 2\alpha x + s(\alpha + \varepsilon)}{(1 - sx)^{\alpha + \varepsilon + 1}}. \quad (2.8)$$

Hence, the points

$$x_{M,m} = \frac{\alpha \pm \sqrt{\alpha^2 + s^2\varepsilon^2 - s^2\alpha^2}}{s(\alpha - \varepsilon)} \quad (2.9)$$

are stationary for the function $g_s(x)$. Since $x_M > 1$, it follows that $g_s(x)$ attains its maximum on the interval $[0, 1]$ at the point

$$x_m = \frac{\alpha - \sqrt{\alpha^2 + s^2\varepsilon^2 - s^2\alpha^2}}{s(\alpha - \varepsilon)} = \frac{s(\alpha + \varepsilon)}{\alpha + \sqrt{\alpha^2 + s^2\varepsilon^2 - s^2\alpha^2}} \in (0, 1). \quad (2.10)$$

By some long but elementary calculations, it follows that

$$g_s(x_m) = \frac{(\alpha + \varepsilon - 1)(2\alpha)^\alpha s \left(\sqrt{\alpha^2 + s^2\varepsilon^2 - s^2\alpha^2} + \varepsilon \right)^\varepsilon}{(\alpha + \varepsilon)^\varepsilon \left(\sqrt{\alpha^2 + s^2\varepsilon^2 - s^2\alpha^2} + \alpha \right)^\alpha}. \quad (2.11)$$

From this and since $f_w(0) = (1 - |w|^2)^\varepsilon$, (2.4) follows. \square

Remark 2.4. Note that

$$\lim_{|w| \rightarrow 1} \|f_w\|_{\mathcal{B}^\alpha} = \lim_{s \rightarrow 1-0} [(1 - s^2)^\varepsilon + g_s(x_m)] = \frac{(\alpha + \varepsilon - 1)2^{\alpha + \varepsilon} \varepsilon^\varepsilon \alpha^\alpha}{(\alpha + \varepsilon)^{\alpha + \varepsilon}}, \quad (2.12)$$

$$\lim_{\varepsilon \rightarrow 0+} \frac{(\alpha + \varepsilon - 1)2^{\alpha + \varepsilon} \varepsilon^\varepsilon \alpha^\alpha}{(\alpha + \varepsilon)^{\alpha + \varepsilon}} = (\alpha - 1)2^\alpha. \quad (2.13)$$

3. Estimates of the essential norm of $uC_\varphi : \mathcal{B}^\alpha$ (or \mathcal{B}_0^α) $\rightarrow H_\mu^\infty$

In this section, we prove the main results in this paper. Before we formulate and prove these results, we prove another auxiliary result.

Lemma 3.1. *Assume $\alpha \in (0, \infty)$, $u \in H(\mathbb{B})$, μ is normal, φ is a holomorphic self-map of \mathbb{B} such that $\|\varphi\|_\infty < 1$, and the operator $uC_\varphi : \mathcal{B}^\alpha$ (or \mathcal{B}_0^α) $\rightarrow H_\mu^\infty$ is bounded. Then, $uC_\varphi : \mathcal{B}^\alpha$ (or \mathcal{B}_0^α) $\rightarrow H_\mu^\infty$ is compact.*

Proof. First note that since $uC_\varphi : \mathcal{B}^\alpha$ (or \mathcal{B}_0^α) $\rightarrow H_\mu^\infty$ is bounded and $f_0(z) \equiv 1 \in \mathcal{B}_0^\alpha \subset \mathcal{B}^\alpha$, it follows that $uC_\varphi(f_0) = u \in H_\mu^\infty$. Now, assume that $(f_k)_{k \in \mathbb{N}}$ is a bounded sequence in \mathcal{B}^α (or \mathcal{B}_0^α) converging to zero on compacts of \mathbb{B} as $k \rightarrow \infty$. Then, we have

$$\|uC_\varphi(f_k)\|_{H_\mu^\infty} \leq \|u\|_{H_\mu^\infty} \sup_{w \in \varphi(\mathbb{B})} |f_k(w)| \rightarrow 0, \quad (3.1)$$

as $k \rightarrow \infty$, since $\varphi(\mathbb{B})$ is contained in the ball $|w| \leq \|\varphi\|_\infty$ which is a compact subset of \mathbb{B} , according to the assumption, $\|\varphi\|_\infty < 1$. Hence, by Lemma 2.2, the operator $uC_\varphi : \mathcal{B}^\alpha$ (or \mathcal{B}_0^α) $\rightarrow H_\mu^\infty$ is compact. \square

Theorem 3.2. *Assume $\alpha > 1$, μ is normal, $u \in H(\mathbb{B})$, $\varphi = (\varphi_1, \dots, \varphi_n)$ is a holomorphic self-map of \mathbb{B} , and $uC_\varphi : \mathcal{B}^\alpha \rightarrow H_\mu^\infty$ is bounded. Then,*

$$\begin{aligned} \frac{2^{-\alpha}}{\alpha-1} \limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(z)|u(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}} &\leq \|uC_\varphi\|_{e, \mathcal{B}_0^\alpha \rightarrow H_\mu^\infty} \\ &\leq \|uC_\varphi\|_{e, \mathcal{B}^\alpha \rightarrow H_\mu^\infty} \\ &\leq C \limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(z)|u(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}}, \end{aligned} \quad (3.2)$$

for some positive constant C .

Proof. Since $uC_\varphi : \mathcal{B}^\alpha \rightarrow H_\mu^\infty$ is bounded, recall that $u \in H_\mu^\infty$. If $\|\varphi\|_\infty < 1$, then, from Lemma 3.1, it follows that $uC_\varphi : \mathcal{B}^\alpha \rightarrow H_\mu^\infty$ is compact which is equivalent with $\|uC_\varphi\|_{e, \mathcal{B}^\alpha \rightarrow H_\mu^\infty} = 0$, and, consequently, $\|uC_\varphi\|_{e, \mathcal{B}_0^\alpha \rightarrow H_\mu^\infty} = 0$. On the other hand, it is clear that in this case the condition $|\varphi(z)| \rightarrow 1$ is vacuous, so that inequalities in (3.2) are vacuously satisfied.

Hence, assume $\|\varphi\|_\infty = 1$. Let $(z_k)_{k \in \mathbb{N}}$ be a sequence in \mathbb{B} such that $\lim_{k \rightarrow \infty} |\varphi(z_k)| = 1$ and $\varepsilon \in (0, 1)$ be fixed. Set

$$f_k^\varepsilon(z) = \frac{(1-|\varphi(z_k)|^2)^\varepsilon}{(1-\langle z, \varphi(z_k) \rangle)^{\alpha+\varepsilon-1}}, \quad k \in \mathbb{N}. \quad (3.3)$$

By Lemma 2.3, it follows that $\sup_{k \in \mathbb{N}} \|f_k^\varepsilon\|_{\mathcal{B}^\alpha} < \infty$, moreover, it is easy to see that $f_k^\varepsilon \in \mathcal{B}_0^\alpha$ for every $k \in \mathbb{N}$ and $f_k^\varepsilon \rightarrow 0$ uniformly on compacts of \mathbb{B} as $k \rightarrow \infty$. Then, by [6, Theorem 7.5], it follows that f_k^ε converges to zero weakly as $k \rightarrow \infty$. Hence, for every compact operator $L : \mathcal{B}_0^\alpha \rightarrow H_\mu^\infty$, we have $\|Lf_k^\varepsilon\|_{H_\mu^\infty} \rightarrow 0$ as $k \rightarrow \infty$.

We have

$$\|f_k^\varepsilon\|_{\mathcal{B}^\alpha} \|uC_\varphi + L\|_{\mathcal{B}_0^\alpha \rightarrow H_\mu^\infty} \geq \|(uC_\varphi + L)(f_k^\varepsilon)\|_{H_\mu^\infty} \geq \|uC_\varphi f_k^\varepsilon\|_{H_\mu^\infty} - \|Lf_k^\varepsilon\|_{H_\mu^\infty} \quad (3.4)$$

for every compact operator $L : \mathcal{B}_0^\alpha \rightarrow H_\mu^\infty$.

Letting $k \rightarrow \infty$ in (3.4) and using the definition of f_k^ε , we obtain

$$C(\varepsilon, \alpha) \|uC_\varphi + L\|_{\mathcal{B}_0^\alpha \rightarrow H_\mu^\infty} = \limsup_{k \rightarrow \infty} \|uC_\varphi f_k^\varepsilon\|_{H_\mu^\infty} \geq \limsup_{k \rightarrow \infty} \frac{\mu(z_k) |u(z_k)|}{(1 - |\varphi(z_k)|^2)^{\alpha-1}}, \quad (3.5)$$

where $C(\varepsilon, \alpha)$ is the quantity in (2.12).

Taking in (3.5) the infimum over the set of all compact operators $L : \mathcal{B}_0^\alpha \rightarrow H_\mu^\infty$, then letting $\varepsilon \rightarrow 0+$ in such obtained inequality, and using (2.13), we obtain

$$2^\alpha (\alpha - 1) \|uC_\varphi\|_{e, \mathcal{B}_0^\alpha \rightarrow H_\mu^\infty} \geq \limsup_{k \rightarrow \infty} \frac{\mu(z_k) |u(z_k)|}{(1 - |\varphi(z_k)|^2)^{\alpha-1}}, \quad (3.6)$$

from which the first inequality in (3.2) follows.

Since the second inequality in (3.2) is obvious, we only have to prove the third one. By Lemma 3.1, we have that for each fixed $\rho \in (0, 1)$ the operator $uC_{\rho\varphi} : \mathcal{B}^\alpha \rightarrow H_\mu^\infty$ is compact.

Let $\delta \in (0, 1)$ be fixed, and let $(\rho_m)_{m \in \mathbb{N}}$ be a sequence of positive numbers which increasingly converges to 1, then for each fixed $m \in \mathbb{N}$, we have

$$\begin{aligned} \|uC_\varphi\|_{e, \mathcal{B}^\alpha \rightarrow H_\mu^\infty} &\leq \|uC_\varphi - uC_{\rho_m\varphi}\|_{\mathcal{B}^\alpha \rightarrow H_\mu^\infty} \\ &= \sup_{\|f\|_{\mathcal{B}^\alpha} \leq 1} \|(uC_\varphi - uC_{\rho_m\varphi})(f)\|_{H_\mu^\infty} \\ &= \sup_{z \in \mathbb{B}} \sup_{\|f\|_{\mathcal{B}^\alpha} \leq 1} \mu(z) |u(z) (f(\varphi(z)) - f(\rho_m\varphi(z)))| \\ &\leq \sup_{|\varphi(z)| \leq \delta} \sup_{\|f\|_{\mathcal{B}^\alpha} \leq 1} \mu(z) |u(z)| |f(\varphi(z)) - f(\rho_m\varphi(z))| \\ &\quad + \sup_{|\varphi(z)| > \delta} \sup_{\|f\|_{\mathcal{B}^\alpha} \leq 1} \mu(z) |u(z)| |f(\varphi(z)) - f(\rho_m\varphi(z))|. \end{aligned} \quad (3.7)$$

By the mean-value theorem, we have

$$\begin{aligned} &\sup_{|\varphi(z)| \leq \delta} \sup_{\|f\|_{\mathcal{B}^\alpha} \leq 1} \mu(z) |u(z) (f(\varphi(z)) - f(\rho_m\varphi(z)))| \\ &\leq (1 - \rho_m) \sup_{|\varphi(z)| \leq \delta} \sup_{\|f\|_{\mathcal{B}^\alpha} \leq 1} \mu(z) |u(z)| |\varphi(z)| \sup_{|w| \leq \delta} |\nabla f(w)| \\ &\leq (1 - \rho_m) \frac{\delta}{(1 - \delta^2)^\alpha} \|u\|_{H_\mu^\infty} \sup_{\|f\|_{\mathcal{B}^\alpha} \leq 1} \sup_{|w| \leq \delta} (1 - |w|^2)^\alpha |\nabla f(w)| \\ &\leq (1 - \rho_m) \frac{\delta}{(1 - \delta^2)^\alpha} \|u\|_{H_\mu^\infty} \rightarrow 0, \end{aligned} \quad (3.8)$$

as $m \rightarrow \infty$.

Moreover, by Lemma 2.1 (case $\alpha > 1$), and known inequality

$$\|f_r\|_{\mathcal{B}^\alpha} \leq \|f\|_{\mathcal{B}^\alpha}, \quad (3.9)$$

where $f_r(z) = f(rz)$, $r \in [0, 1)$, we have

$$\begin{aligned} \mu(z)|u(z)(f(\varphi(z)) - f(\rho_m\varphi(z)))| &\leq C\|f - f_{\rho_m}\|_{\mathcal{B}^\alpha} \frac{\mu(z)|u(z)|}{(1 - |\varphi(z)|^2)^{\alpha-1}} \\ &\leq 2C\|f\|_{\mathcal{B}^\alpha} \frac{\mu(z)|u(z)|}{(1 - |\varphi(z)|^2)^{\alpha-1}} \end{aligned} \quad (3.10)$$

for some positive constant C . Replacing (3.10) in (3.7), letting in such obtained inequality $m \rightarrow \infty$, employing (3.8), and then letting $\delta \rightarrow 1$, the third inequality in (3.2) follows, finishing the proof of the theorem. \square

Corollary 3.3. *Assume $\alpha > 1$, μ is normal, $u \in H(\mathbb{B})$, $\varphi = (\varphi_1, \dots, \varphi_n)$ is a holomorphic self-map of \mathbb{B} , and the operator $uC_\varphi : \mathcal{B}^\alpha$ (or \mathcal{B}_0^α) $\rightarrow H_\mu^\infty$ is bounded. Then, $uC_\varphi : \mathcal{B}^\alpha$ (or \mathcal{B}_0^α) $\rightarrow H_\mu^\infty$ is compact if and only if*

$$\limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(z)|u(z)|}{(1 - |\varphi(z)|^2)^{\alpha-1}} = 0. \quad (3.11)$$

Theorem 3.4. *Assume $u \in H(\mathbb{B})$, μ is normal, $\varphi = (\varphi_1, \dots, \varphi_n)$ is a holomorphic self-map of \mathbb{B} , and $uC_\varphi : \mathcal{B} \rightarrow H_\mu^\infty$ is bounded. Then,*

$$\begin{aligned} C \limsup_{|\varphi(z)| \rightarrow 1} \mu(z)|u(z)| \ln \frac{1 + |\varphi(z)|}{1 - |\varphi(z)|} &\leq \|uC_\varphi\|_{e, \mathcal{B}_0 \rightarrow H_\mu^\infty} \\ &\leq \|uC_\varphi\|_{e, \mathcal{B} \rightarrow H_\mu^\infty} \\ &\leq \limsup_{|\varphi(z)| \rightarrow 1} \mu(z)|u(z)| \ln \frac{1 + |\varphi(z)|}{1 - |\varphi(z)|} \end{aligned} \quad (3.12)$$

for a positive constant C .

Proof. Clearly, $u = (uC_\varphi)(1) \in H_\mu^\infty$. If $\|\varphi\|_\infty < 1$, then the result follows as in Theorem 3.2. Hence, assume $\|\varphi\|_\infty = 1$. We use the following family of test functions

$$h_w(z) = \left(\ln \frac{(1 + |w|)^2}{1 - \langle z, w \rangle} \right)^2 \left(\ln \frac{1 + |w|}{1 - |w|} \right)^{-1}, \quad w \in \mathbb{B}. \quad (3.13)$$

We have

$$\begin{aligned} \sup_{z \in \mathbb{B}} (1 - |z|^2) |\nabla h_w(z)| &= \sup_{z \in \mathbb{B}} 2 \frac{(1 - |z|^2) |w|}{|1 - \langle z, w \rangle|} \left| \ln \frac{(1 + |w|)^2}{1 - \langle z, w \rangle} \right| \left(\ln \frac{1 + |w|}{1 - |w|} \right)^{-1} \\ &\leq 4|w| \left(2 \ln \frac{1 + |w|}{1 - |w|} + 2\pi \right) \left(\ln \frac{1 + |w|}{1 - |w|} \right)^{-1} \leq 4 \left(2 + \frac{2\pi}{\ln 3} \right), \end{aligned} \quad (3.14)$$

when $|w| \geq 1/2$.

From this and since $|h_w(0)| \leq 4(\ln 2)^2 / \ln 3$, for $|w| \geq 1/2$, we have that

$$\|h_w\|_{\mathcal{B}} \leq 4 \left(2 + \frac{2\pi}{\ln 3} + \frac{(\ln 2)^2}{\ln 3} \right) = C_0. \quad (3.15)$$

Assume $(\varphi(z_k))_{k \in \mathbb{N}}$ is a sequence in \mathbb{B} such that $|\varphi(z_k)| \rightarrow 1$ as $k \rightarrow \infty$. Note that $(h_{\varphi(z_k)})_{k \in \mathbb{N}}$ is a bounded sequence in \mathcal{B} (moreover in \mathcal{B}_0) converging to zero uniformly on compacts of \mathbb{B} . Then, by [6, Theorem 7.5], it follows that $h_{\varphi(z_k)}$ converges to zero weakly as $k \rightarrow \infty$. Hence, for every compact operator $L : \mathcal{B}_0 \rightarrow H_\mu^\infty$, we have

$$\lim_{k \rightarrow \infty} \|Lh_{\varphi(z_k)}\|_{H_\mu^\infty} = 0. \quad (3.16)$$

On the other hand, for every compact operator $L : \mathcal{B}_0 \rightarrow H_\mu^\infty$, we have

$$\begin{aligned} \|h_{\varphi(z_k)}\|_{\mathcal{B}} \|uC_\varphi + L\|_{\mathcal{B}_0 \rightarrow H_\mu^\infty} &\geq \|(uC_\varphi + L)h_{\varphi(z_k)}\|_{H_\mu^\infty} \\ &\geq \|uC_\varphi h_{\varphi(z_k)}\|_{H_\mu^\infty} - \|Lh_{\varphi(z_k)}\|_{H_\mu^\infty}. \end{aligned} \quad (3.17)$$

Using (3.15), letting $k \rightarrow \infty$ in (3.17), and applying (3.16), it follows that

$$\begin{aligned} C_0 \|uC_\varphi + L\|_{\mathcal{B}_0 \rightarrow H_\mu^\infty} &\geq \limsup_{k \rightarrow \infty} \|uC_\varphi h_{\varphi(z_k)}\|_{H_\mu^\infty} \\ &\geq \limsup_{k \rightarrow \infty} \mu(z_k) |u(z_k) h_{\varphi(z_k)}(\varphi(z_k))| \\ &= \limsup_{k \rightarrow \infty} \mu(z_k) |u(z_k)| \ln \frac{1 + |\varphi(z_k)|}{1 - |\varphi(z_k)|}. \end{aligned} \quad (3.18)$$

Taking in (3.18) the infimum over the set of all compact operators $L : \mathcal{B}_0 \rightarrow H_\mu^\infty$, we obtain

$$\|uC_\varphi\|_{e, \mathcal{B}_0 \rightarrow H_\mu^\infty} \geq \frac{1}{C_0} \limsup_{k \rightarrow \infty} \mu(z_k) |u(z_k)| \ln \frac{1 + |\varphi(z_k)|}{1 - |\varphi(z_k)|}, \quad (3.19)$$

from which the first inequality in (3.12) follows.

As in Theorem 3.2, we need only to prove the third inequality in (3.12).

Recall that for each $\rho \in (0, 1)$, the operator $uC_{\rho\varphi} : \mathcal{B} \rightarrow H_\mu^\infty$ is compact. Let the sequence $(\rho_m)_{m \in \mathbb{N}}$ be as in Theorem 3.2. Note that inequality (3.7) and relationship (3.8) also hold for $\alpha = 1$. Hence, we should only estimate the quantity

$$I_m^\delta = \sup_{|\varphi(z)| > \delta} \sup_{\|f\|_{\mathcal{B}} \leq 1} \mu(z) |u(z)| |f(\varphi(z)) - f(\rho_m \varphi(z))|. \quad (3.20)$$

On the other hand, by Lemma 2.1 (case $\alpha = 1$) applied to the function $f - f_{\rho_m}$, which belongs to the Bloch space for each $m \in \mathbb{N}$, and inequality (3.9) with $\alpha = 1$, we have

$$\begin{aligned} \mu(z) |u(z)| (f(\varphi(z)) - f(\rho_m \varphi(z))) &\leq \frac{1}{2} \|f - f_{\rho_m}\|_{\mathcal{B}} \mu(z) |u(z)| \ln \frac{1 + |\varphi(z)|}{1 - |\varphi(z)|} \\ &\leq \|f\|_{\mathcal{B}} \mu(z) |u(z)| \ln \frac{1 + |\varphi(z)|}{1 - |\varphi(z)|}. \end{aligned} \quad (3.21)$$

From (3.21), by letting $m \rightarrow \infty$ in (3.7) and using (3.8) (with $\alpha = 1$), and letting $\delta \rightarrow 1$ in such obtained inequality, we obtain

$$\|uC_\varphi\|_{e, \mathcal{B} \rightarrow H_\mu^\infty} \leq \limsup_{|\varphi(z)| \rightarrow 1} \mu(z) |u(z)| \ln \frac{1 + |\varphi(z)|}{1 - |\varphi(z)|}, \quad (3.22)$$

as desired. \square

Corollary 3.5. Assume $u \in H(\mathbb{B})$, μ is normal, $\varphi = (\varphi_1, \dots, \varphi_n)$ is a holomorphic self-map of \mathbb{B} , and the operator $uC_\varphi : \mathcal{B}$ (or \mathcal{B}_0^α) $\rightarrow H_\mu^\infty$ is bounded. Then, $uC_\varphi : \mathcal{B}$ (or \mathcal{B}_0^α) $\rightarrow H_\mu^\infty$ is compact if and only if

$$\limsup_{|\varphi(z)| \rightarrow 1} \mu(z) |u(z)| \ln \frac{1 + |\varphi(z)|}{1 - |\varphi(z)|} = 0. \quad (3.23)$$

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