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Research Article

A Theorem of Nehari Type on Weighted Bergman Spaces of the Unit Ball

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This paper shows that if S is a bounded linear operator acting on the weighted Bergman spaces A_α^2 on the unit ball in $\mathbb{C}^n$ such that $ST_{z_i} = T_{\overline{z}_i}S$ $(i = 1, \dots, n)$, where $T_{z_i} = z_i f$ and $T_{\overline{z}_i} = P(\overline{z}_i f)$; and where P is the weighted Bergman projection, then S must be a Hankel operator.

1. Introduction

Let B_n be the open unit ball in the complex vector space $\mathbb{C}^n$. For $z = (z_1, \dots, z_n)$, $w = (w_1, \dots, w_n) \in \mathbb{C}^n$, let $\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n$, where $\overline{w}_k$ is the complex conjugate of w_k, and $|z| = \sqrt{\langle z, z\rangle}$. For a multi-index $m = (m_1, \dots, m_n)$ and $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, we also write

$$z^m = z_1^{m_1} \dots z_n^{m_n}. \tag{1.1}$$

Let dV be the volume measure on B_n, normalized so that $V(B_n) = 1$. For $\alpha > -1$, the weighted Lebesgue measure dV_α is defined by

$$dV_\alpha(z) = c_\alpha\left(1 - |z|^2\right)^\alpha dV(z), \tag{1.2}$$

where

$$c_\alpha = \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)} \tag{1.3}$$

is a normalizing constant so that dV_α is a probability measure on B_n.

For $p \geq 1$ and $\alpha > -1$, the weighted Bergman space A^p_α consists of holomorphic functions f in $L^p(B_n, dV_\alpha)$, that is,

$$A^p_\alpha = L^p(B_n, dV_\alpha) \cap H(B_n). \tag{1.4}$$

When $\alpha = 0$, A^p_α is the standard (unweighted) Bergman spaces, which is simply denoted by A^p.

The weighted Bergman space A^p_α is a closed subspace of $L^p(B_n, dV_\alpha)$ and the set of all polynomials is dense in A^p_α. See, for example, [1].

With the norm

$$\|f\|_p = \left(\int_{B_n} |f(z)|^p dV_\alpha(z) \right)^{1/p}, \tag{1.5}$$

$L^p(B_n, dV_\alpha)$ and A^p_α become Banach spaces. $L^2(B_n, dV_\alpha)$ is a Hilbert space whose inner product will be denoted by $\langle \cdot, \cdot \rangle_\alpha$. Some other properties of Bergman spaces as well as some recent results on the operators on them, can be found, for example, in [2–13] (see, also the references therein).

For $\varphi \in L^\infty(B_n)$, the Hankel operator H_φ is defined on A^2_α by

$$H_\varphi(f) = P(J(\varphi f)), \tag{1.6}$$

where J is the unitary operator defined on $L^2(B_n, dV_\alpha)$ by

$$J(f(z)) = J(f(z_1, \ldots, z_n)) = f(\overline{z}) = f(\overline{z}_1, \ldots, \overline{z}_n), \tag{1.7}$$

and P is the weighted Bergman projection from $L^2(B_n, dV_\alpha)$ onto A^2_α.

The Toeplitz operator with the symbol $\varphi \in L^\infty(B_n)$ is defined on A^2_α by

$$T_\varphi f = P(f\varphi), \quad f \in A^2_\alpha. \tag{1.8}$$

Toeplitz operators have the following properties: if a and b are complex numbers, and φ and $\psi \in L^\infty(B_n)$, then $T_{a\varphi + b\psi} = aT_\varphi + bT_\psi$, $T^*_\varphi = T_{\overline{\varphi}}$; moreover, if $\varphi \in H^\infty(B_n)$, then $T_\psi T_\varphi = T_{\varphi\psi}$ and $T_{\overline{\varphi}} T_\psi = T_{\overline{\varphi}\psi}$.

The symbol z_i will denote the ith coordinate function ($i = 1, \ldots, n$).

It is easy to see that $H_\varphi T_{z_i} = T_{\overline{z}_i} H_\varphi$. Thus, the Hankel operators H_φ are particular solutions of the operator equation

$$ST_{z_i} = T_{\overline{z}_i} S, \quad i = 1, \ldots, n, \tag{1.9}$$

where S is a bounded linear operator on A^2_α.

It is well known that on the classical Hardy space H^2, Toeplitz operators and Hankel operators are of the same status, and present different operators classes. The authors of [14] regarded Hankel operators as an essential part of Toeplitz operator theory, and many authors studied Hankel operators and their related problems in [14–22].

On the Hardy space H^2, Nehari [19] proved that if S is a bounded linear operator such that $ST_z = T_{\bar{z}}S$, then $S = H_\varphi$ for some $\varphi \in L^\infty$; moreover, φ can be chosen such that $\|H_\varphi\| = \|\varphi\|$. Faour [20] proved a theorem of Nehari type on the Bergman spaces of the unit disk. In [21], the authors gave the characterization of Hankel operators on the generalized H^2 spaces, which is also similar to the Nehari theorem on the Hardy space.

The motivation for this paper is the question whether solutions of the operator (1.9) must be the Hankel operator on the Bergman space A^2_α.

In this paper, we take the weighted Bergman space A^2_α as our domain and prove a Nehari-type theorem. While our method is basically adapted from [20, 21], substantial amount of extra work is necessary for the setting of the weighted Bergman spaces on the unit ball.

2. Nehari-type theorem

To establish a Nehari-type theorem on the weighted Bergman spaces on the unit ball, we recall the atomic decomposition of the weighted Bergman space A^p_α, which plays an important role in this paper. It is shown that every function in the weighted Bergman space A^p_α can be decomposed into a series of nice functions called atoms. These atoms are defined in terms of kernel functions and in some sense act as a basis for A^p_α. The following lemma is Theorem 2.30 in [1].

Lemma 2.1. *Suppose $p > 0$, $\alpha > -1$, and*

$$b > n \max\left(1, \frac{1}{p}\right) + \frac{\alpha + 1}{p}. \tag{2.1}$$

Then there exists a sequence $\{a_k\}$ in B_n such that A^p_α consists exactly of functions of the form

$$f(z) = \sum_{k=1}^{\infty} c_k \frac{\left(1 - |a_k|^2\right)^{(pb-n-1-\alpha)/p}}{\left(1 - \langle z, a_k \rangle\right)^b}, \quad z \in B_n, \tag{2.2}$$

where $\{c_k\}$ belongs to the sequence space l^p and the series converges in the norm topology of A^p_α.

Remark 2.2. By the proof of Theorem 2.30 in [1], it can be seen that the sequence $\{a_k\}$ in Lemma 2.1 is chosen independent of p, α, and b.

Remark 2.3. The proof of Theorem 2.30 in [1] tells us that for any $f \in A^p_\alpha$, we can choose a sequence $\{c_k\}$ in Lemma 2.1 so that

$$\sum_k |c_k|^p \le C \int_{B_n} |f(z)|^p dV_\alpha(z), \tag{2.3}$$

where C is a positive constant independent of f.

The following lemma follows immediately from Lemma 2.1.

Lemma 2.4. *Suppose $\{a_k\}$ is a sequence as in Lemma 2.1, $\alpha > -1$, and $b > n + \alpha + 1$. Let*

$$l_a(z) = \frac{(1 - |a|^2)^{b-n-1-\alpha}}{(1 - \langle z, a\rangle)^b}. \tag{2.4}$$

Then, $A^1_\alpha(B_n)$ consists exactly of the functions of the form

$$f(z) = \sum_{k=1}^{\infty} c_k l_{a_k}, \quad z \in B_n, \tag{2.5}$$

where $\{c_k\}$ belongs to the sequence space l^1 and the series converges in the norm topology of A^1_α.

From now on, we assume that $b > 2(n + \alpha + 1)$ is fixed and $\{a_k\}$ and $l_a(z)$ are defined as in Lemma 2.4.

The following two lemmas follow immediately from Theorem 1.12 in [1].

Lemma 2.5. *Let $\alpha > -1$, $0 < r < 1$, then for every $a \in B_n$, one has*

$$\|l_a(rz)\|_2 \le k(r), \tag{2.6}$$

where $k(r)$ is a constant which only depends on r.

Lemma 2.6. *There exists a constant C such that for every $a \in B_n$, $r \in (0, 1)$,*

$$\|l_a(rz)\|_1 \le C, \tag{2.7}$$

where C is independent of a and r.

Theorem 2.7. *Let S be a bounded linear operator acting on the weighted Bergman space A^2_α such that $ST_{z_i} = T_{\overline{z}_i}S$ $(i = 1, \ldots, n)$. Then, there exists $\varphi \in L^\infty(B_n)$ such that $S = H_\varphi$.*

Proof. Define the linear functional G on A^2_α by $G(f) = \langle Sf, 1\rangle_\alpha$. Clearly, G is a bounded linear functional on A^2_α. Note that $A^2_\alpha \subset A^1_\alpha$. From Lemma 2.4 and Remark 2.3, given $f \in A^2_\alpha$, there exists $\{c_k\}$ in l^1 such that $f = \sum_k c_k l_{a_k}$ converges in A^1_α and $\sum |c_k| \le \beta\|f\|_1$, where β is a positive constant independent f.

For $f \in A^2_\alpha$, let $f^+(z) = \overline{f(\overline{z})} \in A^2_\alpha$. From (1.9), it is easy to see that $ST^k_{z_i} = T^k_{\overline{z}_i}S$ $(i = 1, \ldots, n;\ k = 1, 2, \ldots,)$. If $p = az^k_i, q = bz^m_j$, then we have

$$\begin{aligned}\langle Sp, q\rangle_\alpha &= a\overline{b}\langle ST^k_{z_i}1, T^m_{z_j}1\rangle_\alpha = a\overline{b}\langle T^m_{\overline{z}_j}ST^k_{z_i}1, 1\rangle_\alpha = a\overline{b}\langle ST^m_{z_j}T^k_{z_i}1, 1\rangle_\alpha \\ &= a\overline{b}\langle S(z^m_j z^k_i), 1\rangle_\alpha = \langle S(pq^+), 1\rangle_\alpha.\end{aligned} \tag{2.8}$$

Hence, we establish that $\langle S(pq^+), 1\rangle_\alpha = \langle Sp, q\rangle_\alpha$, where p and q are polynomials in $z = (z_1, \ldots, z_n)$.

Since the set of all polynomials is dense in A^2_α, there are sequences of polynomials $p_n(z)$ and $q_n(z)$ such that

$$\|p_n - l_{a_k}^{1/2}\|_2 \longrightarrow 0, \quad \|q_n - (l_{a_k}^{1/2})^+\|_2 \longrightarrow 0 \quad (\text{as } n \to \infty). \tag{2.9}$$

Furthermore, $\|q_n^+ - l_{a_k}^{1/2}\|_2 \to 0$.

Since

$$\langle S(p_n q_n^+), 1\rangle_\alpha = \langle Sp_n, q_n\rangle_\alpha, \tag{2.10}$$

by using the boundedness of S and the continuity of the scalar product, it follows that

$$\langle Sl_{a_k}^{1/2}, (l_{a_k}^{1/2})^+\rangle_\alpha = \langle Sl_{a_k}, 1\rangle_\alpha. \tag{2.11}$$

Given $r \in (0,1)$, from Lemma 2.5, $f(rz) = \sum_k c_k l_{a_k}(rz)$ converges in A^2_α. Thus, with $f_r(z) = f(rz)$, we see that

$$\begin{aligned}\langle Sf_r, 1\rangle_\alpha &= \left\langle S\left(\sum_k c_k l_{a_k}(rz)\right), 1\right\rangle_\alpha = \sum_k c_k \langle S(l_{a_k}(rz)), 1\rangle_\alpha \\ &= \sum_k c_k \langle S(l_{a_k}^{1/2}(rz)), (l_{a_k}^{1/2}(rz))^+\rangle_\alpha.\end{aligned} \tag{2.12}$$

Note that

$$\begin{aligned}\|l_{a_k}^{1/2}(rz)\|_2 &= \left(\int_{B_n} |(l_{a_k}(rz))| dV_\alpha(z)\right)^{1/2}, \\ \|(l_{a_k}^{1/2}(rz))^+\|_2 &= \left(\int_{B_n} \overline{|(l_{a_k}(r\overline{z}))|} dV_\alpha(z)\right)^{1/2} = \|l_{a_k}^{1/2}(rz)\|_2,\end{aligned} \tag{2.13}$$

and consequently

$$\|l_{a_k}^{1/2}(rz)\|_2 \|(l_{a_k}^{1/2}(rz))^+\|_2 = \|l_{a_k}(rz)\|_1. \tag{2.14}$$

Therefore,

$$|\langle Sf_r, 1\rangle_\alpha| \le \sum_k |c_k| \|S\| \cdot \sup_{a_k} \|l_{a_k}(rz)\|_1. \tag{2.15}$$

Consequently, it follows from Lemma 2.6 that

$$|\langle Sf_r, 1\rangle_\alpha| \le C\beta \|S\| \, \|f\|_1; \tag{2.16}$$

but $f_r \to f$ in $A^2_\alpha(B_n)$. Thus, by the continuity of G it follows that $|G(f)| \le \gamma\|f\|_1$ for some constant γ. Since A^2_α is dense in A^1_α, it follows that G is extended by continuity to an element of $(A^1_\alpha)^*$, and consequently, by the Hahn-Banach theorem to an element of $(L^1(B_n))^* = L^\infty(B_n)$. Therefore, there exists $\varphi \in L^\infty(B_n)$ such that

$$\langle Sf, 1\rangle_\alpha = \langle \varphi f, 1\rangle_\alpha = \langle J(\varphi f), 1\rangle_\alpha = \langle P(J(\varphi f)), 1\rangle_\alpha = \langle H_\varphi f, 1\rangle_\alpha. \tag{2.17}$$

Since

$$\langle H_\varphi(pq^+), 1\rangle_\alpha = \int_{B_n} \varphi(\overline{z})p(\overline{z})\overline{q(z)}dV_\alpha(z) = \langle H_\varphi p, q\rangle_\alpha, \tag{2.18}$$

and by using the fact that $\langle S(pq^+), 1\rangle_\alpha = \langle Sp, q\rangle_\alpha$, where p, q are polynomials in z, it follows that

$$\langle Sp, q\rangle_\alpha = \langle H_\varphi p, q\rangle_\alpha. \tag{2.19}$$

Hence, $S = H_\varphi$, finishing the proof of the theorem. □

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