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Research Article

Porosity of Convex Nowhere Dense Subsets of Normed Linear Spaces

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This paper is devoted to the following question: how to characterize convex nowhere dense subsets of normed linear spaces in terms of porosity? The motivation for this study originates from papers of V. Olevskii and L. Zajíček, where it is shown that convex nowhere dense subsets of normed linear spaces are porous in some strong senses.

1. Introduction

The paper concerns the topic of describing smallness of interesting sets of metric spaces in terms of porosity. The notions of porosity and σ-porosity (a set is σ-porous if it is a countable union of porous sets) can be considered as stronger versions of nowhere density and meagerness—in particular, in any "reasonable" metric space, there exist sets which are nowhere dense and are not σ-porous. Thus it is interesting to know that some sets are not only nowhere dense (meager) but even porous (σ-porous). In such a direction many earlier results were extended, for example, it turned out that the set of all Banach contractions was not only meager but also σ-porous in the space of all nonexpansive mappings (cf. [1, 2]). Since there are various types of porosity (more or less restrictive), the natural problem of finding the most restrictive notion of porosity, which would be suitable for an examined set, is also an interesting task. The reader who is not familiar with porosity is referred to the survey papers [3, 4] on porosity on the real line, metric spaces, and normed linear spaces.

In the paper we try to answer the following question: what is the best approximation of smallness (in terms of porosity) of convex nowhere dense subsets of normed linear spaces? Zajíček [4] observed that such sets are R-ball porous for every $R > 0$, and 0-cone porous (cf. [4, page 518]). In fact, Zajíček's observation is an improvement of the earlier result of Olevskii

[5] (as was shown in [6], Olevskii worked with a much weaker version of porosity than R-ball porosity). Hence for our purpose we need to find some stronger condition, which would imply R-ball porosity for every $R > 0$, and 0-cone porosity.

The paper is organized as follows. In Section 2, we give definitions of some types of porosity, that is, R-ball porosity, 0-angle porosity (a stronger version of 0-cone porosity) and introduce the notion of c-porosity. We also make some basic observations (i.e., c-porosity $\Rightarrow$ 0-angle porosity $\Rightarrow$ R-ball porosity) and demonstrate that c-porosity gives the characterization of smallness of convex nowhere dense sets.

In Section 3, we prove that in any Hilbert space H with $\dim H > 1$, the unit sphere is 0-angle porous and is not a countable union of c-porous sets (i.e., is not σ-c-porous). This observation shows that the notion of 0-angle porosity is quite far from the notion of c-porosity.

The motivation for Section 4 originates from the fact that the notion of c-porosity uses the space of all continuous functionals X^*. In this section we discuss the possibility of finding the best approximation of smallness of a convex nowhere dense sets in terms of porosity *without using* X^*.

In Section 5, we give one example of σ-c-porous subset of the space of continuous functions. For other interesting σ-c-porous sets, we refer the reader to [5] (one of them deals with the Banach-Steinhaus principle).

2. Some Notions of Porosity

In this section we present the definitions of R-ball porosity, ball smallness, (σ-)0-angle porosity, and (σ-)c-porosity. We also make some basic observations, which will be used in the sequel (see Proposition 2.8 and Example 2.9).

Let $(X, \|\cdot\|)$ be a real normed linear space and $M \subset X$. Given $x \in X$ and $r > 0$, we denote by $B(x, r)$ the open ball with center x and radius r. By X^* we denote the space of all continuous linear functionals on X.

Definition 2.1 (see [4, 7]). Let $R > 0$. We say that M is R*-ball porous* if for any $x \in M$ and $\alpha \in (0, 1)$, there exists $y \in X$ such that $\|x - y\| = R$ and $B(y, \alpha R) \cap M = \emptyset$.

Remark 2.2. The definition of R-ball porosity presented in [7], [4, page 516] is slightly different from the above one. Namely, M is R-ball porous if for any $x \in M$ and $\varepsilon \in (0, R)$, there exists $y \in X$ such that $\|x - y\| = R$ and $B(y, R - \varepsilon) \cap M = \emptyset$. However, it is obvious that both definitions are equivalent.

Definition 2.3 (see [4, 7]). We say that M is 0-*angle porous* if for every $x \in M$ and every $r > 0$, there exist $y \in B(x, r)$ and $\phi \in X^* \setminus \{0\}$such that

$$\{z \in X : \phi(z) > \phi(y)\} \cap M = \emptyset. \tag{2.1}$$

Note that 0-angle porosity can be considered as a "global" version of (mentioned in the introduction) 0-cone porosity and, in particular, 0-angle porosity implies 0-cone porosity.

For the definitions of α-cone porosity and α-angle porosity, where $\alpha \in [0, 1)$, see [4, page 516] and [7], respectively.

Definition 2.4. M is called *c-porous* if for any $x \in X$ and every $r > 0$, there are $y \in B(x, r)$ and $\phi \in X^* \setminus \{0\}$ such that

$$\{z \in X : \phi(z) > \phi(y)\} \cap M = \emptyset. \tag{2.2}$$

C-porosity turns out to be the suitable notion to describe the smallness of convex nowhere dense sets (see Proposition 2.5) and is a stronger form of 0-angle porosity ($x \in X$ instead $x \in M$). Indeed, consider the unit sphere S of any nontrivial normed space. S is not c-porous (simply take $x = 0$ and $r = 1/2$) and is 0-angle porous—to see it, use the Hahn-Banach separation theorem (cf. [8]) for sets $\overline{B}(0, 1)$ (the closure of $B(0, 1)$) and $\{(1 + r/2)y\}$, where $y \in S$ and $r > 0$.

If a set M is a countable union of c-porous sets, then we say that M is *σ-c-porous*. In the same way we define *σ- 0-angle porosity*. If $M = \bigcup_{n \in \mathbb{N}} M_n$ and each M_n is R_n-ball porous for some $R_n > 0$, then we say that M is *ball small*.

The next result shows that c-porosity is the best approximation of smallness (in the sense of porosity) of convex nowhere dense sets (in the proof we extend an argument suggested by Zajíček [4, page 518]).

Proposition 2.5. *A subset M of a normed space X is c-porous if and only if* conv M *is nowhere dense.*

Proof. "⇒" It is obvious that for any $\phi \in X^*$ and $y \in X$, we have

$$\{z : \phi(z) > \phi(y)\} \cap M = \emptyset \Longleftrightarrow \{z : \phi(z) > \phi(y)\} \cap \operatorname{conv} M = \emptyset. \tag{2.3}$$

Hence if M is c-porous, then $\operatorname{conv} M$ is also c-porous and, in particular, nowhere dense.

"⇐" Fix any $x \in X$ and $r > 0$. Since $\overline{\operatorname{conv} M}$ (the closure of $\operatorname{conv} M$) is nowhere dense, there exists $y \in B(x, r) \setminus \overline{\operatorname{conv} M}$. Sets $\overline{\operatorname{conv} M}$ and $\{y\}$ satisfy the assumptions of the Hahn-Banach separation theorem, so there exist $\phi \in X^*$ and $c \in \mathbb{R}$ such that $\phi(y) > c$ and for any $z \in \overline{\operatorname{conv} M}, \phi(z) < c$. Then $M \cap \{z : \phi(z) > \phi(y)\} = \emptyset$. □

Corollary 2.6. *Let X be any normed space and let $(*)$ be any condition such that*

$$\text{if } A \subset X \text{ satisfies } (*) \text{ and } B \subset A, \text{ then } B \text{ satisfies } (*). \tag{2.4}$$

If every convex and nowhere dense subset of X satisfies $(*)$, then any c-porous subset of X satisfies $(*)$.

The notions of 0-angle porosity and c-porosity involve the space X^*; however, in its origin the porosity was defined in metric spaces. In the next part of this section we will show what kind of porosity without using X^* is implied by them (see Proposition 2.8). Note that we will use this result in Sections 3 and 4.

We omit the proof of the following result since it is technical and can be easily deduced from the proof of [5, Lemma 1].

Lemma 2.7. *Let $R > 0$, $r \in (0, 1/2)$, $x_0, y_0 \in X$, $\phi \in X^* \setminus \{0\}$. If $\|y_0 - x_0\| < rR$, then there exists $y \in X$ such that $\|y - x_0\| = R$ and*

$$B(y, (1-2r)R) \subset \{x \in X : \phi(x) > \phi(y_0)\}. \tag{2.5}$$

Proposition 2.8. *The following statements hold.*

(i) *If M is 0-angle porous, then M is R-ball porous for every $R > 0$, that is,*

$$\begin{gathered}\text{for every } R > 0,\ x \in M \text{ and every } \alpha \in (0,1),\\ \text{there exists } y \in X \text{ with } \|y - x\| = R \text{ and } B(y, \alpha R) \cap M = \emptyset.\end{gathered} \tag{2.6}$$

(ii) *If M is c-porous, then*

$$\begin{gathered}\text{for every } R > 0,\ x \in X \text{ and every } \alpha \in (0,1),\\ \text{there exists } y \in X \text{ with } \|y - x\| = R \text{ and } B(y, \alpha R) \cap M = \emptyset.\end{gathered} \tag{2.7}$$

Proof. We will prove only (i), since the proof of (ii) is very similar. Fix $R > 0$, $x_0 \in M$ and $\alpha \in (0,1)$. Let $r > 0$ be such that $1 - 2r > \alpha$, and let $y_0 \in X$ and $\phi \in X^* \setminus \{0\}$ be such that $\|y_0 - x_0\| < rR$ and $M \cap \{x : \phi(x) > \phi(y_0)\} = \emptyset$. By Lemma 2.7, we have that there exists $y \in X$ such that $\|y - x_0\| = R$ and $B(y, (1-2r)R) \subset \{x : \phi(x) > \phi(y_0)\}$. Since $B(y, \alpha R) \subset B(y, (1-2r)R)$, the result follows. □

Note that (2.7) is stronger than (2.6). Indeed, the unit sphere in any normed space satisfies (2.6) and does not satisfy (2.7). In the sequel, we will extend this observation (see Theorem 3.2).

The next example shows, in particular, that the converse of the Proposition 2.8 is not true.

Example 2.9. Let $(X, \|\cdot\|)$ be one of the following real Banach spaces: c_0 *or* l_p, $p \in [1, \infty)$. Let us define the set $M := \bigcup_{n \in \mathbb{N}} \{\pm n e_1, \pm n e_2, \dots, \pm n e_n\}$, where

$$e_n(r) := \begin{cases} 1, & r = n, \\ 0, & r \neq n. \end{cases} \tag{2.8}$$

Now we will show that M satisfies the following condition, which is stronger than (2.7) (and, in particular, than (2.6)):

$$\text{for every } R > 0,\ x \in X, \text{ there exists } y \in X \text{ s.t. } \|y - x\| = R \text{ and } B(y, R) \cap M = \emptyset. \tag{2.9}$$

To see it, take any $x \in X$ and $R > 0$. Since $x(n) \to 0$, there exists $n_0 > 3R$ such that $|x(n_0)| < R$. Assume, without loss of generality, that $x(n_0) \geq 0$. Now let $y \in X$ be such that

$$y(k) := \begin{cases} x(k), & k \neq n_0, \\ x(k) + R, & k = n_0. \end{cases} \tag{2.10}$$

Then $\|x - y\| = R$. To see that $B(y, R) \cap M = \emptyset$, take any $z \in M$ and consider three cases.

(i) $z(n_0) = 0$, then $\|y - z\| \geq |x(n_0) + R - z(n_0)| = |x(n_0) + R| \geq R$.

(ii) $z(n_0) \geq n_0$, then $\|y - z\| \geq |x(n_0) + R - z(n_0)| \geq n_0 - x(n_0) - R \geq R$.

(iii) $z(n_0) \leq -n_0$, then $\|y - z\| \geq |x(n_0) + R - z(n_0)| \geq n_0 + x(n_0) + R \geq R$.

Now we will show that M is not 0-angle porous. It is sufficient to show that for any $\phi \in X^* \setminus \{0\}$ and $y \in X$, $M \cap \{x : \phi(x) > \phi(y)\} \neq \emptyset$. Fix any $\phi \in X^* \setminus \{0\}$ and $y \in X$, then there exists a sequence x_0 such that $\phi(x) = \sum_{n=1}^{\infty} x(n)x_0(n)$ for any $x \in X$. Let $n_1 \in \mathbb{N}$ be such that $x_0(n_1) \neq 0$. Assume, without loss of generality, that $x_0(n_1) > 0$. Let $n_2 \geq n_1$ be such that $n_2 x_0(n_1) > \phi(y)$. Then $n_2 e_{n_1} \in M$ and

$$\phi(n_2 e_{n_1}) = n_2 x_0(n_1) > \phi(y). \tag{2.11}$$

Thus M is not 0-angle porous, and hence not c-porous.

3. On c-Porosity

In this section we will show that c-porosity is a much stronger notion of porosity than 0-angle porosity. This will justify introducing this notion.

From now on, if $(H, (\cdot \mid \cdot)_0)$ is a real Hilbert space, then $(\mathbb{R} \times H, (\cdot \mid \cdot))$ denotes the real Hilbert space $\mathbb{R} \times H$ with the inner product $(\cdot \mid \cdot)$ defined as follows:

$$((a, x) \mid (b, y)) := ab + (x \mid y)_0 \text{ for any } (a, x), (b, y) \in \mathbb{R} \times H. \tag{3.1}$$

Denote by $\|\cdot\|_0$ and $\|\cdot\|$ the norms generated by $(\cdot \mid \cdot)_0$ and $(\cdot \mid \cdot)$, respectively.

We will show that in any nontrivial real Hilbert space H with $\dim H > 1$, the unit sphere S is not σ-c-porous. In fact, we will obtain a more general result. If $S = \bigcup_{n \in \mathbb{N}} S_n$, then there exists $n_0 \in \mathbb{N}$ such that S_{n_0} does not satisfy (2.7), and hence (by Proposition 2.8) S is not σ-c-porous.

Lemma 3.1. *Let $(H, (\cdot \mid \cdot)_0)$ be a nontrivial real Hilbert space. For any $\delta \in (0, 1)$, the set*

$$V_\delta := \{(a, y) \in \mathbb{R} \times H : 1 - \delta \leq a \leq 1 \text{ and } \|(a, y)\| = 1\} \tag{3.2}$$

does not satisfy (2.7).

Proof. Take $(1-(1/2)\delta, 0) \in \mathbb{R}\times H$, $R=3$ and

$$\alpha := \frac{1}{R}\max\left\{\sqrt{R^2-4\left(\sqrt{2\delta-\delta^2}-\frac{1}{2}\delta+\frac{1}{16}\delta^2\right)},\sqrt{R^2-\frac{1}{4}\delta^2}\right\}. \tag{3.3}$$

It is easy to see that $\alpha\in(0,1)$. Let $(a_0,y_0)\in\mathbb{R}\times H$ be such that

$$\left\|(a_0,y_0)-\left(1-\frac{1}{2}\delta,0\right)\right\| = R. \tag{3.4}$$

We will show that $B((a_0,y_0),\alpha R)\cap V_\delta\neq\emptyset$. Consider the following three cases.

Case 1 ($a_0\leq 1$ and $y_0\neq 0$). Then

$$a_0<-1 \quad \text{or} \quad \|y_0\|_0>2. \tag{3.5}$$

Indeed, otherwise we would have a contradiction since

$$\begin{aligned} R &= \left\|(a_0,y_0)-\left(1-\frac{1}{2}\delta,0\right)\right\| = \sqrt{\left(a_0-\left(1-\frac{1}{2}\delta\right)\right)^2+\|y_0\|_0^2} \\ &= \sqrt{a_0^2-2a_0\left(1-\frac{1}{2}\delta\right)+\left(1-\frac{1}{2}\delta\right)^2+\|y_0\|_0^2} \leq \sqrt{1+2+1+4}<3=R. \end{aligned} \tag{3.6}$$

Set $\eta := \sqrt{2\delta-\delta^2}/\|y_0\|_0$. It is easy to see that $(1-\delta,\eta y_0)\in V_\delta$. We will show that $(1-\delta,\eta y_0)\in B((a_0,y_0),\alpha R)$. By (3.4), we have

$$\begin{aligned} \|(1-\delta,\eta y_0)-(a_0,y_0)\|^2 &= \left(-\frac{1}{2}\delta-\left(a_0-\left(1-\frac{1}{2}\delta\right)\right)\right)^2+\|(\eta-1)y_0\|_0^2 \\ &= \frac{1}{4}\delta^2+\delta\left(a_0-\left(1-\frac{1}{2}\delta\right)\right)+\left(a_0-\left(1-\frac{1}{2}\delta\right)\right)^2 \\ &\quad +2\delta-\delta^2-2\sqrt{2\delta-\delta^2}\|y_0\|_0+\|y_0\|_0^2 \\ &= R^2-\frac{1}{4}\delta^2+(1+a_0)\delta-2\sqrt{2\delta-\delta^2}\|y_0\|_0, \end{aligned} \tag{3.7}$$

so if $a_0<-1$, then, by (3.3), we infer

$$\|(1-\delta,\eta y_0)-(a_0,y_0)\|^2 < R^2-\frac{1}{4}\delta^2 \leq \alpha^2R^2, \tag{3.8}$$

and if $\|y_0\|_0 > 2$, then, again by (3.3), we get

$$\left\|(1-\delta,\eta y_0)-(a_0,y_0)\right\|^2 < R^2-\frac{1}{4}\delta^2+2\delta-4\sqrt{2\delta-\delta^2} \le \alpha^2R^2. \tag{3.9}$$

Case 2 ($a_0 \le 1$ and $y_0 = 0$). In this case $a_0 = (1-(1/2)\delta)-R$. Set $z \in H$ with $\|z\|_0 = \sqrt{2\delta-\delta^2}$. It is obvious that $(1-\delta,z) \in V_\delta$. We will show that $(1-\delta,z) \in B((a_0,0),\alpha R)$. By (3.3), (3.4), and a fact that $R = 3$, we get

$$\begin{aligned}\|(1-\delta,z)-(a_0,0)\|^2 &= \left(1-\delta-1+\frac{1}{2}\delta+R\right)^2+2\delta-\delta^2 \\ &= R^2-\delta R+\frac{1}{4}\delta^2+2\delta-\delta^2 \\ &= R^2-\delta-\frac{3}{4}\delta^2 < R^2-\frac{3}{4}\delta^2 < \alpha^2R^2.\end{aligned} \tag{3.10}$$

Case 3 ($a_0 > 1$). Take $(1,0) \in V_\delta$. By (3.3) and (3.4) we infer

$$\begin{aligned}\left\|(a_0,y_0)-(1,0)\right\|^2 &= \left\|\left(\left(a_0-\left(1-\frac{1}{2}\delta\right)\right)-\frac{1}{2}\delta,y_0\right)\right\|^2 \\ &= \left(a_0-\left(1-\frac{1}{2}\delta\right)\right)^2-\delta\left(a_0-\left(1-\frac{1}{2}\delta\right)\right)+\frac{1}{4}\delta^2+\|y_0\|_0^2 \\ &= R^2-\delta(a_0-1)-\frac{1}{4}\delta^2 < R^2-\frac{1}{4}\delta^2 \le \alpha^2R^2,\end{aligned} \tag{3.11}$$

so $(1,0) \in V_\delta \cap B((1-(1/2)\delta,0),\alpha R)$.

As a consequence, in all cases we have $B((a_0,y_0),\alpha R) \cap V_\delta \ne \emptyset$, and hence the result follows.

□

Theorem 3.2. *Let* $(H,(\cdot \mid \cdot))$ *be any Hilbert space with* $\dim H > 1$ *and let* S *be the unit sphere in* H*. If* $S = \bigcup_{n\in\mathbb{N}} S_n$*, then there is* $n_0 \in \mathbb{N}$ *such that* S_{n_0} *does not satisfy* (2.7)*. In particular,* S *is not* σ*-c-porous.*

Proof. The second statement follows from the first one by Proposition 2.8. We will prove the first statement. Let $(H,(\cdot \mid \cdot)_0)$ be a Hilbert space with $\dim H > 1$. Since S is complete, by the Baire Category theorem, there exists $n_0 \in \mathbb{N}$ such that S_{n_0} is not nowhere dense in S. Hence there exists a nonempty set U open in S such that $U \subset \overline{S_{n_0}}^S = \overline{S_{n_0}}$ (by $\overline{S_{n_0}}^S$ we denote the closure of S_{n_0} in the space S). Since the closure of a set which satisfies (2.7), also satisfies (2.7), the proof will be completed if we show that U does not satisfy (2.7). Take any $x_0 \in U$ and consider one-dimensional subspace $M := \{ax_0 : a \in \mathbb{R}\}$. It is well known (see, e.g., [8]) that

$$M^\perp := \left\{y \in H : (y \mid x_0)_0 = 0\right\} \tag{3.12}$$

is a closed subspace of H and $H = M \oplus M^{\perp}$. Consider the space $\mathbb{R} \times M^{\perp}$. It is easy to see that the function $H \ni ax_0 + y \overset{F}{\mapsto} (a, y)$ is an isometrical isomorphism between H and $\mathbb{R} \times M^{\perp}$. Since (2.7) is a metric condition, it suffices to show that the set

$$V := F(U) = \left\{ (a, y) \in \mathbb{R} \times M^{\perp} : ax_0 + y \in U \right\} \tag{3.13}$$

does not satisfy (2.7) in $\mathbb{R} \times M^{\perp}$. Since $S_1 := F(S) = \{(a, y) : \|(a, y)\| = 1\}$ and $F_{|S} : S \to S_1$ is a homeomorphism between S and S_1, the set V is open in S_1. Hence and by the fact that the point $(1, 0)$ is in V, we infer there exists $0 < \delta < 1$, such that

$$V_{\delta} \subset V, \tag{3.14}$$

where V_{δ} is defined as in Lemma 3.1. Indeed, since $(1, 0) \in V$ and V is open in S_1, we have that there are $c, d \in \mathbb{R}$ and $r > 0$ such that $0 < c < 1 < d$ and

$$[(c, d) \times B(0, r)] \cap S_1 \subset V. \tag{3.15}$$

Set $\delta := \min\{r^2/2, (1 - c)/2\}$ and take any $(a, y) \in V_{\delta}$, then

$$\begin{gathered} d > 1 \geq a \geq 1 - \delta > 1 - 1 + c = c, \\ \|y\|_0 = \sqrt{1 - a^2} \leq \sqrt{1 - (1 - \delta)^2} = \sqrt{2\delta - \delta^2} < \sqrt{2\delta} \leq r, \end{gathered} \tag{3.16}$$

so $(a, y) \in [(c, d) \times B(0, r)] \cap S_1$ which yields (3.14). Since V_{δ} does not satisfy (2.7) in view of Lemma 3.1, the proof is completed. □

Now we show that for the Euclidean space $\mathbb{R}$, all presented notions of porosity coincide. In [7, page 222] it is given that any ball small subset of $\mathbb{R}$ is countable. Thus and by Proposition 2.8, if $M \subset \mathbb{R}$, then M is σ-c-porous $\Leftrightarrow$ M is a countable union of sets satisfying (2.7) $\Leftrightarrow$ M is σ-0-angle porous $\Leftrightarrow$ M is ball small $\Leftrightarrow$ M is countable.

4. Smallness of Convex Nowhere Dense Sets in Terms of Porosity without Using X^*

In this section we will discuss the problem of finding the best approximation of smallness of a convex nowhere dense subset of a normed space X in terms of porosity without using X^* (as was mentioned, in its origin porosity was defined as a strictly metric condition). By Propositions 2.5 and 2.8, any such set satisfies (2.7). This is a stronger version of the first part of Zajíček's observation, which states that such sets are R-ball porous for every $R > 0$. Indeed, by Theorem 3.2, the unit sphere in Hilbert space is R-ball porous for every $R > 0$ and is not a countable union of sets satisfying (2.7).

Now let M be the set defined in Example 2.9. M satisfies (2.9), hence (2.7), and is not 0-angle porous, so is not c-porous. This shows that, in general, the notion of c-porosity is more restrictive than condition (2.7). On the other hand, as was mentioned, in any nontrivial normed linear space, the unit sphere (which is 0-angle porous) does not satisfy (2.7), and hence, in general, the notion of 0-angle porosity and condition (2.7) are not comparable.

Clearly, condition (2.7) is only one of possible stronger versions of R-ball porosity for every $R > 0$. The other are condition (2.9) and the following weakening of (2.9):

$$\begin{gathered} \text{for any } R > 0 \text{ and any } x \in M, \text{ there exists } y \in X \\ \text{such that } \left(\|x - y\| = R \text{ and } B(y, R) \cap M = \emptyset\right). \end{gathered} \tag{4.1}$$

Now the question arises whether any convex nowhere dense subset of any normed linear space satisfies (4.1) or (2.9)?

Since the closed balls in finite dimensional normed spaces are compact, conditions (2.9) and (2.7) are equivalent in such spaces (note that a similar result is given in [9, Remark 2.4]), and hence any convex nowhere dense subset of such space satisfies (2.9). However, in the remainder of this section we will show that in a very wide class of Banach spaces there are sets, which are convex and nowhere dense, and are not a countable union of sets satisfying (4.1).

Let us focus our attention on nonreflexive spaces.

Proposition 4.1. *Let* $(X, \|\cdot\|)$ *be a real nonreflexive Banach space. Then there exists a closed subspace* $M \subsetneq X$, *which is not a countable union of sets satisfying* (4.1).

Proof. Since X is a nonreflexive Banach space, there exists a closed subspace $M \subsetneq X$ such that for every $x_0 \in M, R > 0$ and every $y \in X$, if $\|y - x_0\| = R$, then $B(y, R) \cap M \neq \emptyset$ (this is a well known fact which follows from the James' theorem [10, page 52]). We will show that M is not a countable union of sets satisfying (4.1). Assume that $M = \bigcup_{n \in \mathbb{N}} M_n$. Since M is complete, by the Baire Category theorem, there exists $n_0 \in \mathbb{N}$ such that M_{n_0} is not nowhere dense in M. Hence for some $x \in M$ and $r > 0$, we have that

$$B(x, r) \cap M \subset \overline{M_{n_0}}^M = \overline{M_{n_0}}. \tag{4.2}$$

Since $x \in \overline{M_{n_0}}$, there exist $x_0 \in M_{n_0}$ and $r_1 > 0$ such that $B(x_0, r_1) \subset B(x, r)$. Then

$$B(x_0, r_1) \cap M \subset \overline{M_{n_0}}. \tag{4.3}$$

Fix any $R > 0$ and let $y \in X$ be such that $\|y - x_0\| = R$. Then there exists $z \in B(y, R) \cap M$, and then the segment $(x_0, z] \subset B(y, R) \cap M$. Hence if $x' \in (x_0, z]$ is such that $\|x_0 - x'\| < r_1$, then $x' \in B(y, R) \cap B(x_0, r_1) \cap M \subset B(y, R) \cap \overline{M_{n_0}}$. Thus $B(y, R) \cap \overline{M_{n_0}} \neq \emptyset$, and hence $B(y, R) \cap M_{n_0} \neq \emptyset$. □

A natural question arises, what happens in reflexive spaces?

Example 4.2. An anonymous referee observed that the Hilbert cube

$$K = \left\{ x \in l_2 : -\frac{1}{n} \le x(n) \le \frac{1}{n}, \ n \in \mathbb{N} \right\} \tag{4.4}$$

does not satisfy (4.1). To see it, recall the concept of the so-called supported points. We say $x \in M \subset X$ is a supported point of M, if there exists $\phi \in X^* \setminus \{0\}$ such that $\phi(x) = \sup\{\phi(y) : y \in M\}$; if such a functional does not exist, then x is called a nonsupported point (cf. [11, page 44]). Now take $x_0 = 0$ and $R = 1$. Then it is easy to see that x_0 is a nonsupported point of K. Now assume that $y \in l_2$ is such that $\|y - x_0\| = \|y\| = 1$ and $B(y,1) \cap K = \emptyset$. Then by the Hahn-Banach separation theorem (cf. [8, page 38]), there exists $\phi \in l_2^* \setminus \{0\}$ with

$$\sup\{\phi(z) : z \in K\} \le \inf\{\phi(z) : z \in B(y,1)\}. \tag{4.5}$$

On the other hand, x_0 is on the boundary of $B(y,1)$, and hence

$$\phi(x_0) \ge \inf\{\phi(z) : z \in B(y,1)\}. \tag{4.6}$$

This gives a contradiction. Hence K does not satisfy (4.1).

By Proposition 4.1 and the previous example, condition (2.7) seems to be quite suitable for describing smallness of convex nowhere dense sets in terms of porosity without using X^*. However, the next example shows that there are sets which satisfy (4.1) (and, in particular, (2.7)) and are not a countable union of c-porous sets.

Example 4.3. Let $X = \mathbb{R}^2$, $\|(x,y)\| =| x | +|y|$ and let

$$M = \left\{ (x,y) : x \ge \frac{1}{\sqrt{2}} \text{ and } x^2 + y^2 = 1 \right\}. \tag{4.7}$$

It is easy to see that M satisfies (4.1). Moreover, using an analogous method as in the proof of Lemma 2.7, it can be easily shown that M is not σ-c-porous.

5. Applications

We will give one example of σ-c-porous set (for other, we refer the reader to [5]).

Let H be a Hilbert space, and let K be a nonempty bounded closed and convex subset of H. Define $C^B(K) := \{A : K \to H : A \text{ is continuous and } A(K) \text{ is bounded in } H\}$. Consider $C^B(K)$ as a Banach space with the norm $\|A\| := \sup_{x\in K}\|A(x)\|$. Let $\Bbbk B$ be the set of all Banach contractions:

$$\Bbbk B = \{A : K \longrightarrow K : \exists_{\alpha\in(0,1)}\forall_{x,y\in K}\|Ax - Ay\| \le \alpha\|x - y\|\}. \tag{5.1}$$

For any $\alpha \in (0,1)$, we also define $\Bbbk B_\alpha := \{A : K \to K : \forall_{x,y\in K}\|Ax - Ay\| \le \alpha\|x - y\|\}$.

Proposition 5.1. *$\Bbbk B$ is a σ-c-porous subset of $C^B(K)$. In particular, $\Bbbk B$ is ball small.*

Proof. De Blasi and Myjak [1] (cf. also [2]) proved that for any $\alpha < 1$, $\Bbbk B_\alpha$ is lower porous (and hence nowhere dense; for the definition of lower porosity, see [4]) subset of the space

$$\Omega := \{A : K \longrightarrow K : \forall_{x,y\in K}\|Ax - Ay\| \le \|x - y\|\}, \tag{5.2}$$

with the metric induced from $(C^B(K), \|\cdot\|)$. Hence $\Bbbk B_\alpha$ is a nowhere dense subset of $C^B(K)$. It is also obvious that $\Bbbk B_\alpha$ is convex. As a consequence, for any $\alpha < 1$, the set $\Bbbk B_\alpha$ is a c-porous subset of $C^B(K)$. Since $\Bbbk B = \bigcup_{n\in\mathbb{N}} \Bbbk B_{(1-1/n)}$, the proof is completed. □

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References

[1] F. S. De Blasi and J. Myjak, "Sur la porosité de l'ensemble des contractions sans point fixe," *Comptes Rendus de l'Académie des Sciences Paris*, vol. 308, no. 2, pp. 51–54, 1989.
[2] S. Reich, "Genericity and porosity in nonlinear analysis and optimization," in *Proceedings of the Computer Methods and Systems (CMS '05)*, pp. 9–15, Cracow, Poland, November 2005, ESI preprint 1756.
[3] L. Zajíček, "Porosity and σ-porosity," *Real Analysis Exchange*, vol. 13, no. 2, pp. 314–350, 1987-1988.
[4] L. Zajíček, "On σ-porous sets in abstract spaces," *Abstract and Applied Analysis*, vol. 2005, no. 5, pp. 509–534, 2005, Proceedings of the International Workshop on Small Sets in Analysis, E. Matoušková, S. Reich and A. Zaslavski, Eds., Hindawi Publishing Corporation, New York, NY, USA.
[5] V. Olevskii, "A note on the Banach-Steinhaus theorem," *Real Analysis Exchange*, vol. 17, no. 1, pp. 399–401, 1991-1992.
[6] F. Strobin, "A comparison of two notions of porosity," *Commentationes Mathematicae*, vol. 48, no. 2, pp. 209–219, 2008.
[7] D. Preiss and L. Zajíček, "Stronger estimates of smallness of sets of Fréchet nondifferentiability of convex functions," *Rendiconti del Circolo Matematico di Palermo, Serie II*, no. 3, supplement, pp. 219–223, 1984.
[8] W. Rudin, *Functional Analysis*, International Series in Pure and Applied Mathematics, McGraw-Hill, New York, NY, USA, 2nd edition, 1991.
[9] J. Duda, "On the size of the set of points where the metric projection exists," *Israel Journal of Mathematics*, vol. 140, pp. 271–283, 2004.
[10] P. Habala, P. Hájek, and V. Zizler, *An Introduction to Banach Space Theory*, vol. 1, MatFiz Press, University of Karlovy, Prague, Czech Republic, 1996.
[11] R. R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, vol. 1364 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1989.