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Research Article

Global Behavior of the Difference Equation $x_{n+1} = (p + x_{n-1})/(qx_n + x_{n-1})$

Taixiang Sun,[1] Hongjian Xi,[2] Hui Wu,[1] and Caihong Han[1]

[1] *College of Mathematics and Information Science, Guangxi University, Nanning, Guangxi 530004, China*
[2] *Department of Mathematics, Guangxi College of Finance and Economics, Nanning 530003, China*

Correspondence should be addressed to Taixiang Sun, stx1963@163.com

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We study the following difference equation $x_{n+1} = (p + x_{n-1})/(qx_n + x_{n-1})$, $n = 0, 1, \ldots,$ where $p, q \in (0, +\infty)$ and the initial conditions $x_{-1}, x_0 \in (0, +\infty)$. We show that every positive solution of the above equation either converges to a finite limit or to a two cycle, which confirms that the Conjecture 6.10.4 proposed by Kulenović and Ladas (2002) is true.

1. Introduction

Kulenović and Ladas in [1] studied the following difference equation:

$$x_{n+1} = \frac{p + x_{n-1}}{qx_n + x_{n-1}}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where $p, q \in (0, +\infty)$ and the initial conditions $x_{-1}, x_0 \in (0, +\infty)$, and they obtained the following theorems.

Theorem A (see [1, Theorem 6.6.2]). *Equation* (1.1) *has a prime period-two solution*

$$\ldots, \phi, \psi, \phi, \psi, \ldots \tag{1.2}$$

if and only if $q > 1 + 4p$. Furthermore, when $q > 1 + 4p$, the prime period-two solution is unique and the values of ϕ and ψ are the positive roots of the quadratic equation

$$t^2 - t + \frac{p}{q - 1} = 0. \tag{1.3}$$

Theorem B (see [1, Theorem 6.6.4]). *Let $\{x_n\}_{n=-1}^{+\infty}$ be a solution of* (1.1). *Let I be the closed interval with end points 1 and p/q and let J and K be the intervals which are disjoint from I and such that*

$$I \cup J \cup K = (0, +\infty). \tag{1.4}$$

Then either all the even terms of the solution lie in J and all odd terms lie in K, or vice-versa, or for some $N \geq 0$,

$$x_n \in I \quad \text{for } n \geq N, \tag{E1}$$

when (E1) *holds, except for the length of the first semicycle of the solution, if $p < q$, the length is one; if $p > q$, the length is at most two.*

Theorem C (see [1, Theorem 6.6.5]). *(a) Assume $q \leq 1 + 4p$. Then the equilibrium $\overline{x} = (1 + \sqrt{1 + 4p(1 + q)})/(2(1 + q))$ of* (1.1) *is global attractor.*

(b) Assume $q > 1 + 4p$. Then every solution of (1.1) *eventually enters and remains in the interval $[p/q, 1]$.*

In [1], they proposed the following conjecture.

Conjecture 1 (see [1, Conjecture 6.10.4]). Assume that $p, q \in (0, +\infty)$. Show that every positive solution of (1.1) either converges to a finite limit or to a two cycle.

Gibbons et al. in [2] trigged off the investigation of the second-order difference equations $x_{n+1} = f(x_n, x_{n-1})$ such that the function $f(x, y)$ is increasing in y and decreasing in x. Motivated by [2], Berg [3] and Stević [4] obtained some important results on the existence of monotone solutions of such equations which was later considerably developed in a series of papers [5–14] (for related papers see also [15–19]). The monotonous character of solutions of the equations was explained by Stević in [20]. For some other papers in the area, see also [1, 17–19, 21–26] and the references cited therein. In this paper, we shall confirm that the Conjecture 1 is true. The main idea used in this paper can be found in papers [24, 26].

2. Global behavior of (1.1)

Theorem 2.1. *Let $\{x_n\}_{n=-1}^{+\infty}$ be a nonoscillatory solution of* (1.1); *then $\{x_n\}_{n=-1}^{+\infty}$ converges to the unique positive equilibrium $\overline{x}$ of* (1.1).

Proof. Since $\{x_n\}_{n=-1}^{+\infty}$ is a nonoscillatory solution of (1.1), we may assume without loss of generality that there exists $N > 0$ such that $x_n \leq \overline{x}$ for any $n \geq N$. We claim $x_{n+1} \geq x_n$ for any $n \geq N$. Indeed, if $x_{n+1} < x_n$ for some $n \geq N$, then

$$\frac{p}{q\overline{x} + \overline{x}} + \frac{1}{q+1} = \frac{p + \overline{x}}{q\overline{x} + \overline{x}} = \overline{x} \geq x_{n+2} = \frac{p + x_n}{qx_{n+1} + x_n} > \frac{p + x_n}{qx_n + x_n} = \frac{p}{qx_n + x_n} + \frac{1}{q+1}, \tag{2.1}$$

which implies $x_n > \overline{x}$; this is a contradiction. Let $\lim_{n\to\infty} x_n = a$; then $a = (p + a)/(qa + a)$ and $a = \overline{x}$. The proof is complete. □

In the sequel, let $q > 1 + 4p$ and $\ldots, \phi, \psi, \phi, \psi, \ldots$ the unique prime period-two solution of (1.1) with $\phi < \psi$. Define $f \in C([\phi, \psi] \times [\phi, \psi], [\phi, \psi])$ by

$$f(x, y) = \frac{p + y}{qx + y} \tag{2.2}$$

for any $x, y \in [\phi, \psi]$ and $g \in C([\phi, \psi], [\phi, \psi])$ by

$$y^* = g(y) = \frac{p + y - y^2}{qy} \tag{2.3}$$

for any $y \in [\phi, \psi]$. Then

$$f(y^*, y) = y. \tag{2.4}$$

Lemma 2.2. *Let $q > 1 + 4p$, then the following statements are true.*

(i) *$f(x, y) > y$ if and only if $x < y^*$.*

(ii) *$x > y$ if and only if $x^* < y^*$.*

(iii) *If $\overline{x} < y < \psi$, then $f(y, y^*) < y^*$ and $y > y^{**}$. If $\phi < y < \overline{x}$, then $f(y, y^*) > y^*$ and $y^{**} > y$.*

Proof. (i) Since f is decreasing in x and $f(y^*, y) = y$, $x < y^*$ if and only if $f(x, y) > f(y^*, y) = y$.

(ii) Since $y^* = g(y)$ is a decreasing function for y, $x > y$ if and only if $x^* < y^*$.

(iii) Since

$$\begin{aligned} f(y, y^*) - y^* &= \frac{p + ((p + y - y^2)/qy)}{qy + ((p + y - y^2)/qy)} - \frac{p + y - y^2}{qy} \\ &= \frac{(q^2 - 1)\left[y - \left(1 - \sqrt{1 + 4p + 4pq}\right)/2(q + 1)\right](y - \phi)(y - \overline{x})(y - \psi)}{qy\left[(q^2 - 1)y^2 + p + y\right]}, \end{aligned} \tag{2.5}$$

it follows that

$$\begin{aligned} &\overline{x} < y < \psi \Longrightarrow f(y, y^*) < y^*, \\ &\phi < y < \overline{x} \Longrightarrow f(y, y^*) > y^*. \end{aligned} \tag{2.6}$$

By (i), we obtain $y > y^{**}$ if $\overline{x} < y < \psi$ and $y^{**} > y$ if $\phi < y < \overline{x}$. The proof is complete. □

Lemma 2.3. *Let $q > 1+4p$ and $\{x_n\}_{n=-1}^{+\infty}$ is a positive solution of* (1.1)*; then $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ do exactly one of the following.*

(i) *Eventually, they are both monotonically increasing.*

(ii) *Eventually, they are both monotonically decreasing.*

(iii) *Eventually, one of them is monotonically increasing and the other is monotonically decreasing.*

Proof. See [20] (also see [27]). □

Remark 2.4. Stević in [20] noticed the relationship between the monotonicity of the subsequences x_{2n} and x_{2n-1} of solution $\{x_n\}_{n=-1}^{+\infty}$ of a second-order difference equation $x_{n+1} = f(x_n, x_{n-1})$ and the monotonicity of the function $f(x, y)$ in variables x and y. A simple observation shows that Stević's proof works in the general case if the function y/x is replaced by $f(x, y)$. The result was later used for many times by Stević and his collaborators (see, e.g., [21, 23–26]).

Lemma 2.5. *Let $q > 1 + 4p$. Assume that there exists some i such that $\psi \geq x_i \geq x_{i+2} > \overline{x} > x_{i+1} \geq \phi$; then $x_{i+1} \geq x_{i+3}$.*

Proof. Since $x_{i+2} = f(x_{i+1}, x_i) \leq x_i = f(x_i^*, x_i)$, it follows that $x_{i+1} \geq x_i^*$. By Lemma 2.2(ii), we get $x_i^{**} \geq x_{i+1}^*$, which with Lemma 2.2(iii) implies $x_i \geq x_i^{**} \geq x_{i+1}^*$. Since $f(x, y)$ is increasing in y $(x, y \in [\phi, \psi])$ and $x_i \geq x_{i+1}^*$, it follows that

$$x_{i+2} = f(x_{i+1}, x_i) \geq f\left(x_{i+1}, x_{i+1}^*\right). \tag{2.7}$$

By Lemma 2.2(iii), we have $x_{i+2} \geq f(x_{i+1}, x_{i+1}^*) \geq x_{i+1}^*$ as $\overline{x} \geq x_{i+1} \geq \phi$. Thus $x_{i+1} = f(x_{i+1}^*, x_{i+1}) \geq f(x_{i+2}, x_{i+1}) = x_{i+3}$. The proof is complete. □

Theorem 2.6. *Let $q > 1+4p$ and $\{x_n\}_{n=-1}^{+\infty}$ be an oscillatory solution of* (1.1)*; then $\{x_n\}_{n=-1}^{+\infty}$ converges to the unique prime period-two solution of* (1.1).

Proof. It follows from Theorem C(b) that there exists $N > 0$ such that for any $n \geq N$,

$$x_n \in \left[\frac{p}{q}, 1\right], \tag{2.8}$$

and $x_N \geq \overline{x}$ and $x_{N+1} < \overline{x}$. We assume without loss of generality that

$$x_n \in \left[\frac{p}{q}, 1\right] \quad \text{for any } n \geq -1, \tag{2.9}$$

and $x_{-1} \geq \overline{x}$ and $x_0 < \overline{x}$. Since

$$h(x, y) = \frac{p + y}{qx + y} \quad \left(x, y \in \left[\frac{p}{q}, 1\right]\right) \tag{2.10}$$

is decreasing in x and increasing in y, it follows that $x_{2n-1} > \overline{x}$ and $x_{2n} < \overline{x}$ for any $n \geq 1$.

If $x_{2n-1} > \overline{x}$ is eventually increasing or $x_{2n} < \overline{x}$ is eventually decreasing, then it follows from Theorem A that $\lim_{n\to\infty} x_{2n-1} = \psi$ and $\lim_{n\to\infty} x_{2n} = \phi$.

If $x_{2n-1} > \overline{x}$ is eventually decreasing and $x_{2n} < \overline{x}$ is eventually increasing, we may assume without loss of generality that $x_{2n} \leq x_{2n+2} < \overline{x} < x_{2n+1} \leq x_{2n-1}$ for any $n \geq 0$. It follows from Lemma 2.5 that $x_{2n} \leq x_{2n+2} \leq \phi < \overline{x} < \psi \leq x_{2n+1} \leq x_{2n-1}$ for any $n \geq 0$. By Theorem A, we obtain $\lim_{n\to\infty} x_{2n-1} = \psi$ and $\lim_{n\to\infty} x_{2n} = \phi$. The proof is complete. □

We confirm from Theorems thm1, 2.6, and C(a) that the Conjecture 1 is true.

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