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Research Article

Remarks on Asymptotic Centers and Fixed Points

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We introduce a class of nonlinear continuous mappings defined on a bounded closed convex subset of a Banach space X. We characterize the Banach spaces in which every asymptotic center of each bounded sequence in any weakly compact convex subset is compact as those spaces having the weak fixed point property for this type of mappings.

1. Introduction

A mapping T on a subset E of a Banach space X is called a nonexpansive mapping if $\|Tx - Ty\| \leq \|x-y\|$ for all $x, y \in E$. Although nonexpansive mappings are widely studied, there are many nonlinear mappings which are more general. The study of the existence of fixed points for those mappings is very useful in solving the problems of equations in science and applied science.

The technique of employing the asymptotic centers and their Chebyshev radii in fixed point theory was first discovered by Edelstein [1], and the compactness assumption given on asymptotic centers was introduced by Kirk and Massa [2]. Recently, Dhompongsa et al. proved in [3] a theorem of existence of fixed points for some generalized nonexpansive mappings on a bounded closed convex subset E of a Banach space with assumption that every asymptotic center of a bounded sequence relative to E is nonempty and compact. However, spaces or sets in which asymptotic centers are compact have not been completely characterized, but partial results are known (see [4, page 93]).

In this paper, we introduce a class of nonlinear continuous mappings in Banach spaces which allows us to characterize the Banach spaces in which every asymptotic center of each bounded sequence in any weakly compact convex subset is compact as those spaces having the weak fixed point property for this type of mappings.

2. Preliminaries

Let E be a nonempty closed and convex subset of a Banach space X and $\{x_n\}$ a bounded sequence in X. For $x \in X$, define the asymptotic radius of $\{x_n\}$ at x as the number

$$r(x, \{x_n\}) = \limsup_{n\to\infty} \|x_n - x\|. \tag{2.1}$$

Let

$$\begin{aligned} r \equiv r(E, \{x_n\}) &:= \inf\{r(x, \{x_n\}) : x \in E\}, \\ A \equiv A(E, \{x_n\}) &:= \{x \in E : r(x, \{x_n\}) = r\}. \end{aligned} \tag{2.2}$$

The number r and the set A are, respectively, called the asymptotic radius and asymptotic center of $\{x_n\}$ relative to E. It is known that $A(E, \{x_n\})$ is nonempty, weakly compact, and convex as E is [4, page 90].

Let $T : E \to E$ be a nonexpansive and $z \in E$. Then for $\alpha \in (0,1)$, the mapping $T_\alpha : E \to E$ defined by setting

$$T_\alpha x = (1-\alpha)z + \alpha T x \tag{2.3}$$

is a contraction mapping. As we have known, Banach contraction mapping theorem assures the existence of a unique fixed point $x_\alpha \in E$. Since

$$\lim_{\alpha\to 1^-} \|x_\alpha - T x_\alpha\| = \lim_{\alpha\to 1^-} (1-\alpha)\|z - T x_\alpha\| = 0, \tag{2.4}$$

we have the following.

Lemma 2.1. *If E is a bounded closed and convex subset of a Banach space and if $T : E \to E$ is nonexpansive, then there exists a sequence $\{x_n\} \subset E$ such that*

$$\lim_{n\to\infty} \|x_n - T x_n\| = 0. \tag{2.5}$$

3. Main Results

Definition 3.1. Let E be a bounded closed convex subset of a Banach space X. We say that a sequence $\{x_n\}$ in X is an asymptotic center sequence for a mapping $T : E \to X$ if, for each $x \in E$,

$$\limsup_{n\to\infty} \|x_n - T x\| \le \limsup_{n\to\infty} \|x_n - x\|. \tag{3.1}$$

We say that $T : E \to X$ is a *D-type mapping* whenever it is continuous and there is an asymptotic center sequence for T.

The following observation shows that the concept of D-type mappings is a generalization of nonexpansiveness.

Proposition 3.2. *Let $T : E \to E$ be a nonexpansive mapping. Then T is a D-type mapping.*

Proof. It is easy to see that T is continuous. By Lemma 2.1, there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.2}$$

For $x \in E$,

$$\|x_n - Tx\| \le \|x_n - Tx_n\| + \|Tx_n - Tx\| \le \|x_n - Tx_n\| + \|x_n - x\|. \tag{3.3}$$

Then

$$\limsup_{n\to\infty} \|x_n - Tx\| \le \limsup_{n\to\infty} \|x_n - x\|. \tag{3.4}$$

This implies that $\{x_n\}$ is an asymptotic center sequence for T. Thus T is a D-type mapping. □

Definition 3.3. We say that a Banach space $(X, \|\cdot\|)$ has the weak fixed point property for D-type mappings if every D-type self-mapping on every weakly compact convex subset of X has a fixed point.

Now we are in the position to prove our main theorem.

Theorem 3.4. *Let X be a Banach space. Then X has the weak fixed point property for D-type mappings if and only if the asymptotic center relative to each nonempty weakly compact convex subset of each bounded sequence of X is compact.*

Proof. Suppose the asymptotic center of any bounded sequence of X relative to any nonempty weakly compact convex subset of X is compact. Let E be a weakly compact convex subset of X and $T : E \to E$ a D-type mapping having $\{x_n\}$ as an asymptotic center sequence. Let r and A, respectively, be the asymptotic radius and the asymptotic center of $\{x_n\}$ relative to E. Since E is weakly compact and convex, A is nonempty weakly compact and convex. For every $x \in A$, since $\{x_n\}$ is an asymptotic center sequence for T, we have

$$r \le \limsup_{n\to\infty} \|x_n - Tx\| \le \limsup_{n\to\infty} \|x_n - x\| = r. \tag{3.5}$$

Hence $T(x) \in A$, which implies that A is T-invariant. By the assumption, A is a compact set. By using Schauder's fixed point theorem, we can conclude that T has a fixed point in A and hence T has a fixed point in E.

Now suppose X has the weak fixed point property for D-type mappings, and suppose there exists a weakly compact convex subset K of X and a bounded sequence $\{x_n\}$ in X whose asymptotic center A relative to K is not compact. By Klee's theorem (see [4, page 203]), there exists a continuous, fixed point free mapping $T : A \to A$. We see that

$\{x_n\}$ is an asymptotic center sequence for T. Indeed, since $Tx \in A$ for each $x \in A$, we have

$$\limsup_{n\to\infty}\|x_n - Tx\| = r = \limsup_{n\to\infty}\|x_n - x\|. \tag{3.6}$$

Then T is a D-type mapping. Thus T should have a fixed point which is a contradiction. □

In 2007, García-Falset et al. [5] introduced another concept of centers of mappings.

Definition 3.5. Let E be a bounded closed convex subset of a Banach space X. A point $x_0 \in X$ is said to be a center for a mapping $T : E \to X$ if, for each $x \in E$,

$$\|Tx - x_0\| \le \|x - x_0\|. \tag{3.7}$$

A mapping $T : E \to X$ is said to be a J-type mapping whenever it is continuous and it has some center $x_0 \in X$.

Definition 3.6. We say that a Banach space X has the J-weak fixed point property if every J-type self-mapping of every weakly compact subset E of X has a fixed point.

Employing the above definitions, the authors proved a characterization of the geometrical property (C) of the Banach spaces introduced in 1973 by Bruck Jr. [6]: a Banach space X has property (C) whenever the weakly compact convex subsets of its unit sphere are compact sets.

Theorem 3.7 (see [5, Theorem 16]). *Let X be a Banach space. Then X has property (C) if and only if X has the J-weak fixed point property.*

It is easy to see that a center $x_0 \in X$ of a mapping $T : E \to X$ can be seen as an asymptotic center sequence $\{x_n\}$ for the mapping T by setting $x_n \equiv x_0$ for all $n \in \mathbb{N}$. This leads to the following conclusion.

Proposition 3.8. *Let $T : E \to X$ be a J-type mapping. Then T is a D-type mapping.*

Consequently, we have the following proposition.

Proposition 3.9. *Let X be a Banach space. If X has the weak fixed point property for D-type mappings, then X has the J-weak fixed point property.*

From Theorems 3.4 and 3.7, and Proposition 3.9, we can conclude this paper by the following result.

Theorem 3.10. *Let X be a Banach space. If the asymptotic center relative to every nonempty weakly compact convex subset of each bounded sequence of X is compact, then X has property (C).*

Dedication

The authors dedicate this work to Professor Sompong Dhompongsa on the occasion of his 60th birthday.

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