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Research Article

New Types of Continuities

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A new concept of quasi slowly oscillating continuity is introduced. Furthermore, it is shown that this kind of continuity implies ordinary continuity, but the converse is not always true.

1. Introduction

Firstly, some definitions and notations will be given in the following. Throughout this paper, $\mathbf{N}$ will denote the set of all positive integers. We will use boldface letters $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}, \ldots$ for sequences $\mathbf{x} = (x_n)$, $\mathbf{y} = (y_n)$, and $\mathbf{z} = (z_n), \ldots$ of terms in $\mathbf{R}$, the set of all real numbers. Also, $\mathbf{s}$ and $\mathbf{c}$ will denote the set of all sequences of points in $\mathbf{R}$ and the set of all convergent sequences of points in $\mathbf{R}$, respectively.

A sequence $\mathbf{x} = (x_n)$ of points in $\mathbf{R}$ is called statistically convergent [1] to an element ℓ of $\mathbf{R}$ if

$$\lim_{n \to \infty} \frac{1}{n} |\{k \le n : |x_k - \ell| \ge \varepsilon\}| = 0, \tag{1.1}$$

for every $\varepsilon > 0$, and this is denoted by $st - \lim_{n \to \infty} x_n = \ell$.

A sequence $\mathbf{x} = (x_n)$ of points in $\mathbf{R}$ is slowly oscillating [2], denoted by $\mathbf{x} \in \mathbf{SO}$, if

$$\lim_{\lambda \to 1^+} \overline{\lim_{n}} \max_{n+1 \le k \le [\lambda n]} |x_k - x_n| = 0, \tag{1.2}$$

where $[\lambda n]$ denotes the integer part of λn. This is equivalent to the following: $x_m - x_n \to 0$ whenever $1 \le m/n \to 1$ as $m, n \to \infty$. In terms of $\varepsilon > 0$ and δ, this is also equivalent to the case when for any given $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon) > 0$ and the positive integer $N = N(\varepsilon)$ such that $|x_m - x_n| < \varepsilon$ if $n \geq N(\varepsilon)$ and $n \leq m \leq (1+\delta)n$.

By a method of sequential convergence, or briefly a method, we mean a linear function G defined on a sublinear space of $\mathbf{s}$, denoted by $c_G(\mathbf{R})$, into $\mathbf{R}$. A sequence $\mathbf{x} = (x_n)$ is said to be G-convergent [3] to ℓ if $\mathbf{x} \in c_G(\mathbf{R})$ and $G(\mathbf{x}) = \ell$. In particular, lim denotes the limit function $\lim \mathbf{x} = \lim_n x_n$ on the linear space $\mathbf{c}$. A method G is called regular if every convergent sequence $\mathbf{x} = (x_n)$ is G-convergent with $G(\mathbf{x}) = \lim \mathbf{x}$. A method G is called subsequentially convergent if whenever $\mathbf{x}$ is G-convergent with $G(\mathbf{x}) = \ell$, then there is a subsequence (x_{n_k}) of $\mathbf{x}$ with $\lim_k x_{n_k} = \ell$. A function f is called G-continuous [3] if $G(f(\mathbf{x})) = f(G(\mathbf{x}))$ for any G-convergent sequence $\mathbf{x}$. Here we note that for special $G = st - \lim$, f is called statistically continuous [3].

Our goal in this paper is to introduce a new concept of quasi slowly oscillating continuity, which cannot be given by means of any G as in [3]. It is proved that quasi slowly oscillating continuity implies ordinary and statistical continuities. We also introduce several other types of continuities and a new type of compactness.

2. Slowly Oscillating Continuity

In [4], the concept of slowly oscillating continuity is defined in the sense that a function f is slowly oscillating continuous, denoted by $f \in \mathbf{SOC}$, if it transforms slowly oscillating sequences to slowly oscillating sequences, that is, $(f(x_n))$ is slowly oscillating whenever (x_n) is slowly oscillating.

We note that slowly oscillating continuity cannot be obtained by any sequential method G.

In the following theorem, we prove that the set of slowly oscillating continuous functions is a closed subset of the space of all continuous functions.

Theorem 2.1. *The set of slowly oscillating continuous functions on a subset E of $\mathbf{R}$ is a closed subset of the set of all continuous functions on E, that is, $\overline{\mathbf{SOC}(E)} = \mathbf{SOC}(E)$, where $\mathbf{SOC}(E)$ is the set of all slowly oscillating continuous functions on E, and $\overline{\mathbf{SOC}(E)}$ denotes the set of all cluster points of $\mathbf{SOC}(E)$.*

Proof. Let f be any element in the closure of $\mathbf{SOC}(E)$. Then there exists a sequence of points in $\mathbf{SOC}(E)$ such that $\lim_{k\to\infty} f_k = f$. To show that f is slowly oscillating continuous, take any slowly oscillating sequence (x_n). Let $\varepsilon > 0$. Since (f_k) converges to f, there exists a positive integer N such that for all $x \in E$ and for all $n \geq N$, $|f(x) - f_n(x)| < \varepsilon/3$. As f_N is slowly oscillating continuous, there exist a positive integer $N_1 \geq N$ and a δ such that $|f_N(x_n) - f_N(x_m)| < \varepsilon/3$ if $n \geq N_1$ and $n \leq m \leq (1+\delta)n$. Hence, for all $n \geq N_1$ and $n \leq m \leq (1+\delta)n$,

$$\begin{aligned} \left|f(x_n) - f(x_m)\right| &\leq \left|f(x_n) - f_N(x_n)\right| + \left|f_N(x_n) - f_N(x_m)\right| + \left|f_N(x_m) - f(x_m)\right| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned} \tag{2.1}$$

This completes the proof of the theorem. □

Corollary 2.2. *The set of all slowly oscillating continuous functions on a subset E of $\mathbf{R}$ is a complete subspace of the space of all continuous functions on E.*

Proof. The proof follows from Theorem 2.1. □

The following theorem shows that on a slowly oscillating compact subset A of $\mathbf{R}$, slowly oscillating continuity implies uniform continuity.

Theorem 2.3. *Let A be a slowly oscillating compact subset of $\mathbf{R}$ and let $f : A \to \mathbf{R}$ be slowly oscillating continuous on A. Then f is uniformly continuous on A.*

Proof. Assume that f is not uniformly continuous on A. Then there exist ϵ_0 and sequences (x_n) and (y_n) in A such that

$$\begin{gathered} |x_n - y_n| < 1/n, \\ |f(x_n) - f(y_n)| \geq \epsilon_0 \end{gathered} \tag{2.2}$$

for all $n \in \mathbf{N}$. Since A is slowly oscillating compact, there is a slowly oscillating subsequence (x_{n_k}) of (x_n). It is clear that the corresponding sequence (y_{n_k}) is also slowly oscillating, since

$$|y_{n_k} - y_{n_m}| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - x_{n_m}| + |x_{n_m} - y_{n_m}|. \tag{2.3}$$

If f is slowly oscillating continuous, then the sequences $(f(x_{n_k}))$ and $(f(y_{n_k}))$ are slowly oscillating. By (2.2), this is not possible. Consequently, we obtain that if f is slowly oscillating continuous on a slowly oscillating compact set A of $\mathbf{R}$, then f is uniformly continuous on A. □

Corollary 2.4. *For any regular subsequential method G, any slowly oscillating continuous function is G-continuous.*

3. Quasi slowly Oscillating Continuity

A sequence $\mathbf{x} = (x_n)$ is called quasi slowly oscillating, denoted by $\mathbf{x} \in \mathbf{QSO}$, if $(\Delta x_n) = (x_n - x_{n+1})$ is a slowly oscillating sequence. The concept of slowly oscillating continuity [4] suggests a new kind of continuity. A function f is quasi slowly oscillating continuous if it transforms quasi slowly oscillating sequences to quasi slowly oscillating sequences, that is, $(f(x_n))$ is quasi slowly oscillating whenever (x_n) is quasi slowly oscillating.

Notice that any slowly oscillating sequence is quasi slowly oscillating, but the converse is not always true. Indeed, let (x_n) be slowly oscillating. Hence, from the definition of slow oscillation of a sequence and the line below

$$\Delta x_n - \Delta x_m = (x_n - x_m) - (x_{n+1} - x_{m+1}), \tag{3.1}$$

we immediately have that (x_n) is quasi slowly oscillating. To see that the converse is not true, we consider the following example. The sequence (x_n) defined by $x_n = \sum_{k=1}^{n}((1/k)\sum_{j=1}^{k}(1/j))$ is quasi slowly oscillating, but not slowly oscillating. We see that composition of two quasi slowly oscillating continuous functions is quasi slowly oscillating continuous. It is clear that the sum of two quasi slowly oscillating continuous functions is quasi slowly oscillating continuous. The product of two quasi slowly oscillating continuous functions needs not be quasi slowly oscillating. For example, for the quasi slowly oscillating continuous functions f and g defined by $f(x) = x$ and $g(x) = x$, the function fg is not quasi slowly oscillating. When both f and g are bounded and quasi slowly oscillating continuous, we have the following theorem.

Theorem 3.1. *If f and g are bounded quasi slowly oscillating continuous functions on a subset E of* $\mathbf{R}$*, then their product fg is quasi slowly oscillating continuous on E.*

This follows from the definition of quasi slowly oscillating continuity.

In connection with slowly oscillating sequences, quasi slowly oscillating sequences, and convergent sequences, the problem arises to investigate the following types of continuity of functions on $\mathbf{R}$.

(**QSO-QSO**) $(x_n) \in \mathbf{QSO} \Rightarrow (f(x_n)) \in \mathbf{QSO}$.

(**QSO-c**) $(x_n) \in \mathbf{QSO} \Rightarrow (f(x_n)) \in \mathbf{c}$.

(**c-c**) For each x_0 in the domain of f, $\lim_{n\to\infty} f(x_n) = f(x_0)$ whenever $\lim_{n\to\infty} x_n = x_0$.

(**c-QSO**) $(x_n) \in \mathbf{c} \Rightarrow (f(x_n)) \in \mathbf{QSO}$.

(**QSO-SO**) $(x_n) \in \mathbf{QSO} \Rightarrow (f(x_n)) \in \mathbf{SO}$.

(**SO-QSO**) $(x_n) \in \mathbf{SO} \Rightarrow (f(x_n)) \in \mathbf{QSO}$.

(**u**) is uniform continuity of f.

It is clear that **(QSO-QSO)** implies **(SO-QSO)**, but **(SO-QSO)** needs not imply **(QSO-QSO)**. Also **(QSO-c)** implies **(c-QSO)** and **(QSO-c)** implies **(c-c)** and we see that **(c-c)** needs not imply **(QSO-c)** since the identity function is an example. We also note that **(u)** implies **(SO-QSO)**.

Theorem 3.2. *If f is quasi slowly oscillating continuous on* $\mathbf{R}$*, then it is continuous in the ordinary sense.*

Proof. Let (x_n) be any convergent sequence with $\lim_{k\to\infty} x_k = x_0$. Then the sequence $(x_1, x_0, x_2, x_0, \ldots, x_n, x_0, \ldots)$ also converges to x_0 and is quasi slowly oscillating. By the hypothesis, the sequence $(f(x_1)-f(x_0), f(x_0)-f(x_2), f(x_2)-f(x_0), f(x_0)-f(x_3), \ldots)$ is slowly oscillating. It follows from the definition of slow oscillation that $\lim_{k\to\infty}(f(x_k) + f(x_{k+1})) = 2f(x_0)$ and $\lim_{k\to\infty}(f(x_k) - f(x_{k+1})) = 0$. Hence, we have $\lim_{k\to\infty} f(x_k) = f(x_0)$. This completes the proof. □

Corollary 3.3. *Any quasi slowly oscillating continuous function is G-continuous for any regular subsequential method G.*

Corollary 3.4. *Any quasi slowly oscillating continuous function is statistically continuous.*

It is well known that uniform limit of a sequence of continuous functions is continuous. This is also true for quasi slowly oscillating continuity, that is, uniform limit of a sequence of quasi slowly oscillating continuous functions is quasi slowly oscillating continuous.

Theorem 3.5. *If (f_n) is a sequence of quasi slowly oscillating continuous functions defined on a subset E of* $\mathbf{R}$ *and let (f_n) is uniformly convergent to a function f, then f is quasi slowly oscillating continuous on E.*

Proof. Let (f_n) be a sequence of quasi slowly oscillating continuous functions defined on a subset E of $\mathbf{R}$ and (f_n) be uniformly convergent to a function f. Let (x_n) be a quasi slowly oscillating sequence and $\varepsilon > 0$. As (f_n) is uniformly convergent to f, there exists a positive integer N such that $|f_n(x) - f(x)| < \varepsilon/5$ for all $x \in E$ whenever $n \geq N$. Since f_N is quasi slowly oscillating continuous, there exist a $\delta > 0$ and a positive integer N_1 such that

$$\left|\Delta f_N(x_m) - \Delta f_N(x_n)\right| < \frac{\varepsilon}{5} \tag{3.2}$$

for $n \geq N_1(\varepsilon)$ and $n \leq m \leq (1+\delta)n$. Now for $n \geq N_1(\varepsilon)$ and $n \leq m \leq (1+\delta)n$, we have

$$\begin{aligned}\left|\Delta f(x_m) - \Delta f(x_n)\right| &\leq \left|f(x_m) - f_N(x_m)\right| + \left|f_N(x_{m+1}) - f(x_{m+1})\right| \\ &\quad + \left|f_N(x_n) - f(x_n)\right| + \left|f(x_{n+1}) - f_N(x_{n+1})\right| \\ &\quad + \left|f_N(x_m) - f_N(x_{m+1}) - f_N(x_n) + f_N(x_{n+1})\right| \\ &\leq \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} = \varepsilon.\end{aligned} \tag{3.3}$$

This completes the proof of the theorem. □

Definition 3.6. A subset F of $\mathbf{R}$ *is called slowly oscillating compact [4] if whenever* $\mathbf{x} = (x_n)$ *is a sequence of points in F, there is a slowly oscillating subsequence* $\mathbf{y} = (y_{n_k})$ *of* $\mathbf{x}$.

Firstly, we see that quasi slowly oscillating compactness cannot be obtained by any G-sequential compactness in the manner of [5].

We note that any compact subset of $\mathbf{R}$ is quasi slowly oscillating compact and any subset of a quasi slowly oscillating compact subset of $\mathbf{R}$ is also quasi slowly oscillating compact. Thus, intersection of two quasi slowly oscillating compact subsets of $\mathbf{R}$ is quasi slowly oscillating compact. More generally any intersection of quasi slowly oscillating compact subsets of $\mathbf{R}$ is quasi slowly oscillating compact.

We also note that for any regular subsequential method G, any G-sequentially compact subset of $\mathbf{R}$ is quasi slowly oscillating compact.

Notice that the union of two quasi slowly oscillating compact subsets of $\mathbf{R}$ is quasi slowly oscillating compact. We see that any finite union of quasi slowly oscillating compact subsets of $\mathbf{R}$ is quasi slowly oscillating compact, but any union of quasi slowly oscillating compact subsets of $\mathbf{R}$ is not always quasi slowly oscillating compact. For example, consider the sets $D_n = \{n\}$ for $n \in \mathbf{N}$. Then for each constant $n \in \mathbf{N}$, the set D_n is quasi slowly oscillating compact, but $\bigcup_{n=1}^{\infty} D_n = \mathbf{N}$ is not quasi slowly oscillating compact.

Theorem 3.7. *Let F be a quasi slowly oscillating compact subset of* $\mathbf{R}$ *and let f be a quasi slowly oscillating continuous function. Then $f(F)$ is quasi slowly oscillating compact.*

Proof. Let $\mathbf{y} = (y_n)$ be any sequence of points in $f(F)$. Then there exists a sequence $\mathbf{x} = (x_n)$ such that $y_n = f(x_n)$ for each $n \in \mathbf{N}$. As F is quasi slowly oscillating compact, there exists a quasi slowly oscillating subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of the sequence $\mathbf{x}$. Since f is quasi slowly oscillating continuous, $f(\mathbf{z}) = (f(z_k)) = (f(x_{n_k}))$ is quasi slowly oscillating. It follows that $f(F)$ is quasi slowly oscillating compact. □

Corollary 3.8. *Quasi slowly oscillating continuous image of any compact subset of* $\mathbf{R}$ *is compact.*

Theorem 3.9. *The set of quasi slowly oscillating continuous functions on a subset E of $\mathbf{R}$ is a closed subset of the set of all continuous functions on E, that is, $\overline{\mathbf{QSOC}(E)} = \mathbf{SOC}(E)$, where $\mathbf{QSOC}(E)$ is the set of all slowly oscillating continuous functions on E, and $\overline{\mathbf{QSOC}(E)}$ denotes the set of all cluster points of $\mathbf{QSOC}(E)$.*

Proof. Let f be any element in the closure of $\mathbf{QSOC}(E)$. Then there exists a sequence of points in $\mathbf{QSOC}(E)$ such that $\lim_{k\to\infty} f_k = f$. To show that f is quasi slowly oscillating continuous, take any quasi slowly oscillating sequence (x_n). Let $\varepsilon > 0$. Since (f_k) converges to f, there exists a positive integer N such that for all $x \in E$ and for all $k \geq N$, $|f(x) - f_k(x)| < \varepsilon/5$. As f_N is quasi slowly oscillating continuous, there exist a positive integer $N_1 \geq N$ and a δ such that $|\Delta f_N(x_n) - \Delta f_N(x_m)| < \varepsilon/5$ if $n \geq N_1$ and $n \leq m \leq (1+\delta)n$. Hence, for all $n \geq N_1$,

$$\begin{aligned}
\left|\Delta f(x_n) - \Delta f(x_m)\right| &\leq \left|\Delta f(x_n) - \Delta f_N(x_n)\right| + \left|\Delta f_N(x_n) - \Delta f_N(x_m)\right| + \left|\Delta f_N(x_m) - \Delta f(x_m)\right| \\
&= \left|f(x_n) - f(x_{n+1}) - f_N(x_n) + f_N(x_{n+1})\right| + \left|\Delta f_N(x_n) - \Delta f_N(x_m)\right| \\
&\quad + \left|f_N(x_m) - f_N(x_{m+1}) - f(x_m) + f(x_{m+1})\right| \\
&\leq \left|f(x_n) - f_N(x_n)\right| + \left|f_N(x_{n+1}) - f(x_{n+1})\right| + \left|\Delta f_N(x_n) - \Delta f_N(x_m)\right| \\
&\quad + \left|f_N(x_m) - f(x_m)\right| + \left|f(x_{m+1}) - f_N(x_{m+1})\right| \\
&\leq \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} = \varepsilon.
\end{aligned} \tag{3.4}$$

This completes the proof of the theorem. □

Finally we note that the following further investigation problems arose.

Problem 1. For further study, we suggest to investigate quasi slowly oscillating sequences of fuzzy points and quasi slowly oscillating continuity for the fuzzy functions. However, due to the change in settings, the definitions and methods of proofs will not always be analogous to those of the present work (see, e.g., [6]).

Problem 2. Investigate a theory in dynamical systems by introducing the following concept: two dynamical systems are called pseudo conjugate if there is 1-1, onto, pseudo continuous h, such that h^{-1} is Δ-pseudo continuous, and h commutes the mappings at each point x.

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