

Research Article

On Semi-Uniform Kadec-Klee Banach Spaces

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Received 30 June 2010; Accepted 31 August 2010

Academic Editor: S. Reich

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Inspired by the concept of U -spaces introduced by Lau, (1978), we introduced the class of semi-uniform Kadec-Klee spaces, which is a uniform version of semi-Kadec-Klee spaces studied by Vlasov, (1972). This class of spaces is a wider subclass of spaces with weak normal structure and hence generalizes many known results in the literature. We give a characterization for a certain direct sum of Banach spaces to be semi-uniform Kadec-Klee and use this result to construct a semi-uniform Kadec-Klee space which is not uniform Kadec-Klee. At the end of the paper, we give a remark concerning the uniformly alternative convexity or smoothness introduced by Kadets et al., (1997).

1. Introduction

Let X be a real Banach space with the unit sphere $S_X = \{x \in X : \|x\| = 1\}$ and the closed unit ball $B_X = \{x \in X : \|x\| \leq 1\}$. In this paper, the strong and weak convergences of a sequence $\{x_n\}$ in X to an element $x \in X$ are denoted by $x_n \rightarrow x$ and $x_n \rightharpoonup x$, respectively. We also let

$$\text{sep}\{x_n\} = \inf\{\|x_n - x_m\| : n < m\}. \quad (1.1)$$

Definition 1.1 (see [1]). We say that a Banach space X is a *Kadec-Klee* space if

$$(KK) : \left. \begin{array}{l} \{x_n\} \subset B_X \\ x_n \rightharpoonup x \\ \text{sep}\{x_n\} > 0 \end{array} \right\} \implies \|x\| < 1. \quad (1.2)$$

A uniform version of the KK property is given in the following definition.

Definition 1.2 (see [2]). We say that a Banach space X is *uniform Kadec-Klee* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$(\text{UKK}) : \left. \begin{array}{l} \{x_n\} \subset B_X \\ x_n \rightarrow x \\ \text{sep}\{x_n\} \geq \varepsilon \end{array} \right\} \implies \|x\| \leq 1 - \delta. \quad (1.3)$$

Two properties above are weaker than the following one.

Definition 1.3 (see [3]). We say that a Banach space X is *uniformly convex* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$(\text{UC}) : \left. \begin{array}{l} x, y \in B_X \\ \|x - y\| \geq \varepsilon \end{array} \right\} \implies \frac{1}{2}\|x + y\| \leq 1 - \delta. \quad (1.4)$$

Let us summarize a relationship between these properties in the following implication diagram:

$$\text{UC} \implies \text{UKK} \implies \text{KK}. \quad (1.5)$$

In the literature, there are some generalizations of UC and KK.

Definition 1.4 (see [4]). We say that a Banach space X is a *U-space* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$(\text{U-space}) : \left. \begin{array}{l} x, y \in B_X \\ \langle x - y, f \rangle \geq \varepsilon \text{ for some } f \in \nabla_x \end{array} \right\} \implies \frac{1}{2}\|x + y\| \leq 1 - \delta. \quad (1.6)$$

Here $\nabla_x = \{f \in S_{X^*} : \langle x, f \rangle = \|x\|\}$.

Definition 1.5 (see [5]). We say that a Banach space X is *semi-Kadec-Klee* if

$$(\text{semi-KK}) : \left. \begin{array}{l} \{x_n\} \subset S_X \\ x_n \rightarrow x \in S_X \end{array} \right\} \implies \langle x, f_n \rangle \rightarrow 1 \quad \forall \{f_n\} \subset S_{X^*} \\ \text{satisfying } f_n \in \nabla_{x_n} \quad \forall n. \quad (1.7)$$

Some interesting results concerning semi-KK property are studied by Megginson [6].

Remark 1.6. It is clear that

$$\text{U-space} \implies \text{semi-KK}. \quad (1.8)$$

Remark 1.7. A Banach space X is semi-KK if and only if

$$\left. \begin{array}{l} \{x_n\} \subset B_X \\ x_n \rightarrow x \\ \inf \{ \langle x_n - x, f_n \rangle : n \in \mathbb{N} \} > 0 \text{ for some } \{f_n\} \subset S_{X^*} \\ \text{satisfying } f_n \in \nabla_{x_n}, \forall n \end{array} \right\} \implies \|x\| < 1. \quad (1.9)$$

We now introduce a property lying between U -space and semi-KK.

Definition 1.8. We say that a Banach space X is *semi-uniform Kadec-Klee* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\text{(semi-UKK)} : \left. \begin{array}{l} \{x_n\} \subset S_X \\ x_n \rightharpoonup x \\ \langle x_n - x, f_n \rangle \geq \varepsilon, \text{ for some } \{f_n\} \subset S_{X^*} \\ \text{satisfying } f_n \in \nabla_{x_n} \quad \forall n \end{array} \right\} \implies \|x\| \leq 1 - \delta. \quad (1.10)$$

In this paper, we prove that semi-UKK property is a nice generalization of U -space and semi-KK property. Moreover, every semi-UKK space has weak normal structure. We also give a characterization of the direct sum of finitely many Banach spaces which is semi-KK and semi-UKK. We use such a characterization to construct a Banach space which is semi-UKK but not UKK. Finally we give a remark concerning the uniformly alternative convexity or smoothness introduced by Kadets et al. [7].

2. Results

2.1. Some Implications

For a sequence $\{x_n\} \subset S_X$ and $\{f_n\} \subset S_{X^*}$ satisfying $f_n \in \nabla_{x_n}$ for all n , we let

$$\text{sep}_{\{f_n\}}\{x_n\} = \inf\{\langle x_n - x_m, f_n \rangle : n < m\}. \quad (2.1)$$

It is clear that $\text{sep}_{\{f_n\}}\{x_n\} \leq \text{sep}\{x_n\}$.

Theorem 2.1. *A Banach space X is semi-UKK if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\left. \begin{array}{l} \{x_n\} \subset S_X \\ x_n \rightharpoonup x \\ \text{sep}_{\{f_n\}}\{x_n\} \geq \varepsilon, \text{ for some } \{f_n\} \subset S_{X^*} \\ \text{satisfying } f_n \in \nabla_{x_n}, \forall n \end{array} \right\} \implies \|x\| \leq 1 - \delta. \quad (2.2)$$

The following theorem shows that our new property is well placed.

Theorem 2.2. *The following implication diagram holds:*

$$\begin{array}{ccccc} UC & \implies & UKK & \implies & KK \\ \Downarrow & & \Downarrow & & \Downarrow \\ U\text{-space} & \implies & \text{semi-UKK} & \implies & \text{semi-KK}. \end{array} \quad (2.3)$$

Remark 2.3. The implication U -space \implies semi-UKK strengthens the result of Vlasov. In fact, it was proved by Vlasov ([5, Theorem 7]) that every uniformly smooth Banach space is semi-KK and by Lau ([4, Corollary 2.5]) that every uniformly smooth Banach space is a U -space.

2.2. Sufficient Conditions for Weak Normal Structure

Recall that a Banach space X has *weak normal structure* (*normal structure*, resp.) if for every weakly compact (bounded and closed, resp.) convex subset C of X containing more than one point there exists a point $x_0 \in C$ such that $\sup\{\|x_0 - z\| : z \in C\} < \text{diam}C$ (see [8]). It is clear that normal structure and weak normal structure coincide whenever the space is reflexive. It was Kirk [9] who proved that if a Banach space X has weak normal structure, then every nonexpansive self-mapping defined on a weakly compact convex subset of X has a fixed point. In this subsection, we present a new and wider class of Banach spaces with weak normal structure.

Lemma 2.4 (Bollobás [10]). *Let X be a Banach space, and let $0 < \varepsilon < 1$. Given $z \in B_X$ and $h \in S_{X^*}$ with $1 - \langle z, h \rangle < \varepsilon^2/4$, then there exist $y \in S_X$ and $g \in \nabla_y$ such that $\|y - z\| < \varepsilon$ and $\|g - h\| < \varepsilon$.*

Theorem 2.5. *If a Banach space X has the following property:
there are two constants $0 < \varepsilon < 1$ and $0 < \delta < 1$ such that*

$$\left. \begin{array}{l} \{x_n\} \subset S_X \\ x_n \rightarrow x \\ \langle x_n - x, f_n \rangle \geq \varepsilon, \text{ for some } \{f_n\} \subset S_{X^*} \\ \text{satisfying } f_n \in \nabla_{x_n}, \forall n \end{array} \right\} \implies \|x\| \leq 1 - \delta, \quad (2.4)$$

then X has weak normal structure.

Proof. Suppose that X does not have weak normal structure. Then there exists a sequence $\{x_n\}$ in X such that the following properties are satisfied (see [11]):

- (i) $\text{diam}\{x_n\} = 1$;
- (ii) $x_n \rightarrow 0$;
- (iii) $\|x_n - x\| \rightarrow 1$ for all $x \in \overline{\text{co}}\{x_n\}$.

In particular, since $0 \in \overline{\text{co}}\{x_n\}$, we have $\|x_n\| \rightarrow 1$.

We now show that for each $0 < \varepsilon < 1$ and $0 < \delta < 1$, there are an element $z \in X$ and sequences $\{z_n\} \subset S_X$ and $\{f_n\} \subset S_{X^*}$ such that

- (i) $z_n \rightarrow z$;
- (ii) $\langle z_n - z, f_n \rangle \geq \varepsilon$ and $\langle z_n, f_n \rangle = 1$ for all n ;
- (iii) $\|z\| > 1 - \delta$.

To see this, let $0 < \delta < 1$ and $0 < \varepsilon < 1$ be given. We may assume that $\|x_1\| > 1 - \delta$. For each n , let $g_n \in \nabla_{x_n - (1/2)x_1}$. This implies $\langle x_n - (1/2)x_1, g_n \rangle = \|x_n - (1/2)x_1\| \rightarrow 1$. We observe that

$$\begin{aligned} 1 &= \left\langle x_n - \frac{1}{2}x_1, g_n \right\rangle \\ &= \frac{1}{2}\langle x_n - x_1, g_n \rangle + \frac{1}{2}\langle x_n, g_n \rangle \\ &\leq \frac{1}{2}\|x_n - x_1\| + \frac{1}{2}\|x_n\| \rightarrow 1. \end{aligned} \quad (2.5)$$

In particular, $\langle x_n - x_1, g_n \rangle \rightarrow 1$ and $\langle x_n, g_n \rangle \rightarrow 1$.

By Lemma 2.4, there are sequences $\{z_n\} \subset S_X$ and $\{f_n\} \subset S_{X^*}$ such that

$$f_n \in \nabla_{z_n}, \quad \forall n, \quad \|z_n - (x_n - x_1)\| \longrightarrow 0, \quad \|f_n - g_n\| \longrightarrow 0. \quad (2.6)$$

Put $z = -x_1$. It is clear that $z_n \rightharpoonup z$. Moreover, we have

$$\begin{aligned} & |\langle z_n - z, f_n \rangle - \langle x_n, g_n \rangle| \\ & \leq |\langle z_n - z, f_n \rangle - \langle z_n - z, g_n \rangle| + |\langle z_n - z, g_n \rangle - \langle x_n, g_n \rangle| \\ & \leq \|z_n - z\| \|f_n - g_n\| + \|z_n - (x_n - x_1)\| \|g_n\| \longrightarrow 0. \end{aligned} \quad (2.7)$$

Consequently, $\langle z_n - z, f_n \rangle \rightarrow 1 > \varepsilon$.

By discarding terms from the beginning of the sequence $\{z_n\}$, we obtain a contradiction with the assumption. This finishes the proof. \square

Corollary 2.6. *A Banach space X has weak normal structure if X is semi-UKK.*

Corollary 2.7 (see [12]). *Every wUKK space has weak normal structure. Recall that a Banach space X is wUKK if there are two constants $\varepsilon > 0$ and $\delta > 0$ such that*

$$(wUKK) : \left. \begin{array}{l} \{x_n\} \subset B_X \\ x_n \rightharpoonup x \\ \text{sep}\{x_n\} \geq \varepsilon \end{array} \right\} \implies \|x\| \leq 1 - \delta. \quad (2.8)$$

Corollary 2.8 (see [13]). *Every U -space has weak normal structure.*

2.3. Stability Results under Taking Finite Direct Sums

In this subsection, we give a necessary and sufficient condition for the direct sum of finitely many Banach spaces to be semi-KK and semi-UKK. Let us recall some definitions.

Let Z be a finite dimensional normed space $(\mathbb{R}^N, \|\cdot\|_Z)$, which has a *monotone* norm; that is,

$$\|(a_1, \dots, a_N)\|_Z \leq \|(b_1, \dots, b_N)\|_Z \quad (2.9)$$

if $0 \leq a_i \leq b_i$ for each $i = 1, \dots, N$. We write $(X_1 \oplus \dots \oplus X_N)_Z$ for the Z -direct sum of the Banach spaces X_1, \dots, X_N equipped with the norm

$$\|(x_1, \dots, x_N)\| = \|(\|x_1\|_{X_1}, \dots, \|x_N\|_{X_N})\|_Z, \quad (2.10)$$

where $x_i \in X_i$ for each $i = 1, \dots, N$.

One should notice that in defining $(X_1 \oplus \dots \oplus X_N)_Z$, we only need to know the behavior of the Z -norm on \mathbb{R}_+^N . Consequently, we can and do assume that the Z -norm is *absolute*; that is,

$$\|(a_1, \dots, a_N)\|_Z = \|(|a_1|, \dots, |a_N|)\|_Z, \quad \forall (a_1, \dots, a_N) \in \mathbb{R}^N. \quad (2.11)$$

The following fact can be proved easily but plays an important role in this paper.

Lemma 2.9. *Suppose that X_1, \dots, X_N are Banach spaces. Then each element f in the dual $(X_1 \oplus \dots \oplus X_N)_Z^*$ of the Z -direct sum $(X_1 \oplus \dots \oplus X_N)_Z$ is identified with the element (x_1^*, \dots, x_N^*) in the Z^* -direct sum $(X_1^* \oplus \dots \oplus X_N^*)_{Z^*}$ such that*

$$\langle (x_1, \dots, x_n), f \rangle = \langle x_1, x_1^* \rangle + \dots + \langle x_N, x_N^* \rangle \quad (2.12)$$

for all $(x_1, \dots, x_N) \in (X_1 \oplus \dots \oplus X_N)_Z$. Moreover,

$$\|f\| = \left\| (\|x_1^*\|_{X_1^*}, \dots, \|x_N^*\|_{X_N^*}) \right\|_{Z^*}. \quad (2.13)$$

Recently, Dowling et al. [14] proved the following theorem.

Theorem 2.10. *Let X_1, \dots, X_N be Banach spaces. Then $(X_1 \oplus \dots \oplus X_N)_Z$ is KK (UKK, resp.) if and only if for each $1 \leq j \leq N$,*

- (1) X_j is KK (UKK, resp.), and
- (2) either X_j is Schur or Z is strictly monotone in the j th coordinate.

Recall that a Banach space X is a *Schur* space if weak and norm sequential convergences coincide in X , and Z is *strictly monotone in the j th coordinate* if

$$\|(a_1, \dots, a_n)\|_Z < \|(b_1, \dots, b_n)\|_Z, \quad (2.14)$$

where $0 \leq a_i \leq b_i$ for each $i = 1, \dots, n$ and $0 \leq a_j < b_j$. Note that by the triangle inequality and the assumption that the Z -norm is absolute, Z is strictly monotone in the j th coordinate if and only if

$$\|(a_1, \dots, a_n)\|_Z < \|(b_1, \dots, b_n)\|_Z, \quad (2.15)$$

where $0 \leq a_i = b_i$ for each $i \neq j$ and $0 \leq a_j < b_j$.

We first define a generalization of Schur spaces.

Definition 2.11. A Banach space X is a *semi-Schur* space if

$$\left. \begin{array}{l} \{x_n\} \subset X \\ \text{(semi-Schur) : } x_n \rightharpoonup x \\ \{f_n\} \subset S_{X^*} \text{ satisfying } f_n \in \nabla_{x_n}, \quad \forall n \end{array} \right\} \implies \|x_n\| - \langle x, f_n \rangle \longrightarrow 0. \quad (2.16)$$

The following two propositions follow easily from the definition of semi-Schur spaces and semi-KK spaces.

Proposition 2.12. *A Banach space X is semi-Schur if and only if*

$$\left. \begin{array}{l} \{x_n\} \subset S_X \\ x_n \rightharpoonup x \\ \{f_n\} \subset S_{X^*} \text{ satisfying } f_n \in \nabla_{x_n}, \forall n \end{array} \right\} \implies \|x_n\| - \langle x, f_n \rangle \longrightarrow 0. \quad (2.17)$$

Proposition 2.13. *A Banach space X satisfies semi-KK property if and only if*

$$\left. \begin{array}{l} \{x_n\} \subset X \\ x_n \rightharpoonup x \\ \|x_n\| \longrightarrow \|x\| \\ \{f_n\} \subset S_{X^*} \text{ satisfying } f_n \in \nabla_{x_n}, \forall n \end{array} \right\} \implies \langle x, f_n \rangle \longrightarrow \|x\|. \quad (2.18)$$

We say that Z has *property (S- j)* where $1 \leq j \leq N$ if whenever $a_1, \dots, a_N, b_1, \dots, b_N \geq 0$ and $0 \leq a'_j < a_j$ satisfy

$$\begin{aligned} \left\| (a_1, \dots, a'_j, \dots, a_N) \right\|_Z &= \left\| (a_1, \dots, a_j, \dots, a_N) \right\|_Z = 1, \\ \left\| (b_1, \dots, b_j, \dots, b_N) \right\|_{Z^*} &= a_1 b_1 + \dots + a_j b_j + \dots + a_N b_N = 1, \end{aligned} \quad (2.19)$$

it follows that $b_j = 0$.

Theorem 2.14. *Suppose that X_1, \dots, X_N are Banach spaces. Then the direct sum $(X_1 \oplus \dots \oplus X_N)_Z$ is semi-KK if and only if for each $j = 1, \dots, N$*

- (a) X_j is semi-KK and
- (b) either X_j is semi-Schur or Z has property (S- j).

Proof. Sufficiency. Suppose that there exists $j \in \{1, \dots, N\}$ such that X_j is not semi-Schur and Z does not have property (S- j). For convenience, we may assume that $j = 1$. Since X_1 is not semi-Schur, there exist sequences $\{x_n^1\} \subset S_{X_1}$, $\{f_n^1\} \subset S_{X_1^*}$, and a number $\varepsilon_0 > 0$ such that

$$x_n^1 \rightharpoonup x^1 \in B_{X_1}, \quad f_n^1 \in \nabla_{x_n^1}, \quad \langle x^1, f_n^1 \rangle \leq 1 - \varepsilon_0, \quad \forall n. \quad (2.20)$$

Since Z does not have property (S-1), there exist numbers $a_1, \dots, a_N, b_1, \dots, b_N \geq 0$ such that the following properties are satisfied:

- (i) $\|(0, a_2, \dots, a_N)\|_Z = \|(a_1, a_2, \dots, a_N)\|_Z = 1$;
- (ii) $\|(b_1, \dots, b_N)\|_{Z^*} = a_1 b_1 + \dots + a_N b_N = 1$;
- (iii) $a_1 > 0$ and $b_1 > 0$.

For $j = 2, \dots, N$, let $x^j \in S_{X_j}$ and $f^j \in \nabla_{x^j}$. Now we put

$$\begin{aligned} \mathbf{x} &= (a_1 x^1, a_2 x^2, \dots, a_N x^N), \\ \mathbf{x}_n &= (a_1 x_n^1, a_2 x_n^2, \dots, a_N x_n^N), \\ \mathbf{f}_n &= (b_1 f_n^1, b_2 f_n^2, \dots, b_N f_n^N). \end{aligned} \quad (2.21)$$

It is clear that $\{\mathbf{x}_n\}$ is a sequence of norm one elements converging weakly to \mathbf{x} and $\mathbf{f}_n \in \nabla_{\mathbf{x}_n}$ for all n . Moreover, by the monotonicity of $\|\cdot\|_Z$, we have $\|\mathbf{x}\| = 1$. Finally, we show that $\langle \mathbf{x}, \mathbf{f}_n \rangle \rightarrow 1$. To see this, we consider

$$\begin{aligned} \langle \mathbf{x}, \mathbf{f}_n \rangle &= a_1 b_1 \langle x^1, f_n^1 \rangle + a_2 b_2 \langle x^2, f_n^2 \rangle + \dots + a_N b_N \langle x^N, f_n^N \rangle \\ &\leq (1 - \varepsilon_0) a_1 b_1 + a_2 b_2 + \dots + a_N b_N. \end{aligned} \quad (2.22)$$

It then follows from $a_1 > 0$ and $b_1 > 0$ that $(1 - \varepsilon_0) a_1 b_1 + \dots + a_N b_N < 1$. This shows that $(X_1 \oplus \dots \oplus X_N)_Z$ is not semi-KK and hence the first half of the proof is done.

Necessity. Suppose that the conditions (a) and (b) hold. Put

$$A = \{j : X_j \text{ is a semi-Schurs pace}\}, \quad B = \{j : Z \text{ has property (S-}j)\}. \quad (2.23)$$

Then, by (b), $A \cup B = \{1, \dots, N\}$. Let $\{\mathbf{x}_n\}$ be a sequence of norm one elements in $(X_1 \oplus \dots \oplus X_N)_Z$ converging weakly to a norm one element $\mathbf{x} \in (X_1 \oplus \dots \oplus X_N)_Z$ and $\{\mathbf{f}_n\}$ a sequence of norm one elements in $((X_1 \oplus \dots \oplus X_N)_Z)^* = (X_1^* \oplus \dots \oplus X_N^*)_{Z^*}$ such that $f_n \in \nabla_{\mathbf{x}_n}$ for all n . For convenience, let us write

$$\mathbf{x} = (x^1, \dots, x^N), \quad \mathbf{x}_n = (x_n^1, \dots, x_n^N), \quad \mathbf{f}_n = (f_n^1, \dots, f_n^N), \quad (2.24)$$

where $x^j, x_n^j \in X_j$ and $f_n^j \in X_j^*$ for all j and n . We prove that

$$\langle x^1, f_n^1 \rangle + \dots + \langle x^N, f_n^N \rangle \rightarrow 1. \quad (2.25)$$

Notice that

$$\begin{aligned} 1 &= \left\| (x_n^1, \dots, x_n^N) \right\| = \langle \mathbf{x}_n, \mathbf{f}_n \rangle = \langle x_n^1, f_n^1 \rangle + \dots + \langle x_n^N, f_n^N \rangle \\ &\leq \left\| x_n^1 \right\| \left\| f_n^1 \right\| + \dots + \left\| x_n^N \right\| \left\| f_n^N \right\| \leq \left\| (x_n^1, \dots, x_n^N) \right\| \left\| (f_n^1, \dots, f_n^N) \right\|^* = 1. \end{aligned} \quad (2.26)$$

Then $\langle x_n^j, f_n^j \rangle = \|x_n^j\| \|f_n^j\|$ and

$$\left\| x_n^1 \right\| \left\| f_n^1 \right\| + \dots + \left\| x_n^N \right\| \left\| f_n^N \right\| = 1 \quad (2.27)$$

for all n . In order to show that (2.25) holds, it suffices to show that

$$\langle x^j, f_n^j \rangle - \|x_n^j\| \|f_n^j\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty \quad (2.28)$$

for all $j = 1, \dots, N$.

Let us note the following facts:

- (i) $f_n^j / \|f_n^j\| \in \nabla_{x_n^j}$ provided that $f_n^j \neq 0$;
- (ii) $x_n^j \rightarrow x^j$ as $n \rightarrow \infty$ for all j .

We first prove that (2.28) holds for all $j \in A$. To see this, we note that if $f_n^j = 0$, then $\langle x^j, f_n^j \rangle = \|x_n^j\| \|f_n^j\| = 0$. Now, we assume that $f_n^j \neq 0$ for all n . It follows then that $f_n^j / \|f_n^j\| \in \nabla_{x_n^j}$ and hence from the semi-Schur property of X_j that $\langle x^j, f_n^j / \|f_n^j\| \rangle - \|x_n^j\| \rightarrow 0$.

Passing to a subsequence, we may assume that the following limits:

$$\lim_k \|x_{n_k}^j\|, \quad \lim_k \|f_{n_k}^j\| \text{ exist.} \quad (2.29)$$

Notice that

$$\begin{aligned} \left\| \left(\lim_k \|x_{n_k}^1\|, \dots, \lim_k \|x_{n_k}^N\| \right) \right\|_Z &= \left\| \left(\lim_k \|f_{n_k}^1\|, \dots, \lim_k \|f_{n_k}^N\| \right) \right\|_{Z^*} = 1, \\ \lim_k \|x_{n_k}^1\| \lim_k \|f_{n_k}^1\| + \dots + \lim_k \|x_{n_k}^N\| \lim_k \|f_{n_k}^N\| &= 1. \end{aligned} \quad (2.30)$$

We next show that (2.28) holds for all $j \in B$. Let us split the proof into two cases.

Case 1. There exists $j \in B$ such that

$$\|x^j\| < \lim_k \|x_{n_k}^j\|. \quad (2.31)$$

In this case, it follows from the property (S- j) that

$$\lim_k \|f_{n_k}^j\| = 0. \quad (2.32)$$

This implies that $\langle x^j, f_{n_k}^j \rangle - \|x_{n_k}^j\| \|f_{n_k}^j\| \rightarrow 0$.

Case 2. $\|x^j\| = \lim_k \|x_{n_k}^j\|$ for all $j \in B$. Again, if $f_n^j = 0$, then $\langle x^j, f_n^j \rangle = \|x^j\| \|f_n^j\| = 0$. Now we may assume that $f_n^j \neq 0$ for all k . This implies that $f_{n_k}^j / \|f_{n_k}^j\| \in \nabla_{x_{n_k}^j}$ and hence it follows from the semi-KK property of X_j that $\langle x^j, f_{n_k}^j / \|f_{n_k}^j\| \rangle - \|x_{n_k}^j\| \rightarrow 0$. In particular, $\langle x^j, f_{n_k}^j \rangle - \|x_{n_k}^j\| \|f_{n_k}^j\| \rightarrow 0$.

From both cases, we have proved that every subsequence of the sequence $\{\langle x, \mathbf{f}_n \rangle\}$ has a further subsequence $\{\langle x, \mathbf{f}_{n_k} \rangle\}$ such that $\langle x, \mathbf{f}_{n_k} \rangle \rightarrow 1$. Hence $\langle x, \mathbf{f}_n \rangle \rightarrow 1$, as desired. \square

Using the proof of the preceding theorem and the fact that the property (S- j) is a uniform property, we obtain the following result.

Theorem 2.15. *Suppose that X_1, \dots, X_N are Banach spaces. Then $(X_1 \oplus \dots \oplus X_N)_Z$ is semi-UKK if and only if for each $j = 1, \dots, N$,*

- (a) X_j is semi-UKK and
- (b) either X_j is semi-Schur or Z has property (S- j).

Finally, we use the characterization above and Theorem 2.10 to construct a Banach space which is semi-UKK but not UKK.

Example 2.16 (A Banach space which is semi-UKK but not UKK). Let Z be a two-dimensional space \mathbb{R}^2 equipped with the norm

$$|(\alpha, \beta)| = \begin{cases} |\beta| & \text{if } |\alpha| \leq |\beta|, \\ \frac{\alpha^2 + \beta^2}{2|\alpha|}, & \text{if } |\alpha| > |\beta|. \end{cases} \quad (2.33)$$

It follows that Z is a (uniformly) smooth space, and its unit sphere consists of

- (i) two half unit circles: the first one is a right half centered at $(1, 0)$ and the second is a left half centered at $(-1, 0)$
- (ii) two horizontal line segments joining the points $(-1, 1)$ and $(1, 1)$ and the points $(-1, -1)$ and $(1, -1)$, respectively.

Furthermore, Z has properties (S-1) and (S-2) but is not strictly monotone in the first-coordinate. Let $X = (\ell_2 \oplus \mathbb{R})_Z$. Then X is semi-UKK but not UKK. The latter follows since Z is not strictly monotone in the first coordinate and ℓ_2 does not have the Schur property.

3. U -Spaces and Uniformly Alternatively Convex or Smooth Spaces

In this section, we discuss some properties of uniformly alternatively convex or smooth spaces which was introduced by Kadets et al. [7].

Definition 3.1. A Banach space X is *uniformly alternatively convex or smooth* if

$$(UACS) : \left. \begin{array}{l} \{x_n\}, \{y_n\} \subset S_X \\ \|x_n + y_n\| \rightarrow 2 \\ \langle x_n, f_n \rangle \rightarrow 1 \text{ for some } \{f_n\} \subset S_{X^*} \end{array} \right\} \Rightarrow \langle y_n, f_n \rangle \rightarrow 1. \quad (3.1)$$

Remark 3.2. It is not hard to see that X is UACS if and only if

$$\left. \begin{array}{l} \{x_n\}, \{y_n\} \subset S_X \\ \|x_n + y_n\| \rightarrow 2 \\ \{f_n\} \subset S_{X^*} \text{ satisfying } f_n \in \nabla_{x_n}, \forall n \end{array} \right\} \Rightarrow \langle y_n, f_n \rangle \rightarrow 1. \quad (3.2)$$

Consequently, X is UACS if and only if it is a U -space.

It is proved by Gao and Lau ([13, Theorem 4.4]) that every UACS space has uniform normal structure which is the same result of Theorem 3.1 of Sirotkin [15]. In fact, this result is recently strengthened by Saejung in [16]. Recall that a Banach space X has *uniform normal structure* if there exists a constant $0 < c < 1$ such that for every bounded closed convex subset C of X containing more than one point there exists a point $x_0 \in C$ such that $\sup\{\|x_0 - z\| : z \in C\} < c \cdot \text{diam } C$.

Moreover, it was Lau ([4, Theorem 2.4]) who proved that X is UACS if and only if its dual space X^* is UACS. By Sirotkin's result ([15, Theorem 2.3]), we have the following theorem.

Theorem 3.3. *Let (S, Σ, μ) be a complete measure space and X be a Banach space. Then the following statements are equivalent:*

- (i) $L^p(\mu, X)$ is UACS for some (and hence all) $1 < p < \infty$;
- (ii) $L^p(\mu, X^*)$ is UACS for some (and hence all) $1 < p < \infty$;
- (iii) X is UACS;
- (iv) X^* is UACS.

In particular, if X is UACS, then both $L^p(\mu, X)$ and $L^p(\mu, X^)$ have uniform normal structure.*

Recall that $L^p(\mu, X)$, where $1 < p < \infty$, is the Lebesgue-Bochner function space of μ -equivalence classes of strongly measurable functions $f : S \rightarrow X$ with $\int_S \|f(t)\|^p d\mu < \infty$, endowed with the norm $\|f\| = (\int_S \|f(t)\|^p d\mu)^{1/p}$ (for more detail, see [17, 18]).

Acknowledgment

The first author is supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.

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