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Research Article

On Stability of Linear Delay Differential Equations under Perron's Condition

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The stability of the zero solution of a system of first-order linear functional differential equations with nonconstant delay is considered. Sufficient conditions for stability, uniform stability, asymptotic stability, and uniform asymptotic stability are established.

1. Introduction

We begin with a classical result for the linear system

$$x' = A(t)x, \tag{L1}$$

where A is an $n \times n$ matrix function defined and continuous on $[0, \infty)$. By $C_B[0, \infty)$, we will denote the set of bounded functions defined and continuous on $[0, \infty)$ and by $|\cdot|$ the Euclidean norm.

In 1930, Perron first formulated the following definition being named after him.

Definition 1.1 (see [1]). System (L1) is said to satisfy Perron's condition (P1) if, for any given vector function $f \in C_B[0, \infty)$, the solution $x(t)$ of

$$x' = A(t)x + f(t), \quad x(0) = 0 \tag{N1}$$

is bounded.

The following theorem by Bellman [2] is well known.

Theorem 1.2 (see [2]). *If* (P1) *holds and* $|A(t)| \le M_1$ *for some positive number* M_1*, then the zero solution of* (L1) *is uniformly asymptotically stable.*

The proof is accomplished by making use of the basic properties of a fundamental matrix, the Banach-Steinhaus theorem, and the adjoint system

$$x' = -A^T(t)x, \tag{1.1}$$

where A^T denotes the transpose of A.

It is shown by an example in [3] that Theorem 1.2 may not be valid if the function f appearing in (N1) is replaced by a constant vector. However, such a theorem is later obtained in [4] under a *Perron-like* condition.

Theorem 1.2 is extended by Halanay [5] to linear delay systems of the form

$$x'(t) = A(t)x(t) + B(t)x(t-\tau), \tag{L2}$$

where A, B are $n \times n$ matrix functions defined and continuous on $[0,\infty)$ and τ is a positive real number.

Definition 1.3. System (L2) is said to satisfy Perron's condition (P2) if for any given vector function $f \in C_B[0,\infty)$, the solution $x(t)$ of

$$x'(t) = A(t)x(t) + B(t)x(t-\tau) + f(t) \tag{N2}$$

satisfying $x(t) = 0$, $t \le 0$, is bounded.

Theorem 1.4 (see [5]). *If* (P2) *holds,* $|A(t)| \le M_1$*, and* $|B(t)| \le M_2$ *for some positive numbers* M_1 *and* M_2*, then the zero solution of* (L2) *is uniformly asymptotically stable.*

The method used to prove Theorem 1.4 is similar to Bellman's except that the adjoint system

$$y'(t) = -A^T(t)y(t) - B^T(t+\tau)y(t+\tau) \tag{1.2}$$

is not constructed with respect to an inner product but the functional

$$F(x,y)(t) = \int_t^{t+\tau} y^T(s)B(s)x(s-\tau)ds + x^T(t)y(t). \tag{1.3}$$

For some extensions to impulsive differential equations, we refer the reader in particular to [6, 7].

In this paper, we consider the more general linear delay system

$$x'(t) = A(t)x(t) + B(t)x\big(g(t)\big), \tag{1.4}$$

where A and B are $n \times n$ matrix functions defined and continuous on $[0, \infty)$ and g is a continuously differentiable increasing function defined on $[0, \infty)$ satisfying $g(t) < t$ and $g'(t) \leq 1$. We set $h := g^{-1}$. Obviously, $h \in C^1[0, \infty)$ and increases on $[0, \infty)$ and $h(t) > t$.

Perron's condition takes the following form.

Definition 1.5. System (1.4) is said to satisfy Perron's condition (P) if, for any given vector function $f \in C_B[0, \infty)$, the solution $x(t)$ of

$$x'(t) = A(t)x(t) + B(t)x\big(g(t)\big) + f(t) \tag{1.5}$$

satisfying $x(t) = 0$, $t \leq 0$ is bounded.

A natural question is whether the zero solution of (1.4) is uniformly asymptotically stable under Perron's condition (P). It turns out that the answer depends on the delay function g.

The paper is organized as follows. In Section 2, we only state our results; the proofs are included in Section 5. We define an adjoint system and give a variation of parameters formula in Section 3 to be needed in proving the main results. Section 4 contains also some lemmas concerning Perron's condition and a relation useful for changing the order of integration.

2. Stability Theorems

The conclusion obtained by Bellman and Halanay for systems (L1) and (L2), respectively, is quite strong. We are only able to prove the stability of the zero solution for more general equation (1.4) under Perron's condition. To get uniform stability or asymptotic stability or uniform asymptotic stability, we impose restrictions on the delay function.

For our purpose, we denote

$$\begin{aligned} h_*(t) &:= h(t) - t, \quad t \geq 0, \\ g_*(t, t_0) &:= \sup_{r \in [h(t_0), t]} \{r - g(r)\}, \quad t, t_0 \geq 0. \end{aligned} \tag{2.1}$$

Theorem 2.1. *Let* (P) *hold. If there are positive numbers M_1 and M_2 such that*

$$|A(t)| \leq M_1, \quad |B(t)| \leq M_2 \quad \forall t \geq 0, \tag{2.2}$$

then the zero solution of (1.4) *is stable.*

Theorem 2.2. *Let* (*P*) *hold. If* (2.2) *is satisfied and if there exists a positive real number M_3 such that*

$$h_*(t) \leq M_3 \quad \forall t \geq 0, \tag{2.3}$$

then the zero solution of (1.4) *is uniformly stable.*

Theorem 2.3. *Let* (P) *hold. If* (2.2) *and*

$$\limsup_{t\to\infty}\frac{g_*(t,t_0)}{t-t_0}=0 \quad \textit{for each } t_0\geq 0 \tag{2.4}$$

are satisfied, then the zero solution of (1.4) *is asymptotically stable.*

Theorem 2.4. *Let* (P) *hold. If* (2.2), (2.3), *and*

$$\limsup_{t\to\infty}\frac{g_*(t,t_0)}{t-t_0}=0 \quad \textit{uniformly for } t_0\geq 0 \tag{2.5}$$

are satisfied, then the zero solution of (1.4) *is uniformly asymptotically stable.*

Remark 2.5. Note that if $g(t)=t-\tau$, then $h(t)=t+\tau$ and hence the conditions (2.3), (2.4), and (2.5) are automatically satisfied. In this case, all theorems become equivalent, that is, the zero solution is uniformly asymptotically stable. Thus, the results obtained by Bellman and Halanay are recovered.

3. Variation of Parameters Formula

To establish a variation of parameters formula to represent the solutions of (1.5), one needs an adjoint system. The following lemma helps to define the adjoint of (1.4).

Lemma 3.1. *Let* $x(t)$ *be a solution of* (1.4). *If* $y(t)$ *is a solution of*

$$y'(t)=-A^T(t)y(t)-B^T(h(t))y(h(t))h'(t), \tag{3.1}$$

then

$$\frac{d}{dt}F\big(x(t),y(t)\big)=0, \tag{3.2}$$

where

$$F(x,y)(t)=\int_t^{h(t)}y^T(s)B(s)x\big(g(s)\big)ds+x^T(t)y(t). \tag{3.3}$$

Proof. Verify directly. □

Definition 3.2. The system (3.1) is said to be adjoint to system (1.4).

It is easy to see that the adjoint of system (3.1) is system (1.4); thus the systems are mutually adjoint to each other.

Lemma 3.3. *Let $Y(t,s)$ be a matrix solution of* (3.1) *for $t<s$ satisfying $Y(s,s)=I$ and $Y(t,s)=0$ for $t>s$. Then $x(t)$ is a solution of* (1.5) *if and only if*

$$x(t)=Y^T(s,t)x(s)+\int_{g(s)}^{s}Y^T(h(\beta),t)B(h(\beta))x(\beta)h'(\beta)d\beta+\int_{s}^{t}Y^T(\beta,t)f(\beta)d\beta. \tag{3.4}$$

Proof. Replacing t by β in (1.5) and then integrating the resulting equation multiplied by $Y^T(\beta,t)$ over $\beta\in[s,t]$, we have

$$\begin{aligned}
&\int_s^t Y^T(\beta,t)A(\beta)x(\beta)d\beta+\int_s^t Y^T(\beta,t)B(\beta)x(g(\beta))d\beta+\int_s^t Y^T(\beta,t)f(\beta)d\beta\\
&\quad=\int_s^t Y^T(\beta,t)x'(\beta)d\beta\\
&\quad=x(t)-Y^T(s,t)x(s)-\int_s^t\left[\frac{\partial}{\partial\beta}Y^T(\beta,t)\right]x(\beta)d\beta\\
&\quad=x(t)-Y^T(s,t)x(s)+\int_s^t\left[Y^T(\beta,t)A(\beta)+Y^T(h(\beta),t)B(h(\beta))h'(\beta)\right]x(\beta)d\beta\\
&\quad=x(t)-Y^T(s,t)x(s)+\int_s^t Y^T(\beta,t)A(\beta)x(\beta)d\beta+\int_{h(s)}^{h(t)}Y^T(\beta,t)B(\beta)x(g(\beta))d\beta.
\end{aligned} \tag{3.5}$$

Comparing both sides and using

$$\int_t^{h(t)}Y^T(\beta,t)B(\beta)x(g(\beta))d\beta=0, \tag{3.6}$$

which is true in view of $Y(\beta,t)=0$ for $\beta>t$, we get

$$x(t)=Y^T(s,t)x(s)-\int_{h(s)}^{s}Y^T(\beta,t)B(\beta)x(g(\beta))d\beta+\int_s^t Y^T(\beta,t)f(\beta)d\beta \tag{3.7}$$

and hence

$$x(t)=Y^T(s,t)x(s)+\int_{g(s)}^{s}Y^T(h(\beta),t)B(h(\beta))x(\beta)h'(\beta)d\beta+\int_{s}^{t}Y^T(\beta,t)f(\beta)d\beta. \tag{3.8}$$

□

It is not difficult to see from (3.4) that if $X(t,s)$ is a matrix solution of (1.4) for $t>s$ satisfying $X(s,s)=I$ and $X(t,s)=0$ for $t<s$, then

$$X(t,s)=Y^T(s,t). \tag{3.9}$$

Using this relation in Lemma 3.3 leads to the following variation of parameters formula.

Lemma 3.4. *Let $X(t,s)$ be a matrix solution of* (1.4) *for $t > s$ satisfying $X(s,s) = I$ and $X(t,s) = 0$ for $t < s$. Then $x(t)$ is a solution of* (1.5) *if and only if*

$$x(t) = X(t,s)x(s) + \int_{g(s)}^{s} X(t,h(\beta))B(h(\beta))x(\beta)h'(\beta)d\beta + \int_{s}^{t} X(t,\beta)f(\beta)d\beta. \tag{3.10}$$

4. Auxiliary Results

Lemma 4.1. *If* (P) *holds, then there is a positive number K_1 such that*

$$\int_0^t |X(t,s)|ds \le K_1 \quad \forall t > 0. \tag{4.1}$$

Proof. The proof follows as in [5]. We provide only the steps for the reader's convenience.

Define

$$\begin{aligned}(Sf)(t) &= \int_0^t X(t,\beta)f(\beta)d\beta, \quad f \in C_B[0,\infty),\\ S_k(f) &= \int_0^{t_k} X(t_k,\beta)f(\beta)d\beta, \quad f \in C_B[0,\infty),\end{aligned} \tag{4.2}$$

for each rational number t_k, $k \in \mathbb{N}$.

In view of (P), the family of continuous linear operators $\{S_k\}$ from $C_B[0,\infty)$ to $C_B[0,\infty)$ is pointwise-bounded. For the space of bounded continuous functions $C_B[0,\infty)$, the usual sup norm $\|\cdot\|$ is used.

By the Banach-Steinhaus theorem, the family is uniformly bounded. Thus, there is a positive number M such that $\|S_k(f)\| \le M\|f\|$ for every $f \in C_B[0,\infty)$.

As the rational numbers are dense in the real numbers, for each t there is t_k such that $t_k \to t$ as $k \to \infty$ and so

$$\left|\int_0^t X(t,\beta)f(\beta)d\beta\right| \le M\|f\| \quad \forall f \in C_B[0,\infty). \tag{4.3}$$

The final step involves choosing a sequence of functions and using a limiting process. □

Lemma 4.2. *If* (2.2) *and* (4.1) *are true, then there is a positive number K_2 such that*

$$|Y(s,t)| \le K_2 \quad \forall\, 0 \le s < t. \tag{4.4}$$

Proof. From (3.1), we have

$$Y(s,t) = I + \int_s^t A^T(\beta)Y(\beta,t)d\beta + \int_s^t B^T(h(\beta))Y(h(\beta),t)h'(\beta)d\beta. \tag{4.5}$$

Hence, by using (4.1), we see that for all $0 \le s < t$,

$$|Y(s,t)| \le 1 + M_1K_1 + M_2K_1 =: K_2. \tag{4.6}$$

□

Lemma 4.3. *Let $G(r,t)$ be a continuous function satisfying $G(r,t) = 0$ for $r > t$. Then*

$$\int_{t_0}^{t}\left[\int_{s}^{h(s)} G(r,t)dr\right]ds = \int_{h(t_0)}^{t}(r - g(r))G(r,t)dr + \int_{t_0}^{h(t_0)}(r - t_0)G(r,t)dr. \tag{4.7}$$

5. Proofs of Theorems

Let $t_0 \ge 0$ be given. For a given continuous vector function ϕ defined on $[g(t_0), t_0]$, let $x(t) = x(t, t_0, \phi)$ denote the solution of (1.4) satisfying

$$x(t) = \phi(t), \quad t \le t_0. \tag{5.1}$$

As usual,

$$\|\phi\|_g = \sup_{t\in[g(t_0),t_0]} |\phi(t)|. \tag{5.2}$$

Proof of Theorem 2.1. From Lemma 3.3, we may write

$$x(t) = Y^T(t_0,t)\phi(t_0) + \int_{g(t_0)}^{t_0} Y^T(h(\beta),t)B(h(\beta))\phi(\beta)h'(\beta)d\beta. \tag{5.3}$$

In view of Lemma 4.2, it follows that

$$|x(t)| \le (K_2 + (h(t_0) - t_0)K_2M_2)\|\phi\|_g. \tag{5.4}$$

Hence, the zero solution is stable. □

Proof of Theorem 2.2. Using (2.3) in (5.4), we get

$$|x(t)| \le K_3\|\phi\|_g, \quad K_3 = K_2 + K_2M_2M_3, \tag{5.5}$$

from which the uniform stability follows. □

Proof of Theorem 2.3. By Theorem 2.1, the zero solution is stable. We need to show the attractivity property.

From Lemma 3.3, for $s \ge t_0$, we can write

$$x(t,t_0,\phi) = Y^T(s,t)x(s,t_0,\phi) + \int_{g(s)}^{s} G(h(\beta),t)x(\beta,t_0,\phi)h'(\beta)d\beta, \tag{5.6}$$

where

$$G(s,t) = Y^T(s,t)B(s). \tag{5.7}$$

Integrating with respect to s from t_0 to t, we have

$$(t-t_0)x(t,t_0,\phi) = \int_{t_0}^{t}\left[Y^T(s,t)x(s,t_0,\phi) + \int_{s}^{h(s)} G(r,t)x(g(r),t_0,\phi)dr\right]ds. \tag{5.8}$$

We change the order of integration by employing Lemma 4.3. After some rearrangements, we obtain

$$\begin{aligned}(t-t_0)x(t,t_0,\phi) &= \int_{t_0}^{t} Y^T(s,t)x(s,t_0,\phi)ds + \int_{h(t_0)}^{t}(s-g(s))G(s,t)x(g(s),t_0,\phi)ds \\ &\quad + \int_{t_0}^{h(t_0)}(s-t_0)G(s,t)x(g(s),t_0,\phi)ds.\end{aligned} \tag{5.9}$$

It follows that

$$(t-t_0)\left|x(t,t_0,\phi)\right| \le K_1K_3\|\phi\|_g + g_*(t,t_0)M_2K_1\|\phi\|_g + h_*(t_0)M_2K_1\|\phi\|_g. \tag{5.10}$$

In view of condition (2.4), we see from (5.10) that

$$\lim_{t\to\infty}\left|x(t,t_0,\phi)\right| = 0. \tag{5.11}$$

□

Proof of Theorem 2.4. By Theorem 2.2, the zero solution is uniformly stable. From (5.10) and (2.3), we have

$$(t-t_0)\left|x(t,t_0,\phi)\right| \le K_1K_3\|\phi\|_g + g_*(t,t_0)M_2K_1\|\phi\|_g + M_3M_2K_1\|\phi\|_g. \tag{5.12}$$

Using condition (2.4) in the above inequality, we see that the zero solution is uniformly asymptotically stable as $t \to \infty$. □

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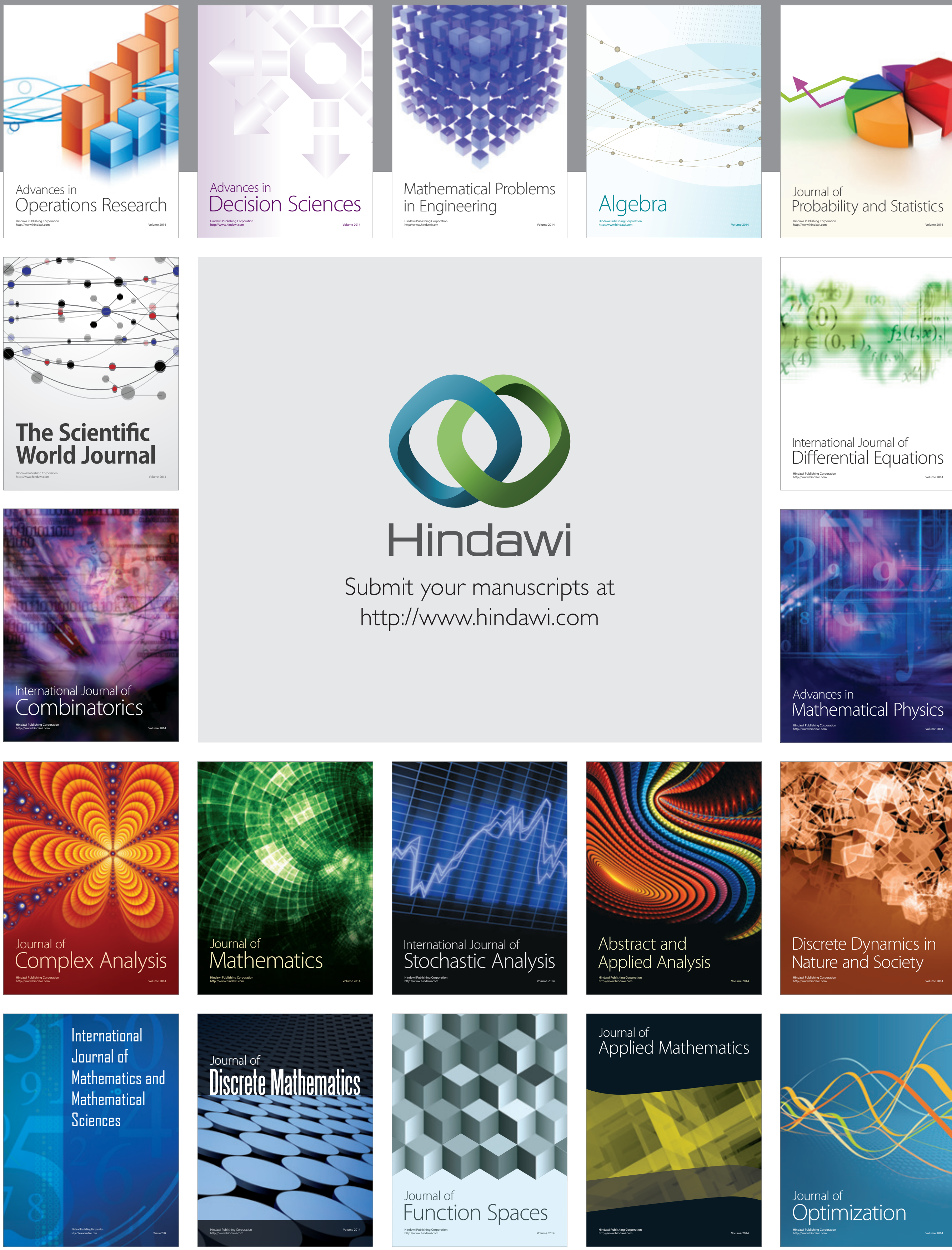
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