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Research Article

A New Class of Meromorphically Analytic Functions with Applications to the Generalized Hypergeometric Functions

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We introduce a new subclass of meromorphically analytic functions, which is defined by means of a Hadamard product (or convolution). A characterization property such as the coefficient bound is obtained for this class. The other related properties, which are investigated in this paper, include the distortion and the radius of starlikeness. We also consider several applications of our main results to the generalized hypergeometric functions.

1. Introduction

Let $\mathcal{A}$ be the class of functions f which are *analytic* in the open unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}. \tag{1.1}$$

As usual, we denote by S the subclass of $\mathcal{A}$, consisting of functions which are also *univalent* in U.

Let w be a fixed point in U and $A(w) = \{f \in H(D) : f(w) = f'(w) - 1 = 0\}$. In [1], Kanas and Ronning introduced the following classes

$$\begin{aligned}
S_w &= \{f \in A(w) : f \text{ is univalent in } U\},\\
ST_w &= \left\{f \in A(w) : \operatorname{Re}\left(\frac{(z-w)f'(z)}{f(z)}\right) > 0,\ z \in U\right\},\\
CV_w &= \left\{f \in A : 1 + \left(\operatorname{Re}\frac{(z-w)f''(z)}{f'(z)}\right) > 0,\ z \in U\right\}.
\end{aligned} \tag{1.2}$$

Later, Acu and Owa [2] studied the classes extensively.

The class ST_w is defined by geometric property that the image of any circular arc centered at w is starlike with respect to $f(w)$, and the corresponding class S_w^c is defined by the property that the image of any circular arc centered at w is convex. We observed that the definitions are somewhat similar to the ones introduced by Goodman in [3, 4] for uniformly starlike and convex functions except that, in this case, the point w is fixed.

Let Σ_w denote the subclass of $A(w)$ consisting of the function of the form

$$f(z) = \frac{1}{z-w} + \sum_{n=1}^{\infty} a_n (z-w)^n. \tag{1.3}$$

The functions f in Σ_w are said to be starlike functions of order β if and only if

$$\operatorname{Re}\left\{-\frac{(z-w)f'(z)}{f(z)}\right\} > \beta \quad ((z-w) \in U), \tag{1.4}$$

for some β $(0 \le \beta < 1)$. We denote by $S_w^*(\beta)$ the class of all starlike functions of order β. Similarly, a function f in S_w is said to be convex of order β if and only if

$$\operatorname{Re}\left(-1 - \frac{(z-w)f''(z)}{f'(z)}\right) > \beta \quad ((z-w) \in U), \tag{1.5}$$

for some β $(0 \le \beta < 1)$. We denote by $C_w(\beta)$ the class of all convex functions of order β.

For the function $f \in \Sigma_w$, we define

$$\begin{aligned}
I_\lambda^0 f(z) &= f(z),\\
I_\lambda^1 f(z) &= (z-w)f'(z) + \frac{2}{z-w},\\
I_\lambda^2 f(z) &= (z-w)\left(I^1 f(z)\right)' + \frac{2}{z-w},
\end{aligned} \tag{1.6}$$

and, for $k = 1, 2, 3, \ldots$, we can write

$$\begin{aligned} I_\lambda^k f(z) &= (z-w)\left(I^{k-1} f(z)\right)' + \frac{2}{z-w} \\ &= \frac{1}{z-w} + \sum_{n=1}^{\infty} [1+\lambda(n-1)]^k a_n (z-w)^n, \end{aligned} \tag{1.7}$$

where $\lambda \geq 1,\ k \geq 0$ and $(z - w \in U)$.

The differential operator I_1^k is studied extensively by Ghanim and Darus [5, 6] and Ghanim et al. [7].

The Hadamard product or convolution of the functions f given by (1.3) with the function g and h given, respectively, by

$$\begin{aligned} g(z) &= \frac{1}{z-w} + \sum_{n=1}^{\infty} b_n (z-w)^n, \\ h(z) &= \frac{1}{z-w} + \sum_{n=1}^{\infty} c_n (z-w)^n, \end{aligned} \tag{1.8}$$

can be expressed as follows:

$$\begin{aligned} (f * g)(z) &= \frac{1}{z-w} + \sum_{n=1}^{\infty} a_n b_n (z-w)^n, \\ (f * h)(z) &= \frac{1}{z-w} + \sum_{n=1}^{\infty} a_n c_n (z-w)^n. \end{aligned} \tag{1.9}$$

Suppose that f and g are two analytic functions in the unit disk U. Then, we say that the function g is subordinate to the function f, and we write

$$g(z) \prec f(z) \quad (z \in U), \tag{1.10}$$

if there exists a Schwarz function $\varpi(z)$ with $\varpi(0) = 0$ and $|\varpi(z)| < 1$ such that

$$g(z) = f(\varpi(z)) \quad (z \in U). \tag{1.11}$$

By applying the above subordination definition, we introduce here a new class $\Sigma_w(A, B, k, \alpha, \lambda)$ of meromorphically functions, which is defined as follows:

Definition 1.1. A function $f \in \Sigma_w$ of the form (1.3) is said to be in the class $\Sigma_w(A, B, k, \alpha, \lambda)$ if it satisfies the following subordination property:

$$\alpha \frac{I_\lambda^k (f * g)(z)}{I_\lambda^k (f * h)(z)} \prec \alpha - \frac{(A-B)(z-w)}{1+B(z-w)} \quad ((z-w) \in U), \tag{1.12}$$

where $-1 \leq B < A \leq 1,\ k \geq 0,\ \alpha > 0,\ \lambda \geq 1$, with condition $I_\lambda^k (f * h)(z) \neq 0$.

The purpose of this paper is to investigate the coefficient estimates, distortion properties, and the radius of starlikeness for the class $\Sigma_w(A, B, k, \alpha, \lambda)$. Some applications of the main results involving generalized hypergeometric functions are also considered.

2. Characterization and Other Related Properties

In this section, we begin by proving a characterization property which provides a necessary and sufficient condition for a function $f \in \Sigma_w$ of the form (1.3) to belong to the class $\Sigma_w(A, B, k, \alpha, \lambda)$ of meromorphically analytic functions.

Theorem 2.1. *The function $f \in \Sigma_w$ is said to be a member of the class $\Sigma_w(A, B, k, \alpha, \lambda)$ if it satisfies*

$$\sum_{n=1}^{\infty}[1+\lambda(n-1)]^k(\alpha b_n(1+B) - c_n(\alpha(1+B)+A-B))\ a_n \le A-B. \tag{2.1}$$

The equality is attained for the function $f_n(z)$ given by

$$f_n(z) = \frac{1}{z-w} + \frac{(A-B)}{[1+\lambda(n-1)]^k(\alpha b_n(1+B) - c_n(\alpha(1+B)+A-B))}(z-w)^n. \tag{2.2}$$

Proof. Let $f \in \Sigma_w(A, B, k, \alpha, \lambda)$, and suppose that

$$\alpha\frac{I_\lambda^k(f*g)(z)}{I_\lambda^k(f*h)(z)} = \alpha - \frac{(A-B)(z-w)}{1+B(z-w)}. \tag{2.3}$$

Then, in view of (2.2), we have

$$\begin{aligned}
&\left|\frac{\alpha\sum_{n=1}^{\infty}[1+\lambda(n-1)]^k a_n(b_n-c_n)(z-w)^{n+1}}{(A-B)-\sum_{n=1}^{\infty}[1+\lambda(n-1)]^k a_n(\alpha B b_n + \{(A-B)-\alpha B\}c_n)(z-w)^{n+1}}\right| \\
&\quad\le \frac{\alpha\sum_{n=1}^{\infty}[1+\lambda(n-1)]^k a_n(b_n-c_n)|z-w|^{n+1}}{(A-B)-\sum_{n=1}^{\infty}[1+\lambda(n-1)]^k a_n(\alpha B b_n + \{(A-B)-\alpha B\}c_n)|z-w|^{n+1}} \\
&\quad\le 1.
\end{aligned} \tag{2.4}$$

Letting $(z-w) \to 1$, we get

$$\sum_{n=1}^{\infty}[1+\lambda(n-1)]^k(\alpha b_n(1+B) - c_n(\alpha(1+B)+A-B))a_n \le (A-B), \tag{2.5}$$

which is equivalent to our condition of the theorem, so that $f \in \Sigma_w(A, B, k, \alpha, \lambda)$. Hence we have the theorem. $\square$

Theorem 2.1 immediately yields the following result.

Corollary 2.2. *If the function* $f \in \Sigma_w$ *belongs to the class* $\Sigma_w(A,B,k,\alpha,\lambda)$, *then*

$$a_n \le \frac{(A-B)}{[1+\lambda(n-1)]^k(\alpha b_n(1+B)-c_n(\alpha(1+B)+A-B))}, \tag{2.6}$$

$n \ge 1$, *where the equality holds true for the functions* $f_n(z)$ *given by* (2.2).

We now state the following growth and distortion properties for the class $\Sigma_w(A, B,k,\alpha,\lambda)$.

Theorem 2.3. *If the function* f *defined by* (1.3) *is in the class* $\Sigma_w(A,B,k,\alpha,\lambda)$, *then for* $0 < |z-w| = r < 1$, *one has*

$$\begin{aligned}
&\frac{1}{r} - \frac{(A-B)}{(\alpha b_1(1+B)-c_1(\alpha(1+B)+A-B))} r \\
&\qquad \le |f(z)| \le \frac{1}{r} + \frac{(A-B)}{(\alpha b_1(1+B)-c_1(\alpha(1+B)+A-B))} r, \\
&\frac{1}{r^2} - \frac{(A-B)}{(\alpha b_1(1+B)-c_1(\alpha(1+B)+A-B))} \\
&\qquad \le |f'(z)| \le \frac{1}{r^2} + \frac{(A-B)}{(\alpha b_1(1+B)-c_1(\alpha(1+B)+A-B))}.
\end{aligned} \tag{2.7}$$

Proof. Since $f \in \Sigma_w(A,B,k,\alpha,\lambda)$, Theorem 2.1 readily yields the inequality

$$\sum_{n=1}^{\infty} a_n \le \frac{(A-B)}{(\alpha b_1(1+B)-c_1(\alpha(1+B)+A-B))}. \tag{2.8}$$

Thus, for $0 < |z-w| = r < 1$ and utilizing (2.8), we have

$$\begin{aligned}
|f(z)| &\leq \frac{1}{|z-w|} + \sum_{n=1}^{m} a_n |(z-w)|^n \\
&\leq \frac{1}{r} + r\sum_{n=1}^{m} a_n \\
&\leq \frac{1}{r} + \frac{(A-B)}{(\alpha b_1(1+B) - c_1(\alpha(1+B) + A - B))} r, \\
|f(z)| &\geq \frac{1}{|z-w|} - \sum_{n=1}^{m} a_n |(z-w)|^n \\
&\geq \frac{1}{r} - r\sum_{n=1}^{m} a_n \\
&\geq \frac{1}{r} - \frac{(A-B)}{(\alpha b_1(1+B) - c_1(\alpha(1+B) + A - B))} r.
\end{aligned} \tag{2.9}$$

Also, from Theorem 2.1, we get

$$\sum_{n=1}^{\infty} n a_n \leq \frac{(A-B)}{(\alpha b_1(1+B) - c_1(\alpha(1+B) + A - B))}. \tag{2.10}$$

Hence

$$\begin{aligned}
|f'(z)| &\leq \frac{1}{|z-w|^2} + \sum_{n=1}^{m} n a_n |(z-w)|^{n-1} \\
&\leq \frac{1}{r} + \sum_{n=1}^{m} n a_n \\
&\leq \frac{1}{r^2} + \frac{(A-B)}{(\alpha b_1(1+B) - c_1(\alpha(1+B) + A - B))}, \\
|f'(z)| &\geq \frac{1}{|z-w|^2} - \sum_{n=1}^{m} n a_n |(z-w)|^{n-1} \\
&\geq \frac{1}{r^2} - \sum_{n=1}^{m} n a_n \\
&\geq \frac{1}{r^2} - \frac{(A-B)}{(\alpha b_1(1+B) - c_1(\alpha(1+B) + A - B))}.
\end{aligned} \tag{2.11}$$

This completes the proof of Theorem 2.3. □

We next determine the radius of meromorphically starlikeness of the class $\Sigma_w(A, B, k, \alpha, \lambda)$, which is given by Theorem 2.4.

Theorem 2.4. *If the function f defined by* (1.3) *is in the class $\Sigma_w(A, B, k, \alpha, \lambda)$, then f is meromorphically starlike of order δ in the disk $|z - w| < r_1$, where*

$$r_1 = \inf_{n\geq 1}\left\{ \frac{(1-\delta)(\alpha b_n(1+B) - c_n(\alpha(1+B) + A - B))}{(n+2-\delta)(A-B)} \right\}^{1/(n+1)}. \tag{2.12}$$

The equality is attained for the function $f_n(z)$ given by (2.2).

Proof. It suffices to prove that

$$\left| \frac{(z-w)\left(I^k f(z)\right)'}{I^k f(z)} + 1 \right| \leq 1 - \delta. \tag{2.13}$$

For $|z - w| < r_1$, we have

$$\begin{aligned}\left| \frac{(z-w)\left(I^k f(z)\right)'}{I^k f(z)} + 1 \right| &= \left| \frac{\sum_{n=1}^{\infty}(n+1)[1+\lambda(n-1)]^k a_n (z-w)^n}{1/(z-w) + \sum_{n=1}^{\infty}[1+\lambda(n-1)]^k a_n (z-w)^n} \right| \\ &= \left| \frac{\sum_{n=1}^{\infty}(n+1)[1+\lambda(n-1)]^k a_n (z-w)^{n+1}}{1 + \sum_{n=1}^{\infty}[1+\lambda(n-1)]^k a_n (z-w)^{n+1}} \right| \\ &\leq \frac{\sum_{n=1}^{\infty}(n+1)[1+\lambda(n-1)]^k a_n |z-w|^{n+1}}{1 - \sum_{n=1}^{\infty}[1+\lambda(n-1)]^k a_n |z-w|^{n+1}}.\end{aligned} \tag{2.14}$$

Hence (2.14) holds true for

$$\sum_{n=1}^{\infty}(n+1)[1+\lambda(n-1)]^k a_n |z-w|^{n+1} \leq (1-\delta)\left(1 - \sum_{n=1}^{\infty}[1+\lambda(n-1)]^k a_n |z-w|^{n+1}\right) \tag{2.15}$$

or

$$\frac{\sum_{n=1}^{\infty}(n+2-\delta)[1+\lambda(n-1)]^k a_n |z-w|^{n+1}}{(1-\delta)} \leq 1. \tag{2.16}$$

With the aid of (2.1) and (2.16), it is true to say that for fixed n

$$\begin{aligned}&\frac{(n+2-\delta)[1+\lambda(n-1)]^k |z-w|^{n+1}}{(1-\delta)} \\ &\quad\leq \frac{[1+\lambda(n-1)]^k(\alpha b_n(1+B) - c_n(\alpha(1+B) + A - B))}{(A-B)} \quad (n \geq 1).\end{aligned} \tag{2.17}$$

Solving (2.17) for $|z-w|$, we obtain

$$|z-w| < \left\{ \frac{(1-\delta)(\alpha b_n(1+B) - c_n(\alpha(1+B)+A-B))}{(n+2-\delta)(A-B)} \right\}^{1/(n+1)}. \tag{2.18}$$

This completes the proof of Theorem 2.4. □

3. Applications Involving Generalized Hypergeometric Functions

Let us define the function $\widetilde{\phi}(a,c;z)$ by

$$\widetilde{\phi}(a,c;z) = \frac{1}{z-w} + \sum_{n=0}^{\infty} \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| a_n (z-w)^n, \tag{3.1}$$

for $c \neq 0,-1,-2,\ldots$, and $a \in \mathbb{C}/\{0\}$, where $(\lambda)n = \lambda(\lambda+1)_{n+1}$ is the Pochhammer symbol. We note that

$$\widetilde{\phi}(a,c;z) = \frac{1}{z-w}\, {}_2F_1(1,a,c;z), \tag{3.2}$$

where

$${}_2F_1(b,a,c;z) = \sum_{n=0}^{\infty} \frac{(b)_n (a)_n}{(c)_n} \frac{(z-w)^n}{n!}. \tag{3.3}$$

Corresponding to the function $\widetilde{\phi}(a,c;z)$ and using the Hadamard product which was defined earlier in the introduction section for $f(z) \in \Sigma$, we define here a new linear operator $L^*(a,c)$ on Σ by

$$L_w^*(a,c)f(z) = \widetilde{\phi}(a,c;z) * f(z) = \frac{1}{z-w} + \sum_{n=1}^{\infty} \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| a_n (z-w)^n. \tag{3.4}$$

For a function $f \in L_w^*(a,c)f(z)$, we define

$$I^0\left(L_w^*(a,c)f(z)\right) = L_w^*(a,c)f(z), \tag{3.5}$$

and, for $k = 1,2,3,\ldots$,

$$\begin{aligned} I^k\left(L_w^*(a,c)f(z)\right) &= z\left(I^{k-1}L^*(a,c)f(z)\right)' + \frac{2}{z-w} \\ &= \frac{1}{z-w} + \sum_{n=1}^{\infty} n^k \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| a_n (z-w)^n. \end{aligned} \tag{3.6}$$

We note $I^k(L_w^*(a,a)f(z))$ studied by Ghanim and Darus [5, 6] and Ghanim et al. [7], and also, $I^k(L_0^*(a,c)f(z))$ studied by Ghanim and Darus [8, 9] and Ghanim et al. [10].

The subordination relation (1.12) in conjunction with (3.4) and (3.6) takes the following form:

$$\alpha\frac{I^k L_w^*(a+1,c)f(z)}{I^k L_w^*(a,c)f(z)} = \alpha - \frac{(A-B)(z-w)}{1+B(z-w)} \tag{3.7}$$

$(0 \le B < A \le 1,\ k \ge 0,\ \alpha > 0)$.

Definition 3.1. A function $f \in \Sigma_w$ of the form (1.3) is said to be in the class $\Sigma_w(A,B,k,\alpha,a,c)$ if it satisfies the subordination relation (3.7) above.

Theorem 3.2. *The function $f \in \Sigma_w$ is said to be a member of the class $\Sigma_w(A,B,k,\alpha,a,c)$ if it satisfies*

$$\sum_{n=1}^{\infty} n^k(\alpha b_n(1+B) - c_n(\alpha(1+B)+A-B))\frac{|(a)_{n+1}|}{|(c)_{n+1}|}a_n \le (A-B). \tag{3.8}$$

The equality is attained for the function $f_n(z)$ given by

$$f_n(z) = \frac{1}{z-w} + \sum_{n=1}^{\infty}\frac{(A-B)|(c)_{n+1}|}{n^k(\alpha b_n(1+B) - c_n(\alpha(1+B)+A-B))|(a)_{n+1}|}(z-w)^n, \tag{3.9}$$

$n \ge 1$.

Proof. By using the same technique employed in the proof of Theorem 2.1 along with Definition 3.1, we can prove Theorem 3.2. □

The following consequences of Theorem 3.2 can be deduced by applying (3.8) and (3.9) along with Definition 3.1.

Corollary 3.3. *If the function $f \in \Sigma_w$ belongs to the class $\Sigma_w(A,B,k,\alpha,a,c)$, then*

$$a_n \le \frac{(A-B)|(c)_{n+1}|}{n^k(\alpha b_n(1+B) - c_n(\alpha(1+B)+A-B))|(a)_{n+1}|}, \tag{3.10}$$

$n \ge 1$, where the equality holds true for the functions $f_n(z)$ given by (3.9).

Corollary 3.4. *If the function f defined by* (1.3) *is in the class $\Sigma_w(A,B,k,\alpha,a,c)$, then f is meromorphically starlike of order δ in the disk $|z-w| < r_3$, where*

$$r_3 = \inf_{n\ge1}\left\{\frac{(1-\delta)(\alpha b_n(1+B) - c_n(\alpha(1+B)+A-B))|(c)_{n+1}|}{(n+2-\delta)(A-B)|(a)_{n+1}|}\right\}^{1/(n+1)}. \tag{3.11}$$

The equality is attained for the function $f_n(z)$ given by (3.9).

A slight background related to the formation of the present operator can be found in [11], and other work can be tackled using this type of operator. Also, the meromorphic functions with the generalized hypergeometric functions were considered recently by Dziok and Srivastava [12, 13], Liu [14], Liu and Srivastava [15], and Cho and Kim [16].

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