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Research Article

Generalized $\psi\rho$-Operations on Fuzzy Topological Spaces

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The aim of this work is to introduce ψ-operations on fuzzy topological spaces and to use them to study fuzzy generalized $\psi\rho$-closed sets and fuzzy generalized $\psi\rho$-open sets. Also, we introduce some characterizations and properties for these concepts. Finally we show that certain results of several publications on the concepts of weakness and strength of fuzzy generalized closed sets are considered as corollaries of the results of this research.

1. Preliminaries

The concept of fuzzy topology was first defined in 1968 by Chang [1] based on the concept of a fuzzy set introduced by Zadeh in [2]. Since then, various important notions in the classical topology such as generalized closed, generalized open set, and weaker and stronger forms of generalized closed and generalized open sets have been extended to fuzzy topological spaces. The purpose of this paper is to introduce and study the concept of ψ-operations, and by using these operations, we will study fuzzy generalized $\psi\rho$-closed sets and fuzzy generalized $\psi\rho$-open sets in fuzzy topological spaces. Also, we show that some results in several papers [3–15] considered as corollaries from the results of this paper. Let (X, τ) be a fuzzy topological space (fts, for short), and let μ be any fuzzy set in X. We define the closure of μ to be $\mathrm{Cl}(\mu) = \wedge\{\lambda \mid \mu \leq \lambda,\ \lambda \text{ is fuzzy closed}\}$ and the interior of μ to be $\mathrm{Int}(\mu) = \vee\{\lambda \mid \lambda \leq \mu,\ \lambda \text{ is fuzzy open}\}$. A fuzzy point x_r [16] is a fuzzy set with support x and value $r \in (0, 1]$. For a fuzzy set μ in X, we write $x_r \in \mu$ if and only if $r \leq \mu(x)$. Evidently, every fuzzy set μ can be expressed as the union of all fuzzy points which belongs to μ. A fuzzy point x_r is said to be quasicoincident [17] with μ denoted by $x_r q \mu$ if and only if $r + \mu(x) > 1$. A

$$\begin{tikzcd}
\text{r-open} \arrow[r] & \theta\text{-open} \arrow[r] & \delta\text{-open} \arrow[r] & \text{Open} \arrow[d] & \\
 & & p\text{-open} \arrow[d] & \alpha\text{-open} \arrow[l] \arrow[d] & \\
\beta\text{-open} \arrow[r] & \text{sp-open} & \gamma\text{-open} \arrow[l] & \text{s-open} \arrow[l] & \text{rs-open} \arrow[l]
\end{tikzcd}$$

Figure 1

fuzzy set μ is said to be quasicoincident with λ, denoted by $\mu q \lambda$, if and only if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. If μ is not quasicoincident with λ, then we write $\lambda \overline{q} \mu$. For a fuzzy set A of a fts (X, τ), $\mathrm{Cl}(A)$ (resp., $\mathrm{Scl}(A)$, $\mathrm{Pcl}(A)$, α-$\mathrm{cl}(A)$, γ-$\mathrm{cl}(A)$, β-$\mathrm{cl}(A)$, $\mathrm{Spcl}(A)$, δ-$\mathrm{cl}(A)$, θ-$\mathrm{cl}(A)$) denote the fuzzy closure (resp., semiclosure, precloure, α-closure, γ-closure, β-closure, semi-preclosure, δ-closure, θ-closure) and $\mathrm{Int}(A)$ (resp., $\mathrm{Sint}(A)$, $\mathrm{Pint}(A)$, α-$\mathrm{int}(A)$, γ-$\mathrm{int}(A)$, β-$\mathrm{int}(A)$, $\mathrm{Spint}(A)$, δ-$\mathrm{int}(A)$, θ-$\mathrm{int}(A)$) denote the fuzzy interior (resp., semi-interior, preinterior, α-interior, γ-interior, β-interior, semi-preinterior, δ-interior, and θ-interior) of A.

Definition 1.1 (see [17]). A fuzzy set A in an fts (X, τ) is said to be q-neighborhood of a fuzzy point x_r if there exists a fuzzy open set U with $x_r q U \leq A$.

Definition 1.2. A fuzzy set μ in an fts (X, τ) is said to be:

(1) fuzzy regular open [18] (ro, for short) if $\mathrm{Int}(\mathrm{Cl}(\mu)) = \mu$,

(2) fuzzy regular closed [18] (rc, for short) if $\mathrm{Cl}(\mathrm{Int}(\mu)) = \mu$,

(3) fuzzy regular semiopen [19](rso, for short) if there exists a fuzzy regular open set λ such that $\lambda \leq \mu \leq \mathrm{Cl}(\lambda)$,

(4) fuzzy α-open [20] (αo, for short) if $\mu \leq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mu))$,

(5) fuzzy α-closed [20] (αc, for short) if $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mu)) \leq \mu$,

(6) fuzzy semiopen [18] (so, for short) if $\mu \leq \mathrm{Cl}(\mathrm{Int}(\mu))$,

(7) fuzzy semiclosed [18] (sc, for short) if $\mathrm{Int}(\mathrm{Cl}(\mu)) \leq \mu$,

(8) fuzzy preopen [20] (po, for short) if $\mu \leq \mathrm{Int}(\mathrm{Cl}(\mu))$,

(9) fuzzy preclosed [20] (pc, for short) if $\mathrm{Cl}(\mathrm{Int}(\mu)) \leq \mu$,

(10) fuzzy γ-open [21] (γo, for short) if $\mu \leq \mathrm{Cl}(\mathrm{Int}(\mu)) \vee \mathrm{Int}(\mathrm{Cl}(\mu))$,

(11) fuzzy γ-closed [21] (γc, for short) if $\mu \leq \mathrm{Cl}(\mathrm{Int}(\mu)) \wedge \mathrm{Int}(\mathrm{Cl}(\mu))$,

(12) fuzzy semi-preopen [22] (spo, for short) if there exists a fuzzy preopen set λ such that $\lambda \leq \mu \leq \mathrm{Cl}(\lambda)$,

(13) fuzzy semi-preclosed [22] (spc, for short) if there exists a fuzzy preclosed set λ such that $\mathrm{Int}(\lambda) \leq \mu \leq \lambda$,

(14) fuzzy β-open [23] (βo, for short) if $\mu \leq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mu)))$,

(15) fuzzy β-closed [23] (βc, for short) if $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mu))) \leq \mu$.

Remark 1.3. From the above definition we have a diagram, Figure 1, showing all relationships between the classes of open sets. None of the implications shown in Figure 1 can be reversed in general.

Definition 1.4 (see [18, 20–23]). (1) The intersection of all fuzzy α-closed (resp., semiclosed, preclosed, γ-closed, semi-preclosed, β-closed) sets containing a fuzzy set A is called a fuzzy α-closure (resp., semiclosure, preclosure, γ-closure, semi-preclosure, β-closure) of A.

(2) The union of all fuzzy α-open (resp., semiopen, preopen, γ-open, semi-preopen, β-open) sets contained in a fuzzy set A is called a fuzzy α-interior (resp., semi-interior, preinterior, γ-interior, semi-preinterior, β-interior) of A.

Definition 1.5. A fuzzy point x_r in an fts X is said to be a fuzzy cluster (resp., θ-cluster [24], δ-cluster [16]) point of a fuzzy set A if and only if for every fuzzy open (resp., open, regular open) q-neighborhood U of x_r, UqA (resp., $\mathrm{Cl}(U)qA$, UqA). The set of all fuzzy cluster (resp., fuzzy θ-cluster, fuzzy δ-cluster) points of A is called the fuzzy closure (resp., θ-closure, δ-closure) of A and is denoted by $\mathrm{Cl}(A)$ (resp., $\theta\text{-cl}(A)$, $\delta\text{-cl}(A)$). A fuzzy set A is fuzzy θ-closed (resp., δ-closed) if and only if $A = \theta\text{-cl}(A)$ (resp., $A = \delta\text{-cl}(A)$). The complement of a fuzzy θ-closed (resp., δ-closed) set is called fuzzy θ-open (resp., δ-open).

Definition 1.6. A fuzzy set A of an fts (X, τ) is said to be:

(1) fuzzy generalized closed [3] (briefly, g-closed) if $\mathrm{Cl(A)} \leq U$, whenever $A \leq U$ and U is a fuzzy open set in X,

(2) fuzzy generalized α-closed [10] (briefly, gα-closed) if $\alpha\text{-cl}(A) \leq U$, whenever $A \leq U$ and U is a fuzzy open set in X,

(3) fuzzy α-generalized closed [13] (briefly, αg-closed) if $\alpha\text{-cl}(A) \leq U$, whenever $A \leq U$ and U is a fuzzy α-open set in X,

(4) fuzzy generalized semiclosed [12] (briefly, gs-closed) if $\mathrm{Scl}(A) \leq U$, whenever $A \leq U$ and U is a fuzzy open set in X,

(5) fuzzy semigeneralized closed [5] (briefly, sg-closed) if $\mathrm{Scl}(A) \leq U$, whenever $A \leq U$ and U is a fuzzy semiopen set in X,

(6) fuzzy generalized preclosed [8] (briefly, gp-closed) if $\mathrm{Pcl}(A) \leq U$, whenever $A \leq U$ and U is a fuzzy open set in X,

(7) fuzzy pregeneralized closed [6] (briefly, pg-closed) if $\mathrm{Pcl}(A) \leq U$, whenever $A \leq U$ and U is a fuzzy preopen set in X,

(8) fuzzy generalized semi-preclosed [11] (briefly, gsp-closed) if $\mathrm{Spcl}(A) \leq U$, whenever $A \leq U$ and U is a fuzzy open set in X,

(9) fuzzy semi-pregeneralized closed [14] (briefly, spg-closed) if $\mathrm{Spcl}(A) \leq U$, whenever $A \leq U$ and U is a fuzzy semi-preopen set in X,

(10) fuzzy regular generalized closed [9] (briefly, rg-closed) if $\mathrm{Cl}(A) \le U$, whenever $A \le U$ and U is a fuzzy regular open set in X,

(11) fuzzy generalized θ-closed [4] (briefly, gθ-closed) if θ-cl$(A) \le U$, whenever $A \le U$ and U is a fuzzy open set in X,

(12) fuzzy θ-generalized closed [7] (briefly, θg-closed) if θ-cl$(A) \le U$, whenever $A \le U$ and U is a fuzzy θ-open set in X,

(13) fuzzy $\delta\theta$-generalized closed [15] (briefly, $\delta\theta$g-closed) if δ-cl$(A) \le U$, whenever $A \le U$ and U is a fuzzy θ-open set in X.

The complement of a fuzzy generalized closed (resp., generalized α-closed, α-generalized closed, generalized semiclosed, semigeneralized closed, generalized preclosed, pre generalized closed, generalized semi-preclosed, semi-pregeneralized closed, regular generalized closed, θ-generalized closed, generalized θ-closed) set is called fuzzy generalized open (g-open, for short) (resp., generalized α-open (gα-open), α-generalized open (αg-open), generalized semiopen (gs-open), semi generalized open (sg-open), generalized preopen (gp-open), pre generalized open (pg-open), generalized semi-preopen (gsp-open), semi-pregeneralized open (spg-open), regular generalized open (rg-open), θ-generalized open (θg-open), generalized θ-open (gθ-open)).

Definition 1.7 (see [25]). A fuzzy point x_r in an fts (X, τ) is called weak (resp., strong) if $r \le 1/2$ (resp., $r > 1/2$).

2. ψ-Operations

In this research, we will denote for a fuzzy open set from type ψ by fuzzy ψ-open and the family of all fuzzy ψ-open sets in an fts (X, τ) by $\psi O(X)$. Also we will denote a fuzzy open (resp., α-open, semiopen, preopen, semi-preopen, γ-open, β-open, δ-open, θ-open, and regular open) set by τ-open (resp., α-open, s-open, p-open, sp-open, γ-open, β-open, δ-open, θ-open, and r-open). Similarly we will denote a fuzzy closed (resp., α-closed, semiclosed, preclosed, semi-preclosed, γ-closed, β-closed, δ-closed, θ-closed, and regular closed) sets by τ-closed (resp., α-closed, s-closed, p-closed, sp-closed, γ-closed, β-closed, δ-closed, θ-closed, and r-closed). Let $\Omega = \{\tau, \alpha, \mathrm{s}, \mathrm{p}, \mathrm{sp}, \gamma, \beta, \delta, \theta, \mathrm{r}\}$.

Definition 2.1. A fuzzy set A in an fts (X, τ) is said to be a fuzzy ψ-q-neighborhood of a fuzzy point x_r if and only if there exists a fuzzy ψ-open set U such that $x_r q U \le A$. The family of all fuzzy $\psi - q$-neighborhoods of a fuzzy point x_r is denoted by $N_{\psi}^{Q}(x_r)$.

Definition 2.2. A fuzzy point x_r in an fts (X, τ) is said to be a fuzzy ψ-cluster point of a fuzzy set A if and only if for every fuzzy $\psi - q$-neighborhood U of a fuzzy point x_r, UqA. The set of all fuzzy ψ-cluster points of a fuzzy set A is called the fuzzy ψ-closure of A and is denoted by $\psi cl(A)$. A fuzzy set A is fuzzy ψ-closed if and only if $A = \psi cl(A)$ and a fuzzy set A is fuzzy ψ-open if and only if its complement is fuzzy ψ-closed.

Theorem 2.3. *For a fuzzy set A in an fts (X, τ),*

$$\psi cl(A) = \wedge \{ F : F \ge A, 1 - F \in \psi O(X) \}. \tag{1}$$

Proof. The proof of this theorem is straightforward, so we omit it. □

Theorem 2.4. *Let A and B be fuzzy sets in an fts (X, τ). Then the following statements are true:*

(1) $\psi cl(0) = 0, \psi cl(1) = 1$;

(2) $A \leq \psi cl(A)$ *for each fuzzy set A of X;*

(3) *if $A \leq B$, then $\psi cl(A) \leq \psi cl(B)$;*

(4) *if A is ψ-closed, then $A = \psi cl(A)$, and if one supposes $\psi cl(A)$ is ψ-closed, then the converse of (4) is true;*

(5) *if $V \in \psi O(X)$, then VqA if and only if $Vq\psi cl(A)$;*

(6) $\psi cl(\psi cl(A)) = \psi cl(A)$;

(7) $\psi cl(A) \vee \psi cl(A) \leq \psi cl(A \vee B)$. *If the intersection of two fuzzy ψ-open sets is fuzzy ψ-open, then $\psi cl(A) \vee \psi cl(A) = \psi cl(A \vee B)$.*

Proof. (1), (2), (3), and (4) are easily proved.

(5) Let $V\overline{q}A$. Then $A \leq 1 - V$, and hence $\psi cl(A) \leq \psi cl(1 - V) = 1 - V$, which implies $V\overline{q}\psi cl(A)$. Hence VqA if and only if $Vq\psi cl(A)$.

(6) Let x_r be a fuzzy point with $x_r \notin \psi cl(A)$. Then there is a fuzzy $\psi - q$-neighborhood U of x_r such that $U\overline{q}A$. From (5) there is a fuzzy $\psi - q$-neighborhood U of x_r such that $U\overline{q}\psi cl(A)$ and hence $x_r \notin \psi cl(\psi cl(A))$. Thus $\psi cl(\psi cl(A)) \leq \psi cl(A)$. But $\psi cl(\psi cl(A)) \geq \psi cl(A)$. Therefore $\psi cl(\psi cl(A)) = \psi cl(A)$.

(7) It is clear. □

Definition 2.5. For a fuzzy set A in an fts (X, τ), we define a fuzzy ψ-interior of A as follows:

$$\psi int(A) = \vee\{U : U \leq A, U \in \psi O(X)\}. \tag{2}$$

Theorem 2.6. *Let A and B be fuzzy sets in an fts (X, τ). Then the following statements are true:*

(1) $\psi int(0) = 0, \psi int(1) = 1$;

(2) $A \geq \psi int(A)$ *for each fuzzy set A of X;*

(3) *if $A \leq B$, then $\psi int(A) \leq \psi int(B)$;*

(4) *if A is ψ-open, then $A = \psi int(A)$, if one supposes, $\psi int(A)$ is ψ-open, then the converse of (4) is true;*

(5) *if $V \in \psi C(X)$, then VqA if and only if $Vq\psi int(A)$;*

(6) $\psi int(\psi int(A)) = \psi int(A)$;

(7) $\psi int(A) \wedge \psi int(A) \geq \psi int(A \wedge B)$. *If the intersection of two fuzzy ψ-open sets is ψ-open, then $\psi int(A) \wedge \psi int(A) = \psi int(A \wedge B)$.*

Proof. It is similar to that of Theorem 2.4. □

Theorem 2.7. *For a fuzzy set A in an fts (X, τ), the following statements are true:*

(1) $\psi cl(1 - A) = 1 - \psi int(A)$;

(2) $\psi int(1 - A) = 1 - \psi cl(A)$.

Proof. It follows from the fact that the complement of a fuzzy ψ-open set is fuzzy ψ-closed and $\vee(1 - A_i) = 1 - \wedge A_i$. □

Definition 2.8. Let A be a fuzzy set of an fts (X, τ). A fuzzy point x_r is said to be ψ-boundary of a fuzzy set A if and only if $x_r \in \psi cl(A) \wedge (1 - \psi cl(A))$. By ψ-Bd(A) one denotes the fuzzy set of all ψ-boundary points of A.

Theorem 2.9. *Let A be a fuzzy set of an fts (X, τ). Then*

$$A \vee \psi\text{-}Bd(A) \leq \psi cl(A). \tag{3}$$

Proof. It follows from Definition 2.8 and Theorem 2.4. □

3. Generalized $\psi\rho$-Closed and Generalized $\psi\rho$-Open Sets

Definition 3.1. Let (X, τ) be an fts. We define the concepts of fuzzy generalized $\psi\rho$-closed and fuzzy generalized $\psi\rho$-open sets, where ψ represents a fuzzy closure operation and ρ represents a notion of fuzzy openness as follows:

(1) A fuzzy set A is said to be generalized $\psi\rho$-closed (g$\psi\rho$-closed, for short) if and only if $\psi cl(A) \leq U$, whenever $A \leq U$ and U is fuzzy ρ-open.

(2) The complement of a fuzzy generalized $\psi\rho$-closed set is said to be fuzzy generalized $\psi\rho$-open (g$\psi\rho$-open, for short).

Remark 3.2. Note that each type of generalized closed set in Definition 2.8 is defined to be generalized $\psi\rho$-closed set for some $\psi \in \Omega \setminus \{r\}$ and $\rho \in \Omega$. Namely, a fuzzy set A is fuzzy g-closed [3] if it is g$\tau\tau$-closed, gα-closed [10] if it is g$\alpha\tau$-closed, αg-closed [13] if it is g$\alpha\alpha$-closed, gs-closed [12] if it is g$s\tau$-closed, sg-closed [5] if it is gs – s-closed, gp-closed [8] if it is gpτ-closed, pg-closed [6] if it is gp – p-closed, gsp-closed [11] if it is gspτ-closed, spg-closed [14] if it is gsp – sp-closed, gθ-closed [4] if it is g$\theta\tau$-closed, θg-closed [7] if it is g$\theta\theta$-closed, and gr-closed [9] if it is gτr-closed.

Theorem 3.3. *A fuzzy set A is generalized $\psi\rho$-open if and only if $\psi int(A) \geq F$, whenever $A \geq F$ and F is fuzzy ρ-closed.*

Proof. It is clear. □

Theorem 3.4. *If A is a fuzzy ψ-closed set in an fts (X, τ), then A is fuzzy generalized $\psi\rho$-closed.*

Proof. Let A be a fuzzy ψ-closed, and let U be a fuzzy ρ-open set in X such that $A \leq U$. Then $\psi cl(A) = A \leq U$, and hence A is fuzzy generalized $\psi\rho$-closed. □

Remark 3.5. In classical topology, if A is a generalized $\psi\rho$-closed set in a topological space X, then $\psi cl(A) \setminus A$ does not contain nonempty ρ-closed. But in fuzzy topology this is not true in general as shown by the following example.

Example 3.6. Let μ, ν, λ, η, and σ be fuzzy subsets of $X = \{x, y\}$ defined as follows:

$$\begin{aligned} \mu(x) &= 0.25, & \mu(y) &= 0.70, \\ v(x) &= 0.65, & v(y) &= 0.35, \\ \lambda(x) &= 0.30, & \lambda(y) &= 0.30, \\ \eta(x) &= 0.30, & \eta(y) &= 0.36, \\ \sigma(x) &= 0.20, & \sigma(y) &= 0.20. \end{aligned} \tag{1}$$

Let $\tau = \{0, \mu, \nu, \mu \wedge \nu, \mu \vee \nu, 1\}$ be a fuzzy topology on X.

One may notice that the following.

(1) η is a fuzzy gθ-closed set and

$$\begin{aligned} \theta\text{-cl}(\eta)(x) &= 0.35, & \theta\text{-cl}(\eta)(y) &= 0.65, \\ (\theta\text{-cl}(\eta) \setminus \eta)(x) &= 0.35, & (\theta\text{-cl}(\eta) \setminus \eta)(y) &= 0.64. \end{aligned} \tag{2}$$

But $(\theta\text{-cl}(\eta) \setminus \eta)$ contains nonempty fuzzy closed $(\mu \vee \nu)^c$.

(2) λ is a fuzzy generalized closed set and

$$\begin{aligned} \text{Cl}(\lambda)(x) &= 0.35, & \text{Cl}(\lambda)(y) &= 0.30, \\ (\text{Cl}(\lambda) \setminus \lambda)(x) &= 0.35, & (\text{Cl}(\lambda) \setminus \lambda)(y) &= 0.30. \end{aligned} \tag{3}$$

But $(\text{Cl}(\lambda) \setminus \lambda)$ contains nonempty closed set $(\mu \vee \nu)^c$.

(3) λ is a fuzzy αg-closed (resp., sg-closed, pg-closed, γg-closed, spg-closed) set and

$$\begin{gathered} \begin{aligned} \alpha\text{-cl}(\lambda)(x) &= 0.30, & \alpha\text{-cl}(\lambda)(y) &= 0.30, \\ (\alpha\text{-cl}(\lambda) \setminus \lambda)(x) &= 0.30, & (\alpha\text{-cl}(\lambda) \setminus \lambda)(y) &= 0.30, \end{aligned} \\ \alpha\text{-cl}(\lambda) = \text{Scl}(\lambda) = \text{Pcl}(\lambda) = \text{Spcl}(\lambda), \end{gathered} \tag{4}$$

and hence

$$(\alpha\text{-cl}(\lambda) \setminus \lambda) = (\text{Scl}(\lambda) \setminus \lambda) = (\text{Pcl}(\lambda) \setminus \lambda) = (\text{Spcl}(\lambda) \setminus \lambda). \tag{5}$$

But (α-cl(λ) $\setminus$ λ) contains nonempty set σ which is fuzzy α-closed and hence is fuzzy semiclosed, preclosed, and semi-preclosed, and so on.

Theorem 3.7. *Let (X, τ) be an fts, and let A be a fuzzy $g\psi\rho$-closed set with $A \leq B \leq \psi cl(A)$. Then B is a fuzzy $g\psi\rho$-closed set.*

Proof. Let H be a fuzzy ρ-open set in X such that $B \leq H$. Then $A \leq H$. Since A is fuzzy g$\psi\rho$-closed, then $\psi cl(A) \leq H$, and hence $\psi cl(B) \leq \psi cl(A)$. Thus $\psi cl(B) \leq H$, and hence B is a fuzzy g$\psi\rho$-closed set □

Theorem 3.8. *Let (X, τ) be an fts, and let A be a fuzzy $g\psi\rho$-open set with $\psi int(A) \leq B \leq A$. Then B is a fuzzy $g\psi\rho$-open set.*

Proof. It is similar to that of Theorem 3.7. □

Theorem 3.9. *Let A be a fuzzy set in an fts, (X, τ) and let $\rho cl(A)$ be ρ-closed for each fuzzy set A. Then A is fuzzy $g\psi\rho$-closed if and only if for each fuzzy point x_r with $x_r q\psi cl(A)$, one has $\rho cl(x_r)qA$.*

Proof. Let $x_r q\psi cl(A)$ and suppose that $\rho cl(x_r)\overline{q}A$. Since $\rho cl(x_r)$ is ρ-closed, then $(\rho cl(x_r))^C$ is fuzzy ρ-open and $A \leq (\rho cl(x_r))^C$. Since A is fuzzy g$\psi\rho$-closed, then $\psi cl(A) \leq (\rho cl(x_r))^C$ and hence $\rho cl(x_r))\overline{q}\psi cl(A)$ which contradict with $x_r q\psi cl(A)$ and hence $\rho cl(x_r)qA$.

Conversely, let B be fuzzy ρ-open set with $A \leq B$ and let $x_r q\psi cl(A)$. By hypothesis $\rho cl(x_r)qA$, and hence there is $y \in X$ such that $\rho cl(x_r)(y) + A(y) > 1$. Put $\rho cl(x_r)(y) = s$. Then $y_s \in \rho cl(x_r)$, $y_s qA$ and hence $y_s qB$. Since $y_s \in \rho cl(x_r)$, B is a fuzzy ρ-open set and $y_s qB$, then $x_r qB$. Hence $\psi cl(A) \leq B$. Thus A is fuzzy g$\psi\rho$-closed. □

Theorem 3.10. *Let (X, τ) be an fts, and let A be a fuzzy set in X. Then the following are equivalent:*

(1) *A is fuzzy $g\psi\rho$-closed;*

(2) *if A is fuzzy ρ-open, then A is fuzzy ψ-closed.*

Proof. (1) → (2). Let A be fuzzy g$\psi\rho$-closed and fuzzy ρ-open with $A \leq A$. Then $\psi cl(A) \leq A$. Since $A \leq \psi cl(A)$, then $A = \psi cl(A)B$. Therefore A is ψ-closed.

(2) → (1). Let A be a fuzzy set with $A \leq B$, where B is fuzzy ρ-open set in X. From (2) we have B is ψ-closed, and hence $\psi cl(A) = A \leq B$. Thus A is fuzzy g$\psi\rho$-closed. □

Theorem 3.11. *Let (X, τ) be an fts and suppose that x_r and y_s are weak and strong fuzzy points, respectively. If x_r is fuzzy $g\psi\rho$-closed and $\rho cl(y_s)$ is fuzzy ρ-closed, then*

$$y_s \in \psi cl(x_r) \Longrightarrow x_r \in \rho cl(y_s). \tag{6}$$

Proof. Let $y_s \in \psi cl(x_r)$ and $x_r \notin \rho cl(y_s)$. Then $r > \rho cl(y_s)$. Since x_r is a weak fuzzy point, then $r \leq 1/2$, and hence $\rho cl(y_s)(x) \leq 1 - r$. Thus $r \leq (\rho cl(y_s))^C(x)$. So $x_r \in (\rho cl(y_s))^C$. Since x_r is fuzzy g$\psi\rho$-closed and $(\rho cl(y_s))^C$ is fuzzy ρ-open, then $\psi cl(x_r) \leq (\rho cl(y_s))^C$, and hence $y_s \in (\rho cl(y_s))^C$, which is a contradiction, since if $y_s \in (\rho cl(y_s))^C$, then $(\rho cl(y_s))(y) \leq 1 - s$. But since $y_s \in (\rho cl(y_s))$, then $s \leq (\rho cl(y_s)(y))$, and hence $s \leq 1 - s$, which implies $s \leq 1/2$. Since y_s is fuzzy strong point, then $s > 1/2$, which is a contradiction. Thus $x_r \in \rho cl(y_s)$. □

Definition 3.12. Let (X, τ) be an fts. A fuzzy point x_r is said to be fuzzy just-ψ-closed if the fuzzy set $\psi cl(x_r)$ is a fuzzy point.

Theorem 3.13. *Let (X, τ) be an fts. If x_r and x_s are two fuzzy points such that $r < s$ and x_s is fuzzy ρ-open, then x_r is fuzzy just-ψ-closed if it is fuzzy $g\psi\rho$-closed.*

Proof. Let $x_r < x_s$, x_s be fuzzy ρ-open, and let x_r be fuzzy $g\psi\rho$-closed. Then $\psi cl(x_r) \leq x_s$, and hence $(\psi cl(x_r))(x) \leq s$ and $(\psi cl(x_r))(z) = 0$ for each $z \in X \setminus \{x\}$. Thus $\psi cl(x_r)$ is a fuzzy point. Therefore x_r is fuzzy just-ψ-closed. □

Definition 3.14. Let (X, τ) be an fts. A fuzzy set U of X is called fuzzy ψ-nearly crisp if $\psi cl(U) \wedge (\psi cl(U))^C = 0$.

Theorem 3.15. *If A is fuzzy $g\psi\rho$-closed and fuzzy ψ-nearly crisp of an fts (X, τ), then $\psi cl(A) \setminus A$ does not contain any nonempty fuzzy ρ-closed set in X.*

Proof. Suppose that A is a fuzzy $g\psi\rho$-closed set in X, and let F be a fuzzy ρ-closed set such that $F \leq \psi cl(A) \setminus A$ and $F \neq 0_X$. Then $A \leq F^C$ and F^C is fuzzy ρ-open. Since A is a fuzzy $g\psi\rho$-closed, then $\psi - cl(A) \leq F^c$, and hence $F \leq (\psi cl(A))^C$, so $F \leq \psi cl(A) \wedge (\psi cl(A)))^C = 0$. Therefore, $F = 0_X$, which is contradiction. Hence $\psi cl(A) \setminus A$ does not contain any nonempty fuzzy ρ-closed set in X. □

Theorem 3.16. *Let (X, τ) be an fts. Then every fuzzy ρ-open set is fuzzy ψ-closed if and only if every fuzzy subset of X is fuzzy $g\psi\rho$-closed.*

Proof. Suppose that U be a fuzzy ρ-open set and A be any fuzzy subset of X such that $A \leq U$. By hypothesis, U is fuzzy ψ-closed, and hence $\psi cl(A) \leq \psi cl(U) \leq U$. Thus A is fuzzy $g\psi\rho$-closed.

Conversely, suppose that every fuzzy subset of X is fuzzy $g\psi\rho$-closed and U is a fuzzy ρ-open set. Since $U \leq U$ and U is fuzzy $g\psi\rho$-closed, then $\psi cl(U) \leq U$ and hence $\psi cl(U) = U$. Thus U is fuzzy ψ-closed. □

Theorem 3.17. *If A is fuzzy $g\psi\rho$-open and fuzzy ψ-nearly crisp of an fts (X, τ), then $G = 1$, where G is a fuzzy ρ-open and $\psi int(A) \vee A^C \leq G$.*

Proof. Suppose that A is a fuzzy $g\psi\rho$-open set in X, and let G be a fuzzy ρ-open set such that $\psi int(A) \vee A^C \leq G$. Then $G^C \leq (\psi int(A) \vee A^C)^C = (\psi int(A))^C \wedge A$. That is $G^C \leq (\psi int(A))^C \setminus A^C$, and hence $G^C \leq \psi cl(A^C) \setminus A^C$. Since G^C is fuzzy ρ-closed and A^C is fuzzy $g\psi\rho$-closed, then by Theorem 3.15, we have $G^C = 0$. Hence $G = 1$. □

Definition 3.18. An fts (X, τ) is said to be fuzzy $\psi\rho$-regular if for each fuzzy point x_r and a fuzzy ρ-closed set F not containing x_r, there is $U, V \in \psi O(X)$ such that $x_r \in U, F \leq V$, and $U\overline{q}V$.

Theorem 3.19. *If (X, τ) is a fuzzy $\psi\rho$-regular space, then for each strong fuzzy x_r and a fuzzy ρ-open set U containing x_r, there is $V \in \psi O(X)$ such that $x_r \in V$ and $\psi cl(V) \leq U$.*

Proof. It is clear. □

Theorem 3.20. *If (X, τ) is a fuzzy $\psi\rho$-regular space, then each strong fuzzy point in X is fuzzy $g\psi\rho$-closed.*

Proof. Let x_r be strong fuzzy point in X, and let U be a fuzzy ρ-open set such that $x_r \leq U$. Then by Theorem 3.19, there is a fuzzy ψ-open set W such that $x_r \in W$ and $\psi cl(W) \leq U$, and hence $\psi cl(x_r) \leq \psi cl(W) \leq U$. Thus x_r is fuzzy $g\psi\rho$-closed. □

Theorem 3.21. *A fuzzy set A in an fts (X, τ) is fuzzy $g\psi\rho$-closed if and only if $A\overline{q}E \Rightarrow \psi cl(A)\overline{q}E$ for each fuzzy ρ-closed set E of X.*

Proof. Let E be a fuzzy ρ-closed set of X and $A\overline{q}E$. Then $A \leq 1 - E$, and $1 - E$ is fuzzy ρ-open in X. Since A is fuzzy $g\psi\rho$-closed, then $\psi cl(A) \leq 1 - E$ and hence $\psi cl(A)\overline{q}E$.

Conversely, let B be a fuzzy ρ-open set of X such that $A \leq H$. Then $A\overline{q}(1 \setminus B)$ and $1 \setminus B$ is fuzzy ρ-closed in X. By hypothesis $\psi cl(A)\overline{q}(1 \setminus B)$, which implies $\psi cl(A) \leq B$. Hence A is fuzzy $g\psi\rho$-closed in X. □

Definition 3.22. An fts (X, τ) is said to be fuzzy quasi-ψ-T_1 if for all fuzzy points x_r and y_s with $x \neq y$, there exist two fuzzy ψ-open sets U, V such that $x_r \in U$ and $y_s \notin U$, $x_r \notin V$, and $y_s \in V$.

Definition 3.23. Let (X, τ) be an fts. A fuzzy point x_r is said to be well ψ-closed if there exists $y_s \in \psi cl(x_r)$ such that $x \neq y$.

Theorem 3.24. *If (X, τ) is an fts and x_r is fuzzy $g\psi\rho$-closed, well ψ-closed fuzzy point, then X is not fuzzy quasi-ψ-T_1.*

Proof. Let X be a fuzzy quasi-$\psi - T_1$ space and x_r is well ψ-closed. Then there exists a fuzzy point y_s with $x \neq y$ such that $y_s \in \psi cl(x_r)$, and hence there exists a fuzzy ψ-open U such that $x_r \in U$ and $y_s \notin U$. Since x_r is fuzzy $g\psi\rho$-closed, then $\psi cl(x_r) \leq U$, and hence $y_s \in U$. This is a contradiction, and hence X is not quasi-$\psi - T_1$. □

Theorem 3.25. *If A is a fuzzy ρ-open and a fuzzy $g\psi\rho$-closed in an fts (X, τ) and ψ-$cl(A)$ is ψ-closed, then A is ψ-closed.*

Proof. Let A be fuzzy ρ-open and fuzzy $g\psi\rho$-closed in X. Then ψ-$cl(A) \leq A$, and hence ψ-$cl(A) = A$. Therefore A is ψ-closed. □

Theorem 3.26. *If $\psi cl(A) \vee \psi cl(A) = \psi cl(A \vee B)$, then the union of two fuzzy $g\psi\rho$-closed sets in an fts (X, τ) is $g\psi\rho$-closed.*

Proof. Let A and B be fuzzy $g\psi\rho$-closed sets in an fts (X, τ), and let U be fuzzy ρ-open such that $A \vee B \leq U$. Then $A \leq U$ and $B \leq U$, and hence $\psi cl(A) \leq U$ and $\psi cl(B) \leq U$. Since $\psi cl(A) \vee \psi cl(A) = \psi cl(A \vee B)$, then $\psi cl(A \vee B) \leq U$, and hence $A \vee B$ is fuzzy $g\psi\rho$-closed in X. □

Remark 3.27. Some results in papers [3–15] can be considered as special results from our results in this paper.

Table 1

	0	1	2	3	4	5	6	7	8	9	10
0	ρ ψ	r	θ	δ	τ	α	s	p	γ	sp	β
1	θ	gθr —	g$\theta\theta$ θg [7]	g$\theta\delta$ —	g$\theta\tau$ gθ [4]	g$\theta\alpha$ —	gθs —	gθp —	g$\theta\gamma$ —	gθsp —	g$\theta\beta$ —
2	δ	gδr —	g$\delta\theta$ $\delta\theta$g [15]	g$\delta\delta$ —	g$\delta\tau$ —	g$\delta\alpha$ —	gδs —	gδp —	g$\delta\gamma$ —	gδsp —	g$\delta\beta$ —
3	τ	gτr rg [9]	g$\tau\theta$ —	g$\tau\delta$ —	g$\tau\tau$ —	g$\tau\alpha$ —	gτs —	gτp —	g$\tau\gamma$ —	gτsp —	g$\tau\beta$ —
4	α	gαr —	g$\alpha\theta$ —	g$\alpha\delta$ —	g$\alpha\tau$ gα [10]	gα αg [13]	gαs —	gαp —	g$\alpha\gamma$ —	gαsp —	g$\alpha\beta$ —
5	s	gs − r —	gsθ —	gsδ —	gsτ gs [12]	gsα —	gs − s sg [5]	gs − p —	gsγ —	gs − sp —	gsβ —
6	p	gp − r —	gpθ —	gpδ —	gpτ gp [8]	gpα —	gp − s —	gp − p pg [6]	gpγ —	gp − sp —	gpβ —
7	γ	gγr —	g$\gamma\theta$ —	g$\gamma\delta$ —	g$\gamma\tau$ —	g$\gamma\alpha$ —	gγs —	gγp —	g$\gamma\gamma$ —	gγsp —	g$\gamma\beta$ —
8	sp	gsp − r —	gspθ —	gspδ —	gspτ gsp [11]	gspα —	gsp − s —	gsp − p —	gspγ —	gsp − sp spg [14]	gspβ —
9	β	gβr —	g$\beta\theta$ —	g$\beta\delta$ —	g$\beta\tau$ —	g$\beta\alpha$ —	gβs —	gβp —	g$\beta\gamma$ —	gβsp —	g$\beta\beta$ —

4. Summary

The results are summarized in the following table. Each cell gives the type of generalized closed set which is g$\psi\rho$-closed, where ψ (closure) is given by the left-hand (zeroth) column and ρ (openness) is given by the top (zeroth) row.

The table highlights some general relationships between certain groups of generalized closed sets. For example, column 2 implies column 1. (Each type of generalized closed set listed in column 2 implies the type of generalized closed set listed in the same row of column 1.) In fact each column in Table 1 implies each of the preceding column apart from columns 6 and 7. Each of these implications, apart from columns 6 and 7, follows immediately from the definitions, since the types of generalized closed sets in any particular row involve the same notion of closure, and these notions of closure decrease in strength from top to down, apart from rows 5 and 6. Similarly each row implies each subsequent row, apart from rows 5 and 6.

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