Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 178568, 19 pages
doi:10.1155/2011/178568

Research Article

New Stability Conditions for Linear Differential Equations with Several Delays

Leonid Berezansky[1] and Elena Braverman[2]

[1] *Department of Mathematics, Ben-Gurion University of the Negev, Beer Sheva 84105, Israel*
[2] *Department of Mathematics and Statistics, University of Calgary, 2500 University Drive Northwest, Calgary, AB, Canada T2N 1N4*

Correspondence should be addressed to Elena Braverman, maelena@math.ucalgary.ca

Received 28 January 2011; Accepted 4 April 2011

Academic Editor: Josef Diblík

New explicit conditions of asymptotic and exponential stability are obtained for the scalar nonautonomous linear delay differential equation $\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(h_k(t)) = 0$ with measurable delays and coefficients. These results are compared to known stability tests.

1. Introduction

In this paper we continue the study of stability properties for the scalar linear differential equation with several delays and an arbitrary number of positive and negative coefficients

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(h_k(t)) = 0, \quad t \geq t_0, \tag{1.1}$$

which was begun in [1–3]. Equation (1.1) and its special cases were intensively studied, for example, in [4–21]. In [2] we gave a review of stability tests obtained in these papers.

In almost all papers on stability of delay-differential equations coefficients and delays are assumed to be continuous, which is essentially used in the proofs of main results. In real-world problems, for example, in biological and ecological models with seasonal fluctuations of parameters and in economical models with investments, parameters of differential equations are not necessarily continuous.

There are also some mathematical reasons to consider differential equations without the assumption that parameters are continuous functions. One of the main methods to investigate impulsive differential equations is their reduction to a nonimpulsive differential equation with discontinuous coefficients. Similarly, difference equations can sometimes be reduced to the similar problems for delay-differential equations with discontinuous piecewise constant delays.

In paper [1] some problems for differential equations with several delays were reduced to similar problems for equations with one delay which generally is not continuous.

One of the purposes of this paper is to extend and partially improve most popular stability results for linear delay equations with continuous coefficients and delays to equations with measurable parameters.

Another purpose is to generalize some results of [1–3]. In these papers, the sum of coefficients was supposed to be separated from zero and delays were assumed to be bounded. So the results of these papers are not applicable, for example, to the following equations:

$$\begin{gathered}\dot{x}(t)+|\sin t|x(t-\tau)=0,\\ \dot{x}(t)+\alpha(|\sin t|-\sin t)x(t-\tau)=0,\\ \dot{x}(t)+\frac{1}{t}x(t)+\frac{\alpha}{t}x\left(\frac{t}{2}\right)=0.\end{gathered} \tag{1.2}$$

In most results of the present paper these restrictions are omitted, so we can consider all the equations mentioned above. Besides, necessary stability conditions (probably for the first time) are obtained for (1.1) with nonnegative coefficients and bounded delays. In particular, if this equation is exponentially stable then the ordinary differential equation

$$\dot{x}(t)+\sum_{k=1}^{m}a_k(t)x(t)=0 \tag{1.3}$$

is also exponentially stable.

2. Preliminaries

We consider the scalar linear equation with several delays (1.1) for $t \geq t_0$ with the initial conditions (for any $t_0 \geq 0$)

$$x(t)=\varphi(t),\quad t<t_0,\qquad x(t_0)=x_0, \tag{2.1}$$

and under the following assumptions:

(a1) $a_k(t)$ are Lebesgue measurable essentially bounded on $[0,\infty)$ functions;

(a2) $h_k(t)$ are Lebesgue measurable functions,

$$h_k(t)\leq t,\qquad \limsup_{t\to\infty}h_k(t)=\infty; \tag{2.2}$$

(a3) $\varphi:(-\infty,t_0)\to R$ is a Borel measurable bounded function.

We assume conditions (a1)–(a3) hold for all equations throughout the paper.

Definition 2.1. A locally absolutely continuous for $t \geq t_0$ function $x : R \to R$ is called *a solution* of problem (1.1), (2.1) if it satisfies (1.1) for almost all $t \in [t_0, \infty)$ and the equalities (2.1) for $t \leq t_0$.

Below we present a solution representation formula for the nonhomogeneous equation with locally Lebesgue integrable right-hand side $f(t)$:

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(h_k(t)) = f(t), \quad t \geq t_0. \tag{2.3}$$

Definition 2.2. A solution $X(t,s)$ of the problem

$$\begin{gathered} \dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(h_k(t)) = 0, \quad t \geq s \geq 0, \\ x(t) = 0, \quad t < s, \qquad x(s) = 1, \end{gathered} \tag{2.4}$$

is called *the fundamental function* of (1.1).

Lemma 2.3 (see [22, 23]). *Suppose conditions (a1)–(a3) hold. Then the solution of* (2.3), (2.1) *has the following form*

$$x(t) = X(t,t_0)x_0 - \int_{t_0}^{t} X(t,s)\sum_{k=1}^{m} a_k(s)\varphi(h_k(s))ds + \int_{t_0}^{t} X(t,s)f(s)ds, \tag{2.5}$$

where $\varphi(t) = 0$, $t \geq t_0$.

Definition 2.4 (see [22]). Equation (1.1) is *stable* if for any initial point t_0 and number $\varepsilon > 0$ there exists $\delta > 0$ such that the inequality $\sup_{t<t_0}|\varphi(t)| + |x(t_0)| < \delta$ implies $|x(t)| < \varepsilon$, $t \geq t_0$, for the solution of problem (1.1), (2.1).

Equation (1.1) is *asymptotically stable* if it is stable and all solutions of (1.1)-(2.1) for any initial point t_0 tend to zero as $t \to \infty$.

In particular, (1.1) is asymptotically stable if the fundamental function is uniformly bounded: $|X(t,s)| \leq K$, $t \geq s \geq 0$ and all solutions tend to zero as $t \to \infty$.

We apply in this paper only these two conditions of asymptotic stability.

Definition 2.5. Equation (1.1) is *(uniformly) exponentially stable*, if there exist $M > 0$, $\mu > 0$ such that the solution of problem (1.1), (2.1) has the estimate

$$|x(t)| \leq Me^{-\mu(t-t_0)}\left(|x(t_0)| + \sup_{t<t_0}\left|\varphi(t)\right|\right), \quad t \geq t_0, \tag{2.6}$$

where M and μ do not depend on t_0.

Definition 2.6. The fundamental function $X(t,s)$ of (1.1) *has an exponential estimation* if there exist $K > 0$, $\lambda > 0$ such that

$$|X(t,s)| \leq Ke^{-\lambda(t-s)}, \quad t \geq s \geq 0. \tag{2.7}$$

For the linear (1.1) with bounded delays the last two definitions are equivalent. For unbounded delays estimation (2.7) implies asymptotic stability of (1.1).

Under our assumptions the exponential stability does not depend on values of equation parameters on any finite interval.

Lemma 2.7 (see [24, 25]). *Suppose $a_k(t) \geq 0$. If*

$$\int_{\max\{h(t),t_0\}}^{t} \sum_{i=1}^{m} a_i(s)ds \leq \frac{1}{e}, \quad h(t) = \min_k\{h_k(t)\}, \quad t \geq t_0, \tag{2.8}$$

or there exists $\lambda > 0$, such that

$$\lambda \geq \sum_{k=1}^{m} A_k e^{\lambda \sigma_k}, \tag{2.9}$$

where

$$0 \leq a_k(t) \leq A_k, \quad t - h_k(t) \leq \sigma_k, \quad t \geq t_0, \tag{2.10}$$

then $X(t,s) > 0$, $t \geq s \geq t_0$, where $X(t,s)$ is the fundamental function of (1.1).

Lemma 2.8 (see [3]). *Suppose $a_k(t) \geq 0$,*

$$\liminf_{t\to\infty} \sum_{k=1}^{m} a_k(t) > 0, \tag{2.11}$$

$$\limsup_{t\to\infty} (t - h_k(t)) < \infty, \quad k = 1, \ldots, m, \tag{2.12}$$

and there exists $r(t) \leq t$ such that for sufficiently large t

$$\int_{r(t)}^{t} \sum_{k=1}^{m} a_k(s)ds \leq \frac{1}{e}. \tag{2.13}$$

If

$$\limsup_{t\to\infty} \sum_{k=1}^{m} \frac{a_k(t)}{\sum_{i=1}^{m} a_i(t)} \left| \int_{h_k(t)}^{r(t)} \sum_{i=1}^{m} a_i(s)ds \right| < 1, \tag{2.14}$$

then (1.1) *is exponentially stable.*

Lemma 2.9 (see [3]). *Suppose* (2.12) *holds and there exists a set of indices $I \subset \{1, \dots, m\}$, such that $a_k(t) \geq 0$, $k \in I$,*

$$\liminf_{t \to \infty} \sum_{k \in I} a_k(t) > 0, \tag{2.15}$$

and the fundamental function of the equation

$$\dot{x}(t) + \sum_{k \in I} a_k(t) x(h_k(t)) = 0 \tag{2.16}$$

is eventually positive. If

$$\limsup_{t \to \infty} \frac{\sum_{k \notin I} |a_k(t)|}{\sum_{k \in I} a_k(t)} < 1, \tag{2.17}$$

then (1.1) *is exponentially stable.*

The following lemma was obtained in [26, Corollary 2], see also [27].

Lemma 2.10. *Suppose for* (1.1) *condition* (2.12) *holds and this equation is exponentially stable. If*

$$\int_0^{\infty} \sum_{k=1}^{n} |b_k(s)| ds < \infty, \qquad \limsup_{t \to \infty} \left(t - g_k(t)\right) < \infty, \quad g_k(t) \leq t, \tag{2.18}$$

then the equation

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t) x(h_k(t)) + \sum_{k=1}^{n} b_k(t) x\left(g_k(t)\right) = 0 \tag{2.19}$$

is exponentially stable.

The following elementary result will be used in the paper.

Lemma 2.11. *The ordinary differential equation*

$$\dot{x}(t) + a(t) x(t) = 0 \tag{2.20}$$

is exponentially stable if and only if there exists $R > 0$ such that

$$\liminf_{t \to \infty} \int_t^{t+R} a(s) ds > 0. \tag{2.21}$$

The following example illustrates that a stronger than (2.21) sufficient condition

$$\liminf_{t,s\to\infty}\frac{1}{t-s}\int_s^t a(\tau)d\tau > 0 \tag{2.22}$$

is not necessary for the exponential stability of the ordinary differential equation (2.20).

Example 2.12. Consider the equation

$$\dot{x}(t)+a(t)x(t)=0, \quad \text{where } a(t)=\begin{cases}1, & t\in[2n,2n+1),\\ 0, & t\in[2n+1,2n+2),\end{cases} \quad n=0,1,2,\ldots \tag{2.23}$$

Then lim inf in (2.22) equals zero, but $|X(t,s)| < ee^{-0.5(t-s)}$, so the equation is exponentially stable. Moreover, if we consider lim inf in (2.22) under the condition $t-s\ge R$, then it is still zero for any $R\le 1$.

3. Main Results

Lemma 3.1. *Suppose* $a_k(t)\ge 0$, (2.11), (2.12) *hold and*

$$\limsup_{t\to\infty}\sum_{k=1}^m\frac{a_k(t)}{\sum_{i=1}^m a_i(t)}\int_{h_k(t)}^t\sum_{i=1}^m a_i(s)ds < 1+\frac{1}{e}. \tag{3.1}$$

Then (1.1) *is exponentially stable.*

Proof. By (2.11) there exists function $r(t)\le t$ such that for sufficiently large t

$$\int_{r(t)}^t\sum_{k=1}^m a_k(s)ds=\frac{1}{e}. \tag{3.2}$$

For this function condition (2.14) has the form

$$\begin{aligned}\limsup_{t\to\infty}&\sum_{k=1}^m\frac{a_k(t)}{\sum_{i=1}^m a_i(t)}\left|\int_{h_k(t)}^t\sum_{i=1}^m a_i(s)ds-\int_{r(t)}^t\sum_{i=1}^m a_i(s)ds\right|\\ &=\limsup_{t\to\infty}\sum_{k=1}^m\frac{a_k(t)}{\sum_{i=1}^m a_i(t)}\left|\int_{h_k(t)}^t\sum_{i=1}^m a_i(s)ds-\frac{1}{e}\right|<1.\end{aligned} \tag{3.3}$$

The latter inequality follows from (3.1). The reference to Lemma 2.8 completes the proof. □

Corollary 3.2. *Suppose* $a_k(t) \geq 0$, (2.11), (2.12) *hold and*

$$\limsup_{t\to\infty} \int_{\min_k\{h_k(t)\}}^{t} \sum_{i=1}^{m} a_i(s)ds < 1 + \frac{1}{e}. \tag{3.4}$$

Then (1.1) *is exponentially stable.*

The following theorem contains stability conditions for equations with unbounded delays. We also omit condition (2.11) in Lemma 3.1.

We recall that $b(t) > 0$ in the space of Lebesgue measurable essentially bounded functions means $b(t) \geq 0$ and $b(t) \neq 0$ almost everywhere.

Theorem 3.3. *Suppose* $a_k(t) \geq 0$, *condition* (3.1) *holds,* $\sum_{k=1}^{m} a_k(t) > 0$ *and*

$$\int_0^{\infty} \sum_{k=1}^{m} a_k(t)dt = \infty, \qquad \limsup_{t\to\infty} \int_{h_k(t)}^{t} \sum_{i=1}^{m} a_i(s)ds < \infty. \tag{3.5}$$

Then (1.1) *is asymptotically stable.*

If in addition there exists $R > 0$ *such that*

$$\liminf_{t\to\infty} \int_{t}^{t+R} \sum_{k=1}^{m} a_k(\tau)d\tau > 0 \tag{3.6}$$

then the fundamental function of (1.1) *has an exponential estimation.*

If condition (2.12) *also holds then* (1.1) *is exponentially stable.*

Proof. Let $s = p(t) := \int_0^t \sum_{k=1}^{m} a_k(\tau)d\tau$, $y(s) = x(t)$, where $p(t)$ is a strictly increasing function. Then $x(h_k(t)) = y(l_k(s))$, $l_k(s) \leq s$, $l_k(s) = \int_0^{h_k(t)} \sum_{i=1}^{m} a_i(\tau)d\tau$ and (1.1) can be rewritten in the form

$$\dot{y}(s) + \sum_{k=1}^{m} b_k(s) y(l_k(s)) = 0, \tag{3.7}$$

where $b_k(s) = a_k(t)/\sum_{i=1}^{m} a_i(t)$, $s - l_k(s) = \int_{h_k(t)}^{t} \sum_{i=1}^{m} a_i(\tau)d\tau$. Since $\sum_{k=1}^{m} b_k(s) = 1$ and $\limsup_{s\to\infty}(s - l_k(s)) < \infty$, then Lemma 3.1 can be applied to (3.7). We have

$$\begin{aligned}
&\limsup_{s\to\infty} \sum_{k=1}^{m} \frac{b_k(s)}{\sum_{i=1}^{m} b_i(s)} \int_{l_k(s)}^{s} \sum_{i=1}^{m} b_i(\tau)d\tau \\
&\quad = \limsup_{s\to\infty} \sum_{k=1}^{m} b_k(s)(s - l_k(s)) \\
&\quad = \limsup_{t\to\infty} \sum_{k=1}^{m} \frac{a_k(t)}{\sum_{i=1}^{m} a_i(t)} \int_{h_k(t)}^{t} \sum_{i=1}^{m} a_i(s)ds < 1 + \frac{1}{e}.
\end{aligned} \tag{3.8}$$

By Lemma 3.1, (3.7) is exponentially stable. Due to the first equality in (3.5) $t \to \infty$ implies $s \to \infty$. Hence $\lim_{t\to\infty} x(t) = \lim_{s\to\infty} y(s) = 0$.

Equation (3.7) is exponentially stable, thus the fundamental function $Y(u,v)$ of (3.7) has an exponential estimation

$$|Y(u,v)| \le Ke^{-\lambda(u-v)}, \quad u \ge v \ge 0, \tag{3.9}$$

with $K > 0$, $\lambda > 0$. Since $X(t,s) = Y(\int_0^t \sum_{k=1}^m a_k(\tau)d\tau, \int_0^s \sum_{k=1}^m a_k(\tau)d\tau)$, where $X(t,s)$ is the fundamental function of (1.1), then (3.9) yields

$$|X(t,s)| \le K\exp\left\{-\lambda \int_s^t \sum_{k=1}^m a_k(\tau)d\tau\right\}. \tag{3.10}$$

Hence $|X(t,s)| \le K$, $t \ge s \ge 0$, which together with $\lim_{t\to\infty} x(t) = 0$ yields that (1.1) is asymptotically stable.

Suppose now that (3.6) holds. Without loss of generality we can assume that for some $R > 0$, $\alpha > 0$ we have

$$\int_t^{t+R} \sum_{k=1}^m a_k(\tau)d\tau \ge \alpha > 0, \quad t \ge s \ge 0. \tag{3.11}$$

Hence

$$\exp\left\{-\lambda \int_s^t \sum_{k=1}^m a_k(\tau)d\tau\right\} \le \exp\left\{\lambda R \sup_{t\ge 0} \sum_{k=1}^m a_k(t)\right\} e^{-\lambda\alpha(t-s)/R}. \tag{3.12}$$

Thus, condition (3.6) implies the exponential estimate for $X(t,s)$.

The last statement of the theorem is evident. □

Remark 3.4. The substitution $s = p(t) := \int_0^t \sum_{k=1}^m a_k(\tau)d\tau$, $y(s) = x(t)$ was first used in [28].

Note that in [10, Lemma 2] this idea was extended to a more general equation

$$\dot{x}(t) + \int_{t_0}^t x(s)d_s r(t,s) = 0. \tag{3.13}$$

The ideas of [10] allow to generalize the results of the present paper to equations with a distributed delay.

Corollary 3.5. *Suppose* $a_k(t) \ge 0$, $\sum_{k=1}^m a_k(t) \equiv \alpha > 0$, *condition* (2.12) *holds and*

$$\limsup_{t\to\infty} \sum_{k=1}^m a_k(t)(t - h_k(t)) < 1 + \frac{1}{e}. \tag{3.14}$$

Then (1.1) *is exponentially stable.*

Corollary 3.6. *Suppose* $a_k(t) = \alpha_k p(t)$, $\alpha_k > 0$, $p(t) > 0$, $\int_0^\infty p(t)dt = \infty$ *and*

$$\limsup_{t\to\infty}\sum_{k=1}^{m}\alpha_k\int_{h_k(t)}^{t}p(s)ds < 1+\frac{1}{e}. \tag{3.15}$$

Then (1.1) *is asymptotically stable.*

If in addition there exists $R > 0$ *such that*

$$\liminf_{t\to\infty}\int_t^{t+R}p(\tau)d\tau > 0, \tag{3.16}$$

then the fundamental function of (1.1) *has an exponential estimation.*

If also (2.12) *holds then* (1.1) *is exponentially stable.*

Remark 3.7. Let us note that similar results for (3.13) were obtained in [10], see Corollary 3.4 and remark after it, Theorem 4 and Corollaries 4.1 and 4.2 in [10], where an analogue of condition (3.16) was applied. This allows to extend the results of the present paper to equations with a distributed delay.

Corollary 3.8. *Suppose* $a(t) \geq 0$, $b(t) \geq 0$, $a(t) + b(t) > 0$,

$$\begin{gathered}\int_0^\infty (a(t)+b(t))dt = \infty, \qquad \limsup_{t\to\infty}\int_{h(t)}^{t}(a(s)+b(s))ds < \infty,\\ \limsup_{t\to\infty}\frac{b(t)}{a(t)+b(t)}\int_{h(t)}^{t}(a(s)+b(s))ds < 1+\frac{1}{e}.\end{gathered} \tag{3.17}$$

Then the following equation is asymptotically stable

$$\dot{x}(t) + a(t)x(t) + b(t)x(h(t)) = 0. \tag{3.18}$$

If in addition there exists $R > 0$ *such that* $\liminf_{t\to\infty}\int_t^{t+R}(a(\tau)+b(\tau))d\tau > 0$ *then the fundamental function of* (3.18) *has an exponential estimation.*

If also $\limsup_{t\to\infty}(t - h(t)) < \infty$ *then* (3.18) *is exponentially stable.*

In the following theorem we will omit the condition $\sum_{k=1}^{m} a_k(t) > 0$ of Theorem 3.3.

Theorem 3.9. *Suppose* $a_k(t) \geq 0$, *condition* (3.4) *and the first inequality in* (3.5) *hold. Then* (1.1) *is asymptotically stable.*

If in addition (3.6) *holds then the fundamental function of* (1.1) *has an exponential estimation.*

If also (2.12) *holds then* (1.1) *is exponentially stable.*

Proof. For simplicity suppose that $m = 2$ and consider the equation

$$\dot{x}(t) + a(t)x(h(t)) + b(t)x\big(g(t)\big) = 0, \tag{3.19}$$

where $a(t) \geq 0$, $b(t) \geq 0$, $\int_0^\infty (a(s)+b(s))ds = \infty$ and there exist $t_0 \geq 0$, $\varepsilon > 0$ such that

$$\int_{\min\{h(t),g(t)\}}^{t} (a(s)+b(s))ds < 1 + \frac{1}{e} - \varepsilon, \quad t \geq t_0. \tag{3.20}$$

Let us find $t_1 \geq t_0$ such that $e^{-h(t)} < \varepsilon/4$, $e^{-g(t)} < \varepsilon/4$, $t \geq t_1$, such t_1 exists due to (a2). Then $\int_{\min\{h(t),g(t)\}}^{t} e^{-s}ds < \varepsilon/2$, $t \geq t_1$. Rewrite (3.19) in the form

$$\dot{x}(t) + \left(a(t)+e^{-t}\right)x(h(t)) + b(t)x\left(g(t)\right) - e^{-t}x(h(t)) = 0, \tag{3.21}$$

where $a(t)+b(t)+e^{-t} > 0$. After the substitution $s = \int_{t_1}^{t}(a(\tau)+b(\tau)+e^{-\tau})d\tau$, $y(s) = x(t)$, (3.21) has the form

$$\dot{y}(s) + \frac{a(t)+e^{-t}}{a(t)+b(t)+e^{-t}}y(l(s)) + \frac{b(t)}{a(t)+b(t)+e^{-t}}y\left(p(s)\right) - \frac{e^{-t}}{a(t)+b(t)+e^{-t}}y(l(s)) = 0, \tag{3.22}$$

where similar to the proof of Theorem 3.3

$$s - l(s) = \int_{h(t)}^{t}\left(a(\tau)+b(\tau)+e^{-\tau}\right)d\tau, \qquad s - p(s) = \int_{g(t)}^{t}\left(a(\tau)+b(\tau)+e^{-\tau}\right)d\tau. \tag{3.23}$$

First we will show that by Corollary 3.2 the equation

$$\dot{y}(s) + \frac{a(t)+e^{-t}}{a(t)+b(t)+e^{-t}}y(l(s)) + \frac{b(t)}{a(t)+b(t)+e^{-t}}y\left(p(s)\right) = 0 \tag{3.24}$$

is exponentially stable. Since $(a(t)+e^{-t})/(a(t)+b(t)+e^{-t}) + b(t)/(a(t)+b(t)+e^{-t}) = 1$, then (2.11) holds. Condition (3.20) implies (2.12). So we have to check only condition (3.4) where the sum under the integral is equal to 1. By (3.20), (3.23) we have

$$\begin{aligned}
\int_{\min\{l(s),p(s)\}}^{s} 1ds = s - \min\{l(s),p(s)\}, \qquad s - l(s) &= \int_{h(t)}^{t}\left(a(\tau)+b(\tau)+e^{-\tau}\right)d\tau \\
= \int_{h(t)}^{t}(a(\tau)+b(\tau))d\tau + \int_{h(t)}^{t} e^{-\tau}d\tau &< 1 + \frac{1}{e} - \varepsilon + \frac{\varepsilon}{2} = 1 + \frac{1}{e} - \frac{\varepsilon}{2}, \quad t \geq t_1.
\end{aligned} \tag{3.25}$$

The same calculations give $s - p(s) < 1 + (1/e) - \varepsilon/2$, thus condition (3.4) holds.

Hence (3.24) is exponentially stable.

We return now to (3.22), $t \geq t_1$. We have $ds = (a(t)+b(t)+e^{-t})dt$, then

$$\int_{t_1}^{\infty}\frac{e^{-t}}{a(t)+b(t)+e^{-t}}ds = \int_{t_1}^{\infty}\frac{e^{-t}}{a(t)+b(t)+e^{-t}}\left(a(t)+b(t)+e^{-t}\right)dt < \infty. \tag{3.26}$$

By Lemma 2.10, (3.22) is exponentially stable. Since $t \to \infty$ implies $s \to \infty$ then $\lim_{t\to\infty} x(t) = \lim_{s\to\infty} y(s) = 0$, which completes the proof of the first part of the theorem. The rest of the proof is similar to the proof of Theorem 3.3. □

Corollary 3.10. *Suppose $a(t) \geq 0$, $\int_0^\infty a(t)dt = \infty$ and*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} a(s)ds < 1 + \frac{1}{e}. \tag{3.27}$$

Then the equation

$$\dot{x}(t) + a(t)x(h(t)) = 0 \tag{3.28}$$

is asymptotically stable. If in addition condition (2.21) *holds then the fundamental function of* (3.28) *has an exponential estimation. If also* $\limsup_{t\to\infty}(t - h(t)) < \infty$ *then* (3.28) *is exponentially stable.*

Now consider (1.1), where only some of coefficients are nonnegative.

Theorem 3.11. *Suppose there exists a set of indices $I \subset \{1, \ldots, m\}$ such that $a_k(t) \geq 0$, $k \in I$,*

$$\int_0^\infty \sum_{k\in I} a_k(t)dt = \infty, \quad \limsup_{t\to\infty} \int_{h_k(t)}^{t} \sum_{i\in I} a_i(s)ds < \infty, \quad k = 1, \ldots, m, \tag{3.29}$$

$$\sum_{k\notin I} |a_k(t)| = 0, \quad t \in E, \qquad \limsup_{t\to\infty,\, t\notin E} \frac{\sum_{k\notin I}|a_k(t)|}{\sum_{k\in I} a_k(t)} < 1, \quad \text{where } E = \left\{t \geq 0, \sum_{k\in I} a_k(t) = 0\right\}. \tag{3.30}$$

If the fundamental function $X_0(t,s)$ of (2.16) *is eventually positive then all solutions of* (1.1) *tend to zero as $t \to \infty$.*

If in addition there exists $R > 0$ such that

$$\liminf_{t\to\infty} \int_t^{t+R} \sum_{k\in I} a_k(\tau)d\tau > 0 \tag{3.31}$$

then the fundamental function of (1.1) *has an exponential estimation.*

If condition (2.12) *also holds then* (1.1) *is exponentially stable.*

Proof. Without loss of generality we can assume $X_0(t,s) > 0$, $t \geq s \geq 0$. Rewrite (1.1) in the form

$$\dot{x}(t) + \sum_{k\in I} a_k(t)x(h_k(t)) + \sum_{k\notin I} a_k(t)x(h_k(t)) = 0. \tag{3.32}$$

Suppose first that $\sum_{k\in I} a_k(t) \neq 0$. After the substitution $s = p(t) := \int_0^t \sum_{k\in I} a_k(\tau)d\tau$, $y(s) = x(t)$ we have $x(h_k(t)) = y(l_k(s))$, $l_k(s) \leq s$, $l_k(s) = \int_0^{h_k(t)} \sum_{i\in I} a_i(\tau)d\tau$, $k = 1,\ldots,m$, and (1.1) can be rewritten in the form

$$\dot{y}(s) + \sum_{k=1}^{m} b_k(s)y(l_k(s)) = 0, \tag{3.33}$$

where $b_k(s) = a_k(t)/\sum_{i\in I} a_i(t)$. Denote by $Y_0(u,v)$ the fundamental function of the equation

$$\dot{y}(s) + \sum_{k\in I} b_k(s)y(l_k(s)) = 0. \tag{3.34}$$

We have

$$\begin{aligned} X_0(t,s) &= Y_0\left(\int_0^t \sum_{k\in I} a_k(\tau)d\tau, \int_0^s \sum_{k\in I} a_k(\tau)d\tau\right), \\ Y_0(u,v) &= X_0\left(p^{-1}(u), p^{-1}(v)\right) > 0, \quad u \geq v \geq 0. \end{aligned} \tag{3.35}$$

Let us check that other conditions of Lemma 2.9 hold for (3.33). Since $\sum_{k\in I} b_k(s) = 1$ then condition (2.15) is satisfied. In addition,

$$\limsup_{s\to\infty,\, p^{-1}(s)\notin E} \frac{\sum_{k\notin I}|b_k(s)|}{\sum_{k\in I} b_k(s)} = \limsup_{t\to\infty,\, t\notin E} \frac{\sum_{k\notin I}|a_k(t)|}{\sum_{k\in I} a_k(t)} < 1. \tag{3.36}$$

By Lemma 2.9, (3.33) is exponentially stable. Hence for any solution $x(t)$ of (1.1) we have $\lim_{t\to\infty} x(t) = \lim_{s\to\infty} y(s) = 0$. The end of the proof is similar to the proof of Theorem 3.9. In particular, to remove the condition $\sum_{k\in I} a_k(t) \neq 0$ we rewrite the equation by adding the term e^{-t} to one of $a_k(t)$, $k \in I$. □

Remark 3.12. Explicit positiveness conditions for the fundamental function were presented in Lemma 2.7.

Corollary 3.13. *Suppose*

$$\begin{gathered} a(t) \geq 0, \quad \int_0^\infty a(t)dt = \infty, \quad \limsup_{t\to\infty} \int_{g_k(t)}^t a(s)ds < \infty, \\ \sum_{k=1}^n |b_k(t)| = 0, \quad t \in E, \qquad \limsup_{t\to\infty,\, t\notin E} \frac{\sum_{k=1}^n |b_k(t)|}{a(t)} < 1, \end{gathered} \tag{3.37}$$

where $E = \{t \geq 0, a(t) = 0\}$. Then the equation

$$\dot{x}(t) + a(t)x(t) + \sum_{k=1}^n b_k(t)x(g_k(t)) = 0 \tag{3.38}$$

is asymptotically stable. If in addition (2.21) *holds then the fundamental function of* (3.38) *has an exponential estimation. If also* $\limsup_{t\to\infty}(t - g_k(t)) < \infty$ *then* (3.38) *is exponentially stable.*

Theorem 3.14. *Suppose* $\int_0^\infty \sum_{k=1}^m |a_k(s)|ds < \infty$. *Then all solutions of* (1.1) *are bounded and* (1.1) *is not asymptotically stable.*

Proof. For the fundamental function of (1.1) we have the following estimation

$$|X(t,s)| \le \exp\left\{\int_s^t \sum_{k=1}^m |a_k(\tau)|d\tau\right\}. \tag{3.39}$$

Then by solution representation formula (2.5) for any solution $x(t)$ of (1.1) we have

$$\begin{aligned}|x(t)| &\le \exp\left\{\int_{t_0}^t \sum_{k=1}^m |a_k(s)|ds\right\}|x(t_0)| + \int_{t_0}^t \exp\left\{\int_s^t \sum_{k=1}^m |a_k(\tau)|d\tau\right\}\sum_{k=1}^m |a_k(s)|\big|\varphi(h_k(s))\big|ds \\ &\le \exp\left\{\int_{t_0}^\infty \sum_{k=1}^m |a_k(s)|ds\right\}\left(|x(t_0)| + \int_{t_0}^\infty \sum_{k=1}^m |a_k(s)|ds\|\varphi\|\right),\end{aligned} \tag{3.40}$$

where $\|\varphi\| = \max_{t<0}|\varphi(t)|$. Then $x(t)$ is a bounded function.

Moreover, $|X(t,s)| \le A := \exp\{\int_0^\infty \sum_{k=1}^m |a_k(s)|ds\}$, $t \ge s \ge 0$. Let us choose $t_0 \ge 0$ such that $\int_{t_0}^\infty \sum_{k=1}^m |a_k(s)|ds < 1/(2A)$, then $X_t'(t,t_0) + \sum_{k=1}^m a_k(t)X(h_k(t),t_0) = 0$, $X(t_0,t_0) = 1$ implies $X(t,t_0) \ge 1 - \int_{t_0}^\infty \sum_{k=1}^m |a_k(s)|A ds > 1 - A(1/(2A)) = 1/2$, thus $X(t,t_0)$ does not tend to zero, so (1.1) is not asymptotically stable. □

Theorems 3.11 and 3.14 imply the following results.

Corollary 3.15. *Suppose* $a_k(t) \ge 0$, *there exists a set of indices* $I \subset \{1,\dots,m\}$ *such that condition* (3.30) *and the second condition in* (3.29) *hold. Then all solutions of* (1.1) *are bounded.*

Proof. If $\int_0^\infty \sum_{k\in I}|a_k(t)|dt = \infty$, then all solutions of (1.1) are bounded by Theorem 3.11. Let $\int_0^\infty \sum_{k\in I}|a_k(t)|dt < \infty$. By (3.30) we have $\int_0^\infty \sum_{k\notin I}|a_k(t)|dt \le \int_0^\infty \sum_{k\in I}|a_k(t)|dt < \infty$. Then $\int_0^\infty \sum_{k=1}^m |a_k(t)|dt < \infty$. By Theorem 3.14 all solutions of (1.1) are bounded. □

Theorem 3.16. *Suppose* $a_k(t) \ge 0$. *If* (1.1) *is asymptotically stable, then the ordinary differential equation*

$$\dot{x}(t) + \left(\sum_{k=1}^m a_k(t)\right)x(t) = 0 \tag{3.41}$$

is also asymptotically stable. If in addition (2.12) *holds and* (1.1) *is exponentially stable, then* (3.41) *is also exponentially stable.*

Proof. The solution of (3.41), with the initial condition $x(t_0) = x_0$, can be presented as $x(t) = x_0 \exp\{-\int_{t_0}^{t} \sum_{k=1}^{m} a_k(s)ds\}$, so (3.41) is asymptotically stable, as far as

$$\int_0^{\infty} \sum_{k=1}^{m} a_k(s)ds = \infty \tag{3.42}$$

and is exponentially stable if (3.6) holds (see Lemma 2.11).

If (3.42) does not hold, then by Theorem 3.14, (1.1) is not asymptotically stable.

Further, let us demonstrate that exponential stability of (1.1) really implies (3.6).

Suppose for the fundamental function of (1.1) inequality (2.7) holds and condition (3.6) is not satisfied. Then there exists a sequence $\{t_n\}$, $t_n \to \infty$, such that

$$\int_{t_n}^{t_n+n} \sum_{k=1}^{m} a_k(\tau)d\tau < \frac{1}{n} < \frac{1}{e}, \quad n \geq 3. \tag{3.43}$$

By (2.12) there exists $n_0 > 3$ such that $t - h_k(t) \leq n_0$, $k = 1, \ldots, m$. Lemma 2.7 implies $X(t, s) > 0$, $t_n \leq s \leq t \leq t_n + n$, $n \geq n_0$. Similar to the proof of Theorem 3.14 and using the inequality $1 - x \geq e^{-x}$, $x > 0$, we obtain

$$X(t_n, t_n + n) \geq 1 - \int_{t_n}^{t_n+n} \sum_{k=1}^{m} a_k(\tau)d\tau \geq \exp\left\{-\int_{t_n}^{t_n+n} \sum_{k=1}^{m} a_k(\tau)d\tau\right\} > e^{-1/n}. \tag{3.44}$$

Inequality (2.7) implies $|X(t_n + n, t_n)| \leq Ke^{-\lambda n}$. Hence $Ke^{-\lambda n} \geq e^{-1/n}$, $n \geq n_0$, or $K > e^{\lambda n - 1/3}$ for any $n \geq n_0$. The contradiction proves the theorem. □

Theorems 3.11 and 3.16 imply the following statement.

Corollary 3.17. *Suppose $a_k(t) \geq 0$ and the fundamental function of* (1.1) *is positive. Then* (1.1) *is asymptotically stable if and only if the ordinary differential equation* (3.41) *is asymptotically stable.*

If in addition (2.12) *holds then* (1.1) *is exponentially stable if and only if* (3.41) *is exponentially stable.*

4. Discussion and Examples

In paper [2] we gave a review of known stability tests for the linear equation (1.1). In this part we will compare the new results obtained in this paper with known stability conditions.

First let us compare the results of the present paper with our papers [1–3]. In all these three papers we apply the same method based on Bohl-Perron-type theorems and comparison with known exponentially stable equations.

In [1–3] we considered exponential stability only. Here we also give explicit conditions for asymptotic stability. For this type of stability, we omit the requirement that the delays are bounded and the sum of the coefficients is separated from zero. We also present some new stability tests, based on the results obtained in [3].

Compare now the results of the paper with some other known results [5–7, 9, 10, 22]. First of all we replace the constant $3/2$ in most of these tests by the constant $1+1/e$. Evidently

$1 + 1/e = 1.3678 \cdots < 3/2$, so we have a worse constant, but it is an open problem to obtain $(3/2)$-stability results for equations with measurable coefficients and delays.

Consider now (3.28) with a single delay. This equation is well studied beginning with the classical stability result by Myshkis [29]. We present here several statements which cover most of known stability tests for this equation.

Statement 1 (see [5]). Suppose $a(t) \geq 0$, $h(t) \leq t$ are continuous functions and

$$\limsup_{t \to \infty} \int_{h(t)}^{t} a(s)ds \leq \frac{3}{2}. \tag{4.1}$$

Then all solutions of (3.28) are bounded.

If in addition

$$\liminf_{t \to \infty} \int_{h(t)}^{t} a(s)ds > 0, \tag{4.2}$$

and the strict inequality in (4.1) holds then (3.28) is exponentially stable.

Statement 2 (see [7]). Suppose $a(t) \geq 0$, $h(t) \leq t$ are continuous functions, the strict inequality (4.1) holds and $\int_0^\infty a(s)ds = \infty$. Then all solutions of (3.28) tend to zero as $t \to \infty$.

Statement 3 (see [9, 10]). Suppose $a(t) \geq 0$, $h(t) \leq t$ are measurable functions, $\int_0^\infty a(s)ds = \infty$, $A(t) = \int_0^t a(s)ds$ is a strictly monotone increasing function and

$$\limsup_{t \to \infty} \int_{h(t)}^{t} a(s)ds < \sup_{0<\omega<\pi/2} \left(\omega + \frac{1}{\Phi(\omega)} \right) \approx 1.45\ldots, \tag{4.3}$$

$\Phi(\omega) = \int_0^\infty u(t,\omega)dt$, where $u(t,\omega)$ is a solution of the initial value problem

$$\dot{y}(t) + y(t-\omega) = 0, \quad y(t) = 0, \quad t < 0, \quad y(0) = 1. \tag{4.4}$$

Then (3.28) is asymptotically stable.

Note that instead of the equation $\dot{y}(t) + y(t-\omega) = 0$ with a constant delay, the equation

$$\dot{y}(t) + y(t - \tau(t)) = 0 \tag{4.5}$$

can be used as the model equation. For example, the following results are valid.

Statement 4 (see [10]). Equation (4.5) is exponentially stable if $|\tau(t) - \omega| \leq k/\chi(\omega)$, where $k \in [0,\omega)$, $0 \leq \omega < \pi/2$ and $\chi(\omega) = \int_0^\infty |u(t,\omega)|dt$.

Obviously in this statement the delay can exceed 2.

Statement 5 (see [10]). Let $\tau(t) \leq k + \omega\{t/\omega\}$, where $k \in (0,1)$, $0 < \omega < 1$, $\{q\}$ is the fractional part of q. Then (4.5) is exponentially stable.

Here the delay $\tau(t)$ can be in the neighbourhood of ω which is close to 1.

Example 4.1. Consider the equation

$$\dot{x}(t) + \alpha(|\sin t| - \sin t)x(h(t)) = 0, \quad h(t) \le t, \tag{4.6}$$

where $h(t)$ is an arbitrary measurable function such that $t - h(t) \le \pi$ and $\alpha > 0$.

This equation has the form (3.28) where $a(t) = \alpha(|\sin t| - \sin t)$. Let us check that the conditions of Corollary 3.10 hold. It is evident that $\int_0^\infty a(s)ds = \infty$. We have

$$\limsup_{t\to\infty} \int_{h(t)}^{t} a(s)ds \le \limsup_{t\to\infty} \int_{t-\pi}^{t} a(s)ds \le -\alpha \int_{\pi}^{2\pi} 2\sin s\, ds = 4\alpha. \tag{4.7}$$

If $\alpha < 0.25(1 + 1/e)$, then condition (3.27) holds, hence all solutions of (4.6) tend to zero as $t \to \infty$.

Statements 1–3 fail for this equation. In Statements 1 and 2 the delay should be continuous. In Statement 3 function $A(t) = \int_0^t a(s)ds$ should be strictly increasing.

Consider now the general equation (1.1) with several delays. The following two statements are well known for this equation.

Statement 6 (see [6]). Suppose $a_k(t) \ge 0$, $h_k(t) \le t$ are continuous functions and

$$\limsup_{t\to\infty} a_k(t) \limsup_{t\to\infty}(t - h_k(t)) \le 1. \tag{4.8}$$

Then all solutions of (1.1) are bounded and 1 in the right-hand side of (4.8) is the best possible constant.

If $\sum_{k=1}^{m} a_k(t) > 0$ and the strict inequality in (4.8) is valid then all solutions of (1.1) tend to zero as $t \to \infty$.

If $a_k(t)$ are constants then in (4.8) the number 1 can be replaced by $3/2$.

Statement 7 (see [7]). Suppose $a_k(t) \ge 0$, $h_k(t) \le t$ are continuous, $h_1(t) \le h_2(t) \le \cdots \le h_m(t)$ and

$$\limsup_{t\to\infty} \int_{h_1(t)}^{t} \sum_{k=1}^{m} a_k(s)ds \le \frac{3}{2}. \tag{4.9}$$

Then any solution of (1.1) tends to a constant as $t \to \infty$.

If in addition $\int_0^\infty \sum_{k=1}^{m} a_k(s)ds = \infty$, then all solutions of (1.1) tend to zero as $t \to \infty$.

Example 4.2. Consider the equation

$$\dot{x}(t) + \frac{\alpha}{t}x\left(\frac{t}{2} - \sin t\right) + \frac{\beta}{t}x\left(\frac{t}{2}\right) = 0, \quad t \ge t_0 > 0, \tag{4.10}$$

where $\alpha > 0$, $\beta > 0$. Denote $p(t) = 1/t$, $h(t) = t/2 - \sin t$, $g(t) = t/2$.

We apply Corollary 3.6. Since $\lim_{t\to\infty}[\ln(t/2) - \ln(t/2 - \sin t)] = 0$, then

$$\limsup_{t\to\infty}\left(\alpha\int_{h(t)}^{t} p(s)ds + \beta\int_{g(t)}^{t} p(s)ds\right) \le (\alpha+\beta)\ln 2. \tag{4.11}$$

Hence if $\alpha + \beta < (1/\ln 2)(1 + 1/e)$ then (4.10) is asymptotically stable. Statement 4 fails for this equation since the delays are unbounded. Statement 5 fails for this equation since neither $h(t) \le g(t)$ nor $g(t) \le h(t)$ holds.

Stability results where the nondelay term dominates over the delayed terms are well known beginning with the book of Krasovskii [30]. The following result is cited from the monograph [22].

Statement 8 (see [22]). Suppose $a(t)$, $b_k(t)$, $t - h_k(t)$ are bounded continuous functions, there exist δ, k, $\delta > 0$, $0 < k < 1$, such that $a(t) \ge \delta$ and $\sum_{k=1}^{m} |b_k(t)| < k\delta$. Then the equation

$$\dot{x}(t) + a(t)x(t) + \sum_{k=1}^{m} b_k(t)x(h_k(t)) = 0 \tag{4.12}$$

is exponentially stable.

In Corollary 3.13 we obtained a similar result without the assumption that the parameters of the equation are continuous functions and the delays are bounded.

Example 4.3. Consider the equation

$$\dot{x}(t) + \frac{1}{t}x(t) + \frac{\alpha}{t}x\left(\frac{t}{2}\right) = 0, \quad t \ge t_0 > 0. \tag{4.13}$$

If $\alpha < 1$ then by Corollary 3.13 all solutions of (4.13) tend to zero. The delay is unbounded, thus Statement 8 fails for this equation.

In [31] the authors considered a delay autonomous equation with linear and nonlinear parts, where the differential equation with the linear part only has a positive fundamental function and the linear part dominates over the nonlinear one. They generalized the early result of Győri [32] and some results of [33].

In Theorem 3.11 we obtained a similar result for a linear nonautonomous equation without the assumption that coefficients and delays are continuous.

In all the results of the paper we assumed that all or several coefficients of equations considered here are nonnegative. Stability results for (3.28) with oscillating coefficient $a(t)$ were obtained in [34, 35].

We conclude this paper with some open problems.

(1) Is the constant $1 + 1/e$ sharp? Prove or disprove that in Corollary 3.10 the constant $1 + 1/e$ can be replaced by the constant $3/2$.
Note that all known proofs with the constant $3/2$ apply methods which are not applicable for equations with measurable parameters.

(2) Suppose (2.11), (2.12) hold and

$$\limsup_{t\to\infty}\sum_{k=1}^{m}\frac{|a_k(t)|}{\sum_{i=1}^{m}a_i(t)}\int_{h_k(t)}^{t}\sum_{i=1}^{m}a_i(s)ds<1+\frac{1}{e}. \tag{4.14}$$

Prove or disprove that (1.1) is exponentially stable.
The solution of this problem will improve Theorem 3.3.

(3) Suppose (1.1) is exponentially stable. Prove or disprove that the ordinary differential equation (3.41) is also exponentially (asymptotically) stable, without restrictions on the signs of coefficients $a_k(t) \geq 0$, as in Theorem 3.16. The solution of this problem would improve Theorem 3.16.

Acknowledgments

L. Berezansky is partially supported by Israeli Ministry of Absorption. E. Braverman is partially supported by the NSERC Research Grant.

References

[1] L. Berezansky and E. Braverman, "On stability of some linear and nonlinear delay differential equations," *Journal of Mathematical Analysis and Applications*, vol. 314, no. 2, pp. 391–411, 2006.
[2] L. Berezansky and E. Braverman, "On exponential stability of linear differential equations with several delays," *Journal of Mathematical Analysis and Applications*, vol. 324, no. 2, pp. 1336–1355, 2006.
[3] L. Berezansky and E. Braverman, "Explicit exponential stability conditions for linear differential equations with several delays," *Journal of Mathematical Analysis and Applications*, vol. 332, no. 1, pp. 246–264, 2007.
[4] T. Yoneyama and J. Sugie, "On the stability region of scalar delay-differential equations," *Journal of Mathematical Analysis and Applications*, vol. 134, no. 2, pp. 408–425, 1988.
[5] T. Yoneyama, "The 3/2 stability theorem for one-dimensional delay-differential equations with unbounded delay," *Journal of Mathematical Analysis and Applications*, vol. 165, no. 1, pp. 133–143, 1992.
[6] T. Krisztin, "On stability properties for one-dimensional functional-differential equations," *Funkcialaj Ekvacioj*, vol. 34, no. 2, pp. 241–256, 1991.
[7] J. W.-H. So, J. S. Yu, and M.-P. Chen, "Asymptotic stability for scalar delay differential equations," *Funkcialaj Ekvacioj*, vol. 39, no. 1, pp. 1–17, 1996.
[8] I. Győri, F. Hartung, and J. Turi, "Preservation of stability in delay equations under delay perturbations," *Journal of Mathematical Analysis and Applications*, vol. 220, no. 1, pp. 290–312, 1998.
[9] I. Győri and F. Hartung, "Stability in delay perturbed differential and difference equations," in *Topics in Functional Differential and Difference Equations (Lisbon, 1999)*, vol. 29 of *Fields Inst. Commun.*, pp. 181–194, American Mathematical Society, Providence, RI, USA, 2001.
[10] S. A. Gusarenko and A. I. Domoshnitskiĭ, "Asymptotic and oscillation properties of first-order linear scalar functional-differential equations," *Differential Equations*, vol. 25, no. 12, pp. 1480–1491, 1989.
[11] T. Wang, "Inequalities and stability for a linear scalar functional differential equation," *Journal of Mathematical Analysis and Applications*, vol. 298, no. 1, pp. 33–44, 2004.
[12] J. H. Shen and J. S. Yu, "Asymptotic behavior of solutions of neutral differential equations with positive and negative coefficients," *Journal of Mathematical Analysis and Applications*, vol. 195, no. 2, pp. 517–526, 1995.
[13] X. Wang and L. Liao, "Asymptotic behavior of solutions of neutral differential equations with positive and negative coefficients," *Journal of Mathematical Analysis and Applications*, vol. 279, no. 1, pp. 326–338, 2003.
[14] Z. Zhang and Z. Wang, "Asymptotic behavior of solutions of neutral differential equations with positive and negative coefficients," *Annals of Differential Equations*, vol. 17, no. 3, pp. 295–305, 2001.

[15] Z. Zhang and J. Yu, "Asymptotic behavior of solutions of neutral difference equations with positive and negative coefficients," *Mathematical Sciences Research Hot-Line*, vol. 2, no. 6, pp. 1–12, 1998.
[16] N. V. Azbelev and P. M. Simonov, *Stability of Differential Equations with Aftereffect*, vol. 20 of *Stability and Control: Theory, Methods and Applications*, Taylor & Francis, London, UK, 2003.
[17] A. Ivanov, E. Liz, and S. Trofimchuk, "Halanay inequality, Yorke 3/2 stability criterion, and differential equations with maxima," *The Tohoku Mathematical Journal*, vol. 54, no. 2, pp. 277–295, 2002.
[18] E. Liz, V. Tkachenko, and S. Trofimchuk, "A global stability criterion for scalar functional differential equations," *SIAM Journal on Mathematical Analysis*, vol. 35, no. 3, pp. 596–622, 2003.
[19] X. H. Tang, "Asymptotic behavior of delay differential equations with instantaneously terms," *Journal of Mathematical Analysis and Applications*, vol. 302, no. 2, pp. 342–359, 2005.
[20] V. V. Malygina, "Some criteria for stability of equations with retarded argument," *Differential Equations*, vol. 28, no. 10, pp. 1398–1405, 1992.
[21] V. V. Malygina, "Stability of solutions of some linear differential equations with aftereffect," *Russian Mathematics*, vol. 37, no. 5, pp. 63–75, 1993.
[22] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional-Differential Equations*, vol. 99 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1993.
[23] N. V. Azbelev, L. Berezansky, and L. F. Rahmatullina, "A linear functional-differential equation of evolution type," *Differential Equations*, vol. 13, no. 11, pp. 1331–1339, 1977.
[24] I. Győri and G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, NY, USA, 1991.
[25] L. Berezansky and E. Braverman, "On non-oscillation of a scalar delay differential equation," *Dynamic Systems and Applications*, vol. 6, no. 4, pp. 567–580, 1997.
[26] L. Berezansky and E. Braverman, "Preservation of exponential stability for linear non-autonomous functional differential systems," *Automatica*, vol. 46, no. 12, pp. 2077–2081, 2010.
[27] L. Berezansky and E. Braverman, "Preservation of the exponential stability under perturbations of linear delay impulsive differential equations," *Zeitschrift für Analysis und ihre Anwendungen*, vol. 14, no. 1, pp. 157–174, 1995.
[28] G. Ladas, Y. G. Sficas, and I. P. Stavroulakis, "Asymptotic behavior of solutions of retarded differential equations," *Proceedings of the American Mathematical Society*, vol. 88, no. 2, pp. 247–253, 1983.
[29] A. D. Myshkis, *Differential Equations with Retarded Argument*, Nauka, Moscow, Russia, 1951.
[30] N. Krasovskii, *Stability of Motion*, Nauka, Moscow, Russia, 1959, translation, Stanford University Press, 1963.
[31] E. Liz and M. Pituk, "Exponential stability in a scalar functional differential equation," *Journal of Inequalities and Applications*, Article ID 37195, 10 pages, 2006.
[32] I. Győri, "Interaction between oscillations and global asymptotic stability in delay differential equations," *Differential and Integral Equations*, vol. 3, no. 1, pp. 181–200, 1990.
[33] T. Faria and W. Huang, "Special solutions for linear functional differential equations and asymptotic behaviour," *Differential and Integral Equations*, vol. 18, no. 3, pp. 337–360, 2005.
[34] I. Győri and M. Pituk, "Stability criteria for linear delay differential equations," *Differential and Integral Equations*, vol. 10, no. 5, pp. 841–852, 1997.
[35] L. Berezansky and E. Braverman, "On exponential stability of a linear delay differential equation with an oscillating coefficient," *Applied Mathematics Letters*, vol. 22, no. 12, pp. 1833–1837, 2009.