Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 230851, 21 pages
doi:10.1155/2011/230851

Research Article

Bistable Wave Fronts in Integrodifference Equations

Shuxia Pan[1] and Ping-An Zhang[2]

[1] *Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou, Gansu 730050, China*
[2] *Department of Applied Mathematics, Xidian University, Xi'an, Shaanxi 710071, China*

Correspondence should be addressed to Shuxia Pan, shxpan@yeah.net

Received 14 April 2011; Accepted 28 June 2011

Academic Editor: Elena Braverman

This paper is concerned with the bistable wave fronts of integrodifference equations. The existence, uniqueness, and asymptotic stability of bistable wave fronts for such an equation are proved by the squeezing technique based on comparison principle.

1. Introduction

In 1982, Weinberger [1] proposed a famous discrete-time recursion to model a class of biological processes, to which a number of evolutionary systems can be reduced. A typical recursion takes the following form

$$v_{n+1}(x) = Q[v_n](x), \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where $x \in \mathscr{H} \subset \mathbb{R}^m$, $v \in \mathbb{R}^k$, and Q is an $\mathbb{R}^k$-valued mapping, herein $\mathscr{H}$ admits proper geometric structure such that it is a habitat (e.g., the integer lattice in $\mathbb{R}^m$; see [1]). Such a recursion can be defined for both discrete and continuous temporal-spatial variables, for example, the reaction-diffusion equations, lattice dynamical systems, and integrodifference equations [1–3]. And these models can be dealt with by the same theory scheme established for the abstract discrete-time recursions; see, for example, Liang and Zhao [2], Weinberger et al. [3]. In particular, Weinberger [1] derived a concrete population model as follows:

$$v_{n+1}(\mathbf{x}) = Q[v_n](\mathbf{x}) = \int_{\mathbb{R}^2} m(\mathbf{x}-\mathbf{y})g(v_n(\mathbf{y}))d\mathbf{y}, \quad n = 0, 1, 2, \ldots, \tag{1.2}$$

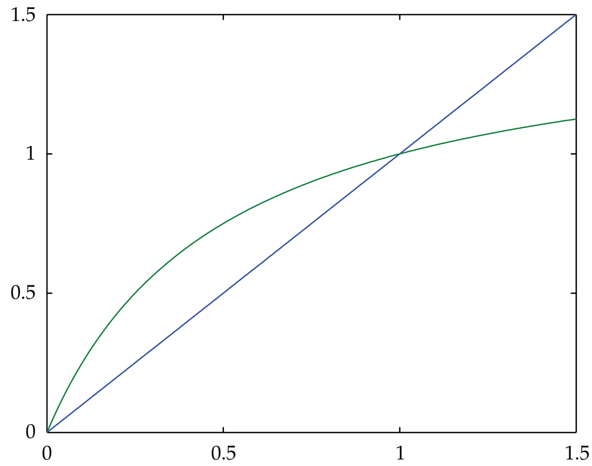

Figure 1: $g(u) = 3u/(1+2u)$.

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, $v_n \in \mathbb{R}$ denotes the gene fraction in the newly born individuals of the n-th generation, m is the probability function describing the migration of the individuals, and $g(v_n)$ means the gene fraction before the migration.

In the past decades, the traveling wave solutions and the spreading speeds of discrete-time recursions have been widely studied if they are monostable (analogous to reaction-diffusion equations), and we refer to [1–13]. As mentioned in Weinberger [1, Theorem 6.6 and Remark 11.4], maybe the traveling wave solutions of *bistable* recursions (analogous to reaction-diffusion equations) are totally different from those of the monostable equations, and we may also compare the spatial propagation modes in Kot et al. [14] and Wang et al. [15] to understand the differences between the monostable and the bistable. When (1.2) is concerned, an important $g(u)$ is the so-called Hill function

$$g(u) = \frac{\lambda u^p}{1 + (\lambda - 1)u^p}, \quad \lambda > 1, \ p > 0, \ u \geq 0. \tag{1.3}$$

If $p = 1$ and $\lambda > 1$, then it is the famous Beverton-Holt stock recruitment curve, and it is monostable (see Kot [16]); if $p = 2$ and $\lambda > 2$, then it is bistable (see Wang et al. [15]). More precisely, $g(u) = u$ has three equilibria $0, 1/(\lambda-1), 1$, and $0, 1$ are locally stable, while $1/(\lambda-1)$ is unstable in the sense of the corresponding difference equation. In population dynamics, the case $p = 2$ with $\lambda > 2$ describes the famous Allen effect (see [15]). For the reader's convenience in understanding the difference, we may see the following Figures 1 and 2 for $p = 1, 2$ with $\lambda = 3$, respectively.

For other special examples of the abstract discrete-time recursions, there are also some significant differences between the monostable system and the bistable one from the viewpoint of the traveling wave solutions such as reaction-diffusion equations [17–21] and lattice differential equations [22–25].

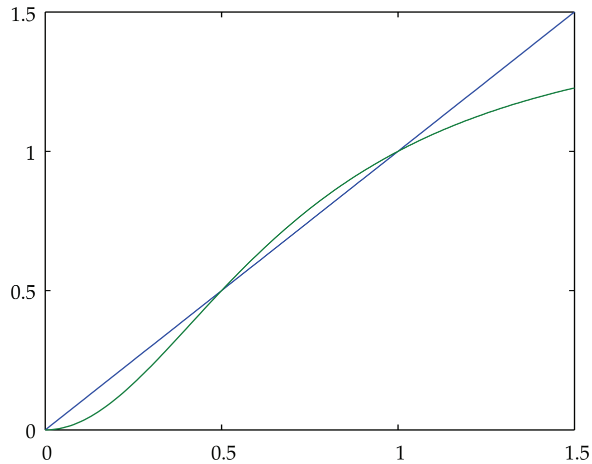

Figure 2: $g(u) = 3u^2/(1+2u^2)$.

If the spatial variable is one-dimensional, Lui [26, 27] considered the bistable wave fronts of the following integrodifference equation:

$$v_{n+1}(x) = \int_{\mathbb{R}} k(x-y)g(v_n(y))dy, \quad x \in \mathbb{R}, \tag{1.4}$$

where $k : \mathbb{R} \to \mathbb{R}^+$ is a probability function. In particular, the author proved that for a large class of initial data, the solution of (1.4) will be trapped between two suitable translates of the wave, and the author conjectured that the bistable wave front of (1.4) is stable indeed, namely, these translates would convergence to the same limit. For the case that the wave speed is 0, some results on the stability of the traveling wave solutions were given by Lui [28]. Creegan and Lui [29] further considered the existence and uniqueness of bistable wave fronts of an integrodifference equation in $\mathbb{R}^n$. In this paper, we will prove the conjecture in Lui [26, 27] for the asymptotic stability of the bistable wave fronts in a position.

For the sake of simplicity, we first assume that k takes the form of the Gaussian in (1.4), namely; we will investigate the bistable wave fronts of the following integrodifference equation:

$$u_{n+1}(x) = Q[u_n](x) = \frac{1}{\sqrt{4\pi d}} \int_{\mathbb{R}} e^{-(x-y)^2/4d} g(u_n(y))dy, \quad n = 0, 1, 2, \ldots, \tag{1.5}$$

where $x, y \in \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, and $d > 0$ is a constant. We should note that there are some significant differences in investigating the traveling wave solutions of the bistable and monostable evolution equations; on the one hand, the process of constructing upper and lower solutions was completed by different techniques. On the other hand, it seems that the characteristic equation of the monostable equation plays a more important role than that of the bistable one, and we refer to [20–25]. So, we do not hope that the techniques used for monostable integrodifference equation in [9] can be applied to a bistable equation, and we will apply the squeezing technique based on comparison principle and upper and lower solutions, which was earlier used in Chen [22] and Fife and McLeod [30] and was further developed by Ma and Wu [31], Ma and Zou [24], Smith and Zhao [18], and Wang et al. [20]. In particular, our discussion is independent of the characteristic equation, so we also relax some requirements on g which were used in Lui [26, 27] (see Remark 2.1). Moreover, Yagisita [32] established an abstract scheme to establish the existence of bistable wave fronts of monotone semiflows, and from Yagisita [32], Corollary 5 can be applied to our model if we can verify [32, Hypotheses 2-3]. In this paper, we could construct proper functions satisfying [32], Hypotheses 2-3 for (1.5), and so, their assumptions are reasonable and achievable, at least for (1.5).

Comparing our results with those in [1, 2, 9], a bistable integrodifference equation is also significantly different from a monostable one in the sense of the traveling wave solutions, such as the wave speed of bistable wave fronts of a bistable equation is unique while a monostable equation has infinite many wave speeds such that the system has a monostable wave front with each speed, and the wave profile of bistable wave front of bistable equation is unique in the sense of phase shift, while there are many different wave profiles of monostable wave fronts with different wave speeds of monostable equation.

The remainder of this paper is organized as follows. In Section 2, we will do some preliminaries for later sections. The corresponding initial value problem will be investigated in Section 3. In Section 4, the stability and uniqueness of the bistable wave fronts will be established. In Section 5, we show the existence of traveling wave solutions for (1.5). This paper ends up with the discussion on the bistable wave fronts of integrodifference equations.

2. Preliminaries

Throughout the paper, we will use the standard order and the interval notations in $\mathbb{R}$. We also denote $C(\mathbb{R},\mathbb{R})$ as follows:

$$C(\mathbb{R},\mathbb{R}) = \{u \mid u : \mathbb{R} \longrightarrow \mathbb{R} \text{ is uniformly continuous and bounded}\}, \tag{2.1}$$

which is a Banach space equipped with the super norm $|\cdot|$. Assume that $a < b$ with $a, b \in \mathbb{R}$, then $C_{[a,b]}$ will be interpreted as

$$C_{[a,b]} = \{u : u \in C(\mathbb{R},\mathbb{R}),\ a \le u(x) \le b, \quad \forall x \in \mathbb{R}\}. \tag{2.2}$$

For convenience, we list the assumptions on g, which will be imposed in what follows:

(g1) for $u \in [0,1]$, $g(u) = u$ exactly has three distinct zeros $0, 1, a \in (0,1)$,

(g2) if $u \in [0,1]$, then $g(u) \in C^2$; if $u \in (0,1)$, then $g'(u) > 0$,

(g3) there exist $\rho > 0, \varrho > 0$ such that $g'(u) < 1-\varrho$ for $u \in [0,\rho]\cup[1-\rho,1]$ and $g'(u) > 1+\varrho$ for $u \in [a-\rho, a+\rho]$.

Let $g(u) = \lambda u^2/(1+(\lambda-1)u^2)$ with $\lambda > 2$ (see Wang et al. [15]) or $g(u) = (su^2+u)/(1+su^2+\sigma(1-u)^2)$ with $0 < \sigma \le s$ (see Weinberger [1]), then (g1)–(g3) hold.

Remark 2.1. In Lui [26, 27], $g'(a) = \max_{x\in[0,1]} g'(x)$ was required. Moreover, Lui [27] assumed that $g'(0) > 0$, since the characteristic equation was involved. It is clear that $g(u) = \lambda u^2/(1+(\lambda-1)u^2)$ does not satisfy these assumptions if $\lambda \neq 4$. But Lui [26, 27] have less requirements on the kernel function.

Definition 2.2. A *traveling wave solution* of (1.5) is a special solution with the form $u_n(x) = \phi(x+cn)$, in which $-c \in \mathbb{R}$ is the wave speed that the wave profile $\phi \in C(\mathbb{R},\mathbb{R})$ spreads in $\mathbb{R}$. In particular, if $\phi(t)$ is monotone in $t \in \mathbb{R}$, then it is a *traveling wave front*.

By Definition 2.2, a traveling wave solution of (1.5) must satisfy

$$\phi(t+c) = \frac{1}{\sqrt{4\pi d}} \int_{\mathbb{R}} e^{-(t-y)^2/4d} g(\phi(y))\,dy, \quad t \in \mathbb{R}, \tag{2.3}$$

and we are interested in the following asymptotic boundary conditions:

$$\lim_{t\to-\infty} \phi(t) = 0, \qquad \lim_{t\to\infty} \phi(t) = 1. \tag{2.4}$$

Remark 2.3. Analogues to a parabolic system [19], we also call a traveling wave front satisfying (2.3)-(2.4) as a *bistable wave front* if (g1) and (g3) hold. Moreover, the smoothness effect of the Gaussian kernel (see [33]) implies that $\phi(t) \in C^1(\mathbb{R},\mathbb{R})$ if $\phi(t) \in C(\mathbb{R},\mathbb{R})$.

3. Initial Value Problem

In this part, we will show some results of the following Cauchy type problem:

$$\begin{gathered} u_{n+1}(x) = Q[u_n](x), \quad n = 0,1,2,3\ldots, \\ u_0(x) = u(x), \end{gathered} \tag{3.1}$$

in which $u(x) \in C_{[0,1]}$ and Q is given by (1.5). We also refer to Liang and Zhao [2], Weinberger [1]. In particular, the following result is obvious.

Theorem 3.1. *For all $n = 0, 1, \ldots, u_n(x) \in C_{[0,1]}$. Moreover, $u_n(x)$ is uniformly continuous for all $n \geq 1$ and $x \in \mathbb{R}$.*

Definition 3.2. Assume that $v_n(x) \in C_{[0,1]}$ for all $n = 0, 1, \ldots, N$. Then, $v_n(x)$ is called an *upper solution* (a *lower solution*) of (3.1) if

$$v_{n+1}(x) \geq (\leq) Q[v_n](x), \quad n = 0,1,\ldots,N-1, \ v_0(x) \geq (\leq) u(x). \tag{3.2}$$

By the upper and lower solutions, we can establish the following comparison principle.

Theorem 3.3. *Assume that $\overline{u}_n(x)$ and $\underline{u}_n(x), n = 0,1,\ldots,N$, are upper and lower solutions of* (3.1), *respectively. Then, the following items hold:*

(i) $\overline{u}_n(x) \geq \underline{u}_n(x), n = 0,1,\ldots,N,$

(ii) $\overline{u}_n(x) \geq u_n(x) \geq \underline{u}_n(x), n = 0,1,\ldots,N,$

(iii) $\overline{u}_{n+1}(x) - \underline{u}_{n+1}(x) \geq \dfrac{1}{\sqrt{4\pi d}} \int_{\mathbb{R}} e^{-(x-y)^2/4d} [g(\overline{u}_n(y)) - g(\underline{u}_n(y))]\,dy, n = 0,1,\ldots,N-1.$

Assume that $\phi(t)$ is a traveling wave front of (1.5) and satisfies (2.3)-(2.4). Then, the following result is clear by the item (iii) and the condition (g2).

Lemma 3.4. *$\phi(t)$ is strictly monotone such that $\phi'(t) > 0$ and $0 < \phi(t) < 1$ for all $t \in \mathbb{R}$.*

Lemma 3.5. *Assume that $\phi(x + cn)$ is a bistable wave front of* (1.5)*. Then, there exist positive numbers β, σ and $\delta_0 < min\{a, 1 - a\}$ such that for any $\delta \in (0, \delta_0]$ and $\xi^+, \xi^- \in \mathbb{R}$*

$$\begin{aligned} \overline{u}_n(x) &= \min\left\{\phi\left(x + cn + \xi^+ - \sigma\delta e^{-\beta n}\right) + \delta e^{-\beta n}, 1\right\}, \\ \underline{u}_n(x) &= \max\left\{\phi\left(x + cn + \xi^- + \sigma\delta e^{-\beta n}\right) - \delta e^{-\beta n}, 0\right\} \end{aligned} \tag{3.3}$$

are the upper and lower solutions of (3.1) *if $\overline{u}_0(x) \geq u(x) \geq \underline{u}_0(x), x \in \mathbb{R}$.*

Proof. By Definition 3.2, it suffices to prove that

$$\overline{u}_{n+1}(x) \geq Q[\overline{u}_n](x), \underline{u}_{n+1}(x) \leq Q[\underline{u}_n](x), \quad n = 0, 1, \ldots. \tag{3.4}$$

If $\overline{u}_{n+1}(x) = 1$ holds, then the result is clear by the assumptions (g1) and (g2). If $\overline{u}_{n+1}(x) < 1$, by Definition 2.2, we only need to prove that

$$\begin{aligned} &\phi\left(x + c(n+1) + \xi^+ - \sigma\delta e^{-\beta(n+1)}\right) + \delta e^{-\beta(n+1)} - \phi\left(x + c(n+1) + \xi^+ - \sigma\delta e^{-\beta n}\right) \\ &\quad \geq \frac{1}{\sqrt{4\pi d}} \int_{\mathbb{R}} e^{-(x-y)^2/4d}\left[g(\overline{u}_n(y)) - g\left(\phi\left(y + cn + \xi^+ - \sigma\delta e^{-\beta n}\right)\right)\right] dy. \end{aligned} \tag{3.5}$$

Define $\iota = \max_{u\in[0,1]} g'(u)$. Let $M > 0$ be large enough, then (g1)–(g3) imply that there exists $\rho' \in (0, 1)$ such that

$$\begin{aligned} 0 \leq g(\overline{u}_n(y)) - g\left(\phi\left(y + cn + \xi^+ - \sigma\delta e^{-\beta n}\right)\right) &\leq 1 - \rho'^{-\beta n}, \\ (1 - \rho') + \frac{\iota}{\sqrt{4\pi d}} \int_{-\infty}^{-M} e^{-y^2/4d} dy &< 1, \end{aligned} \tag{3.6}$$

for all $|y + cn + \xi^+ - \sigma\delta e^{-\beta n}| > M$ and $\delta \in (0, \delta_0]$. Thus, it is clear that

$$\begin{aligned} &\frac{1}{\sqrt{4\pi d}} \int_{\mathbb{R}} e^{-(x-y)^2/4d}\left[g(\overline{u}_n(y)) - g\left(\phi\left(y + cn + \xi^+ - \sigma\delta e^{-\beta n}\right)\right)\right] dy \\ &\quad \leq \left[(1 - \rho') + \frac{\iota}{\sqrt{4\pi d}} \int_{-\infty}^{-M} e^{-y^2/4d} dy\right] \delta e^{-\beta n}, \end{aligned} \tag{3.7}$$

for $|x + c(n+1) + \xi^+ - \sigma\delta e^{-\beta(n+1)}| > 2M$ or $|x + c(n+1) + \xi^+ - \sigma\delta e^{-\beta n}| \geq 2M$ with $M > 0$ large enough. Let $\beta > 0$ be small enough such that

$$e^{-\beta} > (1-\rho') + \frac{\iota}{\sqrt{4\pi d}} \int_{-\infty}^{-M} e^{-y^2/4d} dy. \tag{3.8}$$

Then, the result is clear if $|x+c(n+1)+\xi^+ - \sigma\delta e^{-\beta(n+1)}| > 2M$ or $|x+c(n+1)+\xi^+ - \sigma\delta e^{-\beta n}| \geq 2\mathrm{M}$ hold.

If $|x + c(n+1) + \xi^+ - \sigma\delta e^{-\beta(n+1)}| \leq 2M$ and $|x + c(n+1) + \xi^+ - \sigma\delta e^{-\beta n}| \leq 2M$, define $\upsilon = \min_{s\in[-2M,2M]} \phi'(s) > 0$, which is well defined by Lemma 3.4. Then, (3.5) holds if

$$\upsilon\sigma\delta e^{-\beta n}\left[1 - e^{-\beta}\right] + \delta e^{-\beta(n+1)} \geq \iota\delta e^{-\beta n}. \tag{3.9}$$

Let $\sigma > 0$ be large enough, then the above inequality is clear. Thus, $\overline{u}_n(x)$ is an upper solution of (1.5) if $\beta > 0, (1/\sigma) > 0$ are small enough.

Similarly, we can prove that $\underline{u}_n(x)$ is a lower solution of (1.5) if $\beta > 0$ is small enough while $\sigma > 0$ is large enough. The proof is complete. □

Remark 3.6. We should note that for any fixed $\delta_0 < \min\{a, 1-a\}$, σ and β are uniform for all $\delta \in (0, \delta_0]$, which is very important in the following discussion.

Let $\chi(x) \in C_{[0,1]}$ be a fixed function such that

$$\begin{aligned} &\chi(x) = 0 \quad \text{if } x \leq 0, \qquad \chi(x) = 1 \quad \text{if } x \geq 4, \\ &0 < \chi'(x) < 1, \qquad \left|\chi''(x)\right| \leq 1 \quad \text{if } x \in (0,4). \end{aligned} \tag{3.10}$$

Lemma 3.7. *For any $\delta \in (0, \delta_0]$ with $2\delta_0 < \min\{a, 1-a, \rho\}$ (ρ is defined by (g3)), there exist two positive numbers $\epsilon = \epsilon(\delta)$ and $C = C(\delta)$ such that for every $\xi^+, \xi^- \in \mathbb{R}$, define the continuous functions*

$$\begin{aligned} u_n^+(x) &= \min\left\{(1+\delta) - \left[1 - (a-2\delta)e^{-\epsilon n}\right]\chi(-\epsilon(x - \xi^+ + Cn)), 1\right\}, \\ u_n^-(x) &= \max\left\{-\delta + \left[1 - (1-a-2\delta)e^{-\epsilon n}\right]\chi\left(\epsilon\left(x - \xi^- + Cn\right)\right), 0\right\}. \end{aligned} \tag{3.11}$$

Then, $u_n^+(x)$ and $u_n^-(x)$ are the upper and lower solutions of (3.1) *if $u_0^+(x) \geq u(x) \geq u_0^-(x)$.*

Proof. By Definition 3.2, it suffices to prove that

$$u_{n+1}^+(x) \geq Q[u_n^+](x), \qquad u_{n+1}^-(x) \leq Q\left[u_n^-\right](x), \quad n = 0, 1, \ldots. \tag{3.12}$$

If $\chi(-\epsilon(x - \xi^+ + Cn)) \in [0,1)$, then $-\epsilon(x - \xi^+ + Cn) \in (-\infty, 4)$. Let $C > 0$ be large enough such that $C\epsilon \geq 4$, then it is clear that $\chi(-\epsilon(x - \xi^+ + C(n+1))) = 0$ such that $1 = u_{n+1}^+(x) \geq Q[u_n^+](x)$.

If $\chi(-\epsilon(x-\xi^+ + Cn)) = 1$, then $u_n^+(x) = \delta + (a-2\delta)e^{-\epsilon n} \leq a - \delta$. For any given $\delta > 0$, there exists $\epsilon' > 0$ such that $g(u) \leq (1-\epsilon')u, u \in [0, a - (\delta/2)]$. Moreover, there exists $M > 0$ such that $(1-\epsilon')+(\iota/\sqrt{4\pi d})\int_M^\infty e^{-y^2/4d}dy < 1$. Similar to the discussion of (3.5), $u_{n+1}^+(x) \ge Q[u_n^+](x)$ can be verified if $\epsilon = \epsilon(\delta) > 0$ is small enough and $C = C(\delta) > 0$ is large enough.

In a similar way, we can prove that $u_{n+1}^-(x) \le Q[u_n^-](x), x \in \mathbb{R}, n \ge 0$. The proof is complete. □

4. Asymptotic Stability of Traveling Wave Fronts

Lemma 3.5 implies a rough result on the long-time behavior between a traveling wave front and the solution of the initial value problem (3.1) if the initial value satisfies proper assumptions, which indicates that the solution can be controlled by two different phase shifts of the same traveling wave front. In this section, we will prove these phase shifts will convergence to the same limit, and this will affirm the stability of a bistable wave front. We first show the following result, which means that the phase shifts in Lemma 3.5 will be smaller if n is larger.

Lemma 4.1. *Assume that there exist* $n^* \ge 0, \xi \in \mathbb{R}, \delta \in (0, \delta_0/2]$ *and* $h > 0$ *such that*

$$\max\{\phi(x+cn^*+\xi)-\delta,0\} \le u_{n^*}(x) \le \min\{\phi(x+cn^*+\xi+h)+\delta,1\}, \quad x \in \mathbb{R}. \tag{4.1}$$

Then, for every $n \ge n^*+1$, *there exist* $\hat{\xi}(n), \hat{\delta}(n), \hat{h}(n)$ *satisfying*

$$\begin{aligned} &\hat{\xi}(n) \in [\xi-\sigma_1\delta, \xi+h+\sigma_1\delta], \\ &\hat{\delta}(n) \le e^{-\beta(n-n^*-1)}[\delta+\varepsilon^*\min\{h,1\}], \\ &\hat{h}(n) \le [h-\sigma_1\varepsilon^*\min\{h,1\}]+2\sigma_1\delta, \end{aligned} \tag{4.2}$$

such that (4.1) *remains true if we replace* (n^*, ξ, δ, h) *by* $(n, \hat{\xi}(n), \hat{\delta}(n), \hat{h}(n))$, *herein* $\beta > 0, \sigma = \sigma_1$ *such that Lemma 3.5 holds.*

Proof. Without loss of generality, we assume that $\xi = n^* = 0$, and denote $u_0(x) = u(x)$. By Theorem 3.3, we may obtain

$$\begin{aligned} &\max\left\{\phi\left(x+cn-\sigma_1\delta\left(1-e^{-\beta n}\right)\right)-\delta e^{-\beta n},0\right\} \le u_n(x) \\ &\quad \le \min\left\{\phi\left(x+cn+\sigma_1\delta\left(1-e^{-\beta n}\right)+h\right)+\delta e^{-\beta n},1\right\}, \quad x \in \mathbb{R},\ n \ge 0. \end{aligned} \tag{4.3}$$

Moreover, it is clear that there exist $\vartheta \in (0,1/2)$ and an interval $I \subset \mathbb{R}$ such that $u(y), \phi(y) \in [\vartheta, 1-\vartheta]$ for all $y \in I$ (since $\limsup_{x\to-\infty} u(x) \le \delta_0/2$, $\limsup_{x\to\infty} u(x) \ge 1-(\delta_0/2)$ and (2.4) hold). Without loss of generality, we further assume that $I = [0,1]$. Define constants

$$\overline{h} = \min\{h,1\}, \qquad \varepsilon_1 = \frac{1}{2}\min_{\xi\in[0,2]}\phi'(\xi) > 0. \tag{4.4}$$

Then,

$$\int_0^1 \left[\phi\left(y+\overline{h}\right)-\phi(y)\right]dy=\int_0^1 \left[\phi\left(y+\overline{h}\right)-u(y)+u(y)-\phi(y)\right]dy \geq 2\varepsilon_1\overline{h}, \tag{4.5}$$

which implies that at least one of the following is true:

$$\text{(i)} \int_0^1 [u(y)-\phi(y)]dy \geq \varepsilon_1\overline{h}; \qquad \text{(ii)} \int_0^1 \left[\phi\left(y+\overline{h}\right)-u(y)\right]dy \geq \varepsilon_1\overline{h}. \tag{4.6}$$

Let $M>0$ be large enough such that

$$\phi'(t)<\frac{1}{2\sigma_1}, \quad |t|>M. \tag{4.7}$$

If the case (i) holds, then $|x|\leq M+1+|c|$ with $M>0$ large enough indicates that

$$\begin{aligned} &u_1(x)-\max\left\{\phi(x+\xi_1)-\delta e^{-\beta},0\right\} \\ &\quad\geq \frac{\eta}{\sqrt{4\pi \mathrm{d}}}e^{(M+2+|c|)^2/-4d}\int_0^1 [u(y)-\max\{\phi(y)-\delta,0\}]dy \\ &\quad> \frac{\eta}{\sqrt{4\pi d}}e^{(M+2+|c|)^2/-4d}\int_0^1 [u(y)-\phi(y)]dy \geq \frac{\eta\varepsilon_1\overline{h}}{\sqrt{4\pi d}}e^{(M+2+|c|)^2/-4d}, \end{aligned} \tag{4.8}$$

where $\xi_1=c-\sigma_1\delta(1-e^{-\beta})$ and $\eta=\min_{u\in[\vartheta,1-\vartheta]}g'(u)>0$. Set

$$\varepsilon^*=\min\left\{\frac{\delta_0}{2},\frac{1}{2\sigma_1},\min_{|x|\leq M+2+|c|}\frac{\eta\varepsilon_1}{2\sigma_1\phi'(x)\sqrt{4\pi d}}e^{(M+2+|c|)^2/-4d}\right\}. \tag{4.9}$$

Then, there exists $\theta\in[x+\xi_1,x+\xi_1+2\sigma_1\varepsilon^*\overline{h}]$ such that

$$\phi\left(x+\xi_1+2\sigma_1\varepsilon^*\overline{h}\right)-\phi(x+\xi_1)=2\phi'(\theta)\sigma_1\varepsilon^*\overline{h}\leq\frac{\eta\varepsilon_1\overline{h}}{\sqrt{4\pi d}}e^{(M+2+|c|)^2/-4d}, \tag{4.10}$$

for all $|x|\leq M+1+|c|$. Thus, (4.8) indicates that

$$u_1(x)\geq\max\left\{\phi\left(x+\xi_1+2\sigma_1\varepsilon^*\overline{h}\right)-\delta e^{-\beta},0\right\}, \quad |x|\leq M+1+|c|. \tag{4.11}$$

If the case (i) is false while the case (ii) is true, then we can also establish a similar result and the discussion is omitted.

If $|x|\geq M+1+|c|$, then the definition of M implies that

$$\phi(x+\xi_1)\geq\phi\left(x+\xi_1+2\sigma_1\varepsilon^*\overline{h}\right)-\varepsilon^*\overline{h}. \tag{4.12}$$

Combining this with (3.10), it is clear that

$$u_1(x) \ge \max\left\{\phi\left(x+\xi_1+2\sigma_1\varepsilon^*\overline{h}\right)-\left[\delta e^{-\beta}+\varepsilon^*\overline{h}\right],0\right\},\quad x\in\mathbb{R}. \tag{4.13}$$

Since $\delta e^{-\beta}+\varepsilon^*\overline{h}\le\delta_0$ holds, then Theorem 3.3 and Lemma 3.4 indicate that

$$\begin{aligned} u_{n+1}(x) & \\ &\ge \max\left\{\phi\left(x+cn+\xi_1+2\sigma_1\varepsilon^*\overline{h}-\sigma_1\left(\delta e^{-\beta}+\varepsilon^*\overline{h}\right)\left(1-e^{-\beta n}\right)\right)-\left(\delta e^{-\beta}+\varepsilon^*\overline{h}\right)e^{-\beta n},0\right\} \\ &\ge \max\left\{\phi\left(x+c+cn+\sigma_1\varepsilon^*\overline{h}-\sigma_1\delta\right)-\left(\delta+\varepsilon^*\overline{h}\right)e^{-\beta n},0\right\}. \end{aligned} \tag{4.14}$$

Therefore, the left side of (4.1) with $(n,\widehat{\xi}(n),\widehat{\delta}(n),\widehat{h}(n))$ is true.

In a similar way, we can verify the right side of (4.1) if we replace (n^*,ξ,δ,h) by $(n,\widehat{\xi}(n),\widehat{\delta}(n),\widehat{h}(n))$. The proof is complete. □

The following theorem is the main result of this section, which further implies the asymptotic stability of a bistable wave front of (1.5).

Theorem 4.2. *Assume that $\phi(x+cn)$ is a monotone solution of* (2.3)-(2.4) *and $u(x)\in C_{[0,1]}$ satisfies*

$$\liminf_{x\to\infty} u(x) > a,\qquad \limsup_{x\to-\infty} u(x) < a. \tag{4.15}$$

Let $u_n(x)$ be defined by (3.1)*. Then, there exist $\xi=\xi(u(x))\in\mathbb{R}, K=K(u(x))>0$ and $\kappa>0$ (κ is independent of the initial value) such that*

$$\sup_{x\in\mathbb{R}}\left|u_n(x)-\phi(x+cn+\xi)\right|\le Ke^{-\kappa n},\quad n\ge 0. \tag{4.16}$$

Proof. By what we have done (Lemmas 3.5 and 4.1), the key of proof is the precise estimate of phase shifts mentioned above.

For any given $\delta>0$, Lemmas 3.7 and 4.1 imply that

$$\max\left\{\phi\left(x+cN-\frac{H}{2}\right)-\delta,0\right\}\le u_N(x)\le\min\left\{\phi\left(x+cN+\frac{H}{2}\right)+\delta,1\right\},\quad x\in\mathbb{R}, \tag{4.17}$$

for $N>0$ and $H>0$ large enough. Define constants

$$\delta^* := \min\left\{\frac{\delta_0}{2},\frac{\varepsilon^*}{4}\right\},\qquad \kappa^* := \sigma_1\varepsilon^*-2\sigma_1\delta^*\ge\frac{\sigma_1\varepsilon^*}{2}>0. \tag{4.18}$$

We further choose $n^* \geq 2$ such that

$$e^{-\beta(n^*-1)}\left[1+\frac{\varepsilon^*}{\delta^*}\right] \leq 1-\kappa^*. \tag{4.19}$$

In (4.17), let $\delta = \delta^*$, and denote the corresponding constants H and N by h_0 and N_0.

Claim 1. There exists a natural number $N_1 > N_0$ such that (4.1) holds for $n = N_1, \delta = \delta^*, h < 1$ and some $\xi \in \mathbb{R}$.

In fact, letting $n = N_0, \xi = -h_0/2, h = h_0$, and $\delta = \delta^*$ in (4.1), then Lemma 4.1 implies that (4.1) holds with $n = N_0+n^*$, some $\xi \in [(-h_0/2)-\sigma_1\delta^*, h_0/2+\sigma_1\delta^*]$, $\delta = \delta^*$, and $h = h_0-\kappa^*$, since the definitions of n^* and κ^* imply that

$$e^{-\beta(n^*-1)}[\delta^*+\varepsilon^*] \leq \delta^*, \qquad h_0-\sigma_1\varepsilon^*+2\sigma_1\delta^* \leq h_0-\kappa^*. \tag{4.20}$$

We now repeat the above process N times (see (4.4), so we can repeat the process for any $h \geq 1$) such that

$$h_0-(N-1)\kappa^* \geq 1, \qquad h_0-N\kappa^* < 1. \tag{4.21}$$

Then, (4.1) holds for $n = N_0+Nn^*, \delta = \delta^*, h = h_0-N\kappa^* < 1$, and some $\xi = \xi_0 \in \mathbb{R}$. Therefore, the claim is true.

Claim 2. For every $k \geq 0$, (4.1) holds for some $\xi_k \in \mathbb{R}$ and

$$n = N_k := N_1+kn^*, \qquad \delta = \delta^k := (1-\kappa^*)^k\delta^*, \qquad h = h^k := (1-\kappa^*)^k. \tag{4.22}$$

In fact, the result with $k = 0$ is clear by what we have done. Assume that the claim holds for some $k \geq 0$, and we now prove the claim for $k+1$. By applying Lemma 4.1, (4.1) holds with (n^*, ξ, δ, h) replaced by $(N_{k+1}, \widehat{\xi}, \widehat{\delta}, \widehat{h})$, herein

$$\begin{aligned}
&\widehat{\xi} \in \left[\xi_k-\sigma_1\delta^k, \xi_k+\sigma_1\delta^k\right],\\
&\widehat{\delta} \leq e^{-\beta(n^*-1)}\left(\delta^k+\varepsilon^*h^k\right) = (1-\kappa^*)^k\delta^*e^{-\beta(n^*-1)}\left(1+\frac{\varepsilon^*}{\delta^*}\right) \leq (1-\kappa^*)^{k+1}\delta^*,\\
&\widehat{h} \leq h^k-\sigma_1\varepsilon^*h^k+2\sigma_1\delta^k = (1-\kappa^*)^k(1-\sigma_1\varepsilon^*+2\sigma_1\delta^*) = (1-\kappa^*)^{k+1},
\end{aligned} \tag{4.23}$$

by the definitions of δ^*, κ^* and n^*. Thus, (4.1) holds for $k+1$. The proof of Claim 2 is completed by the mathematical induction.

By what we have done, (4.1) holds if (n^*, ξ, δ, h) is replaced by $(N_k, \xi^k, \delta^k, h^k)$ for all $k = 0, 1, \ldots$. The comparison principle further indicates that (4.1) holds for all $n \geq N_k, \delta = \delta_n$, $h = h^n+2\sigma_1\delta^k, \xi = \xi^k-\sigma_1\delta^k$ with $k = 0, 1, \ldots$.

Define $\delta(n) = \delta^k$, $\xi(n) = \xi^k-\sigma_1\delta^k$, and $h(n) = h^k+2\sigma_1\delta^k$ for all $N_k \leq n < N_{k+1}$ and $k = 0, 1, \ldots$. Then,

$$\phi(x+cn+\xi(n))-\delta(n) \leq u_n(x) \leq \phi(x+cn+\xi(n)+h(n))+\delta(n), \tag{4.24}$$

for all $x \in \mathbb{R}, n \geq N_1$. The definitions of $\delta(n)$ and $h(n)$ also imply that

$$\begin{aligned} \delta(n) &= \delta^n = (1-\kappa^*)^n \delta^* \leq \delta^* e^{(((n-N_1)/n^*)-1)\ln(1-\kappa^*)}, \quad n \geq N_1, \\ h(n) &= h^n + 2\sigma_1 \delta^k \leq (1+2\sigma_1\delta^*) e^{(((n-N_1)/n^*)-1)\ln(1-\kappa^*)}, \quad n \geq N_1, \end{aligned} \tag{4.25}$$

in which k is the largest integer no bigger than $(n-N_1)/n^*$.

Furthermore, $n \geq n^* \geq N_1$ and $\xi(n) \in [\xi(n^*) - \sigma_1\delta(n^*), \xi(n^*) + h(n^*) + \sigma_1\delta(n^*)]$ imply that

$$|\xi(n) - \xi(n^*)| \leq h(n) + 2\sigma_1\delta(n^*), \tag{4.26}$$

which indicates that $\xi(n)$ is bounded and $\lim_{n\to\infty}\xi(n)$ exists (since $h(n) \to 0$, $\delta(n) \to 0$ if $n \to \infty$). Let $\kappa = -(\ln(1-\kappa^*))/n^*$ be fixed and $K > 0$ be large enough, then the result is clear. This completes the proof. □

By Lemma 3.5 and Theorem 4.2, the following result is evident, which indicates that the wave speed of the bistable wave fronts of (1.5) is unique and the wave profile is also unique in the sense of phase shift.

Theorem 4.3. *Assume that $\phi(x+cn)$ is a bistable wave front of* (1.5). *If $\phi_1(x+c_1 n)$ is another traveling wave solution of* (1.5) *satisfying* (2.3)-(2.4), *then $c = c_1$ holds and there exists a constant $h \in \mathbb{R}$ such that $\phi(\xi) = \phi_1(\xi + h)$, $\xi \in \mathbb{R}$.*

Due to our estimates in Lemma 3.5, the proof is similar to that of Smith and Zhao [18, Theorem 3.4], and we omit it here.

5. Existence of Traveling Wave Fronts

We first list the main result of this section as follows.

Theorem 5.1. *Assume that (g1)–(g3) hold. Then* (1.5) *admits a traveling wave front. Namely, there exist $c \in \mathbb{R}$ and $\phi \in C_{[0,1]}$ such that* (2.3)-(2.4) *hold.*

We will split the proof of Theorem 5.1 into several lemmas, of which the motivation is the stability of traveling wave front. Namely, a bistable wave front can be approached by the solution of (1.5) if its initial value satisfies proper assumptions. For this purpose, we first consider the initial value problem

$$w_{n+1}(x) = Q[w_n](x), \quad w_0(x) = \chi(x), \quad n = 0, 1, \ldots, \tag{5.1}$$

herein $\chi(x)$ is same to that in Section 3.

Lemma 5.2. *For any given $\alpha \in (0,1)$ and $n = 0,1,\ldots$, there exists $z(\alpha,n)$ such that $w_n(z(\alpha,n)) = \alpha$.*

Proof. The proof is essentially based on the implicit function theorem. For the case of $n = 0$, the result is clear by the strict monotonicity of $\chi(x),\ x \in (0,4)$. We claim that $w_n(x),\ n \geq 1$ is strictly monotone in $x \in \mathbb{R}$. For any $x > y$, it is clear that

$$w_1(x) - w_1(y) = \frac{1}{\sqrt{4\pi d}} \int_{\mathbb{R}} e^{-z^2/4d} [g(\chi(x+z)) - g(\chi(y+z))] dz > 0, \tag{5.2}$$

by the definition of χ and the assumption (g2). Then, $w_1(x)$ is strictly monotone in $x \in \mathbb{R}$. Assume that $w_n(x)$ is strictly monotone for some $n \geq 1$, then we can prove that $w_{n+1}(x)$ is also strictly monotone in $x \in \mathbb{R}$, and the proof is similar to the discussion of w_1. By mathematical induction, $w_n(x), n \geq 1$ is strictly monotone in $x \in \mathbb{R}$. Moreover, it is also clear that $\lim_{x\to-\infty} w_n(x) = 0$, $\lim_{x\to\infty} w_n(x) = 1$ for any fixed $n \geq 0$. Combining the implicit function theorem with the strict monotonicity of $w_n(x)$, then the lemma is obvious. The proof is complete. □

Lemma 5.3. *Let $H(x)$ be the Heaviside function. Then, the following results hold.*

(i) *Assume that v_n^1 and w_n^1 are defined by the following linear equations:*

$$\begin{aligned} v_{n+1}^1(x) &= \frac{g'(a)}{\sqrt{4\pi d}} \int_{\mathbb{R}} e^{-(x-y)^2/4d} v_n^1(y) dy, \quad n = 0,1,\ldots, \qquad v_0^1(x) = H(x), \\ w_{n+1}^1(x) &= \frac{g'(a)}{\sqrt{4\pi d}} \int_{\mathbb{R}} e^{-(x-y)^2/4d} w_n^1(y) dy, \quad n = 0,1,\ldots, \qquad w_0^1(x) = -1 + 2H(x). \end{aligned} \tag{5.3}$$

Then, there exist $x_1 \in \mathbb{R}, n_1 \geq 1$ such that

$$v_{n_1}^1(x_1) \geq 3, \qquad w_{n_1}^1(x_1) \leq -3. \tag{5.4}$$

(ii) *There exists $\delta_1 > 0$ small enough such that the solutions to*

$$\begin{aligned} v_{n+1}^2(x) &= Q\left[v_n^2\right](x), \quad n = 0,1,\ldots, \qquad v_0^2(x) = a + \delta_1 H(x), \\ w_{n+1}^2(x) &= Q\left[w_n^2\right](x), \quad n = 0,1,\ldots, \qquad w_0^2(x) = a + \delta_1 [H(x) - H(-x)] \end{aligned} \tag{5.5}$$

satisfy

$$v_{n_1}^2(x_1) \geq a + 2\delta_1, \qquad w_{n_1}^2(x_1) \leq a - 2\delta_1. \tag{5.6}$$

(iii) *There exists $h_1 > 0$ such that the solutions to*

$$\begin{aligned} v_{n+1}^3(x) &= Q\left[v_n^3\right](x), \qquad v_0^3(x) = a + \delta_1 H(x) - aH(-h_1 - x), \\ w_{n+1}^3(x) &= Q\left[w_n^3\right](x), \qquad w_0^3(x) = a + \delta_1 [H(x) - H(-x)] + (1 - a - \delta_1) H(x - h_1) \end{aligned} \tag{5.7}$$

satisfy

$$v_{n_1}^3(x_1) \geq a + \delta_1, \qquad w_{n_1}^3(x_1) \leq a - \delta_1. \tag{5.8}$$

(iv) *Let* $u_n(x)$ *be the solution of* (1.5) *with uniformly continuous and nondecreasing initial data* $u(x)$ *satisfying* $0 \leq u(x) \leq 1$ *and for some finite* $\xi_-(0)$ *and* $\xi_+(0)$,

$$u(\xi_-(0)) \leq a - \delta_1, \qquad u(\xi_+(0)) \geq a + \delta_1. \tag{5.9}$$

Then, for each $n \geq 1$, *there exist* $\xi_-(n)$, $\xi_+(n)$ *such that*

$$\begin{gathered} u_n(\xi_-(n)) = a - \delta_1, \qquad u_n(\xi_+(n)) = a + \delta_1, \\ \xi_+(n) - \xi_-(n) \leq \max\left\{\xi_+(0) - \xi_-(0) + 8\epsilon^{-1}(\delta_1) + 2C(\delta_1)n_1, 2h_1\right\}, \end{gathered} \tag{5.10}$$

where $\epsilon^{-1}(\delta_1), C(\delta_1)$ *are as in Lemma 3.7.*

Proof. For each fixed $n > 0$, it is clear that v_n^i, w_n^i, $i = 1, 2, 3$ are uniformly continuous, and the comparison principle remains true even if the initial value is a discontinuous function. More precisely, if $0 \leq u(y) \leq v(y) \leq 1$, $y \in \mathbb{R}$ such that $Q[u](x)$, $Q[v](x)$, $x \in \mathbb{R}$ are well defined, then $0 \leq Q[u](x) \leq Q[v](x) \leq 1$, $x \in \mathbb{R}$. We now prove the items (i)–(iv) one by one.

Proof of (i). For each fixed $n > 0$, we know that $v_n(x) \in C(\mathbb{R}, \mathbb{R})$ and

$$\lim_{x \to -\infty} v_n^1(x) = 0, \qquad \lim_{x \to \infty} v_n^1(x) = \left[g'(a)\right]^n. \tag{5.11}$$

Let $n_1 > 0$ such that $[g'(a)]^{n_1} \geq 9$ and $[g'(a)]^{n_1 - 1} < 9$. Then, there exists $x_1 \in \mathbb{R}$ such that $v_{n_1}^1(x_1) = 3$. Since $w_n^1 = -[g'(a)]^n + 2v_n^1$, then $w_{n_1}^1(x_1) \leq -3$. □

Proof of (ii). Let $K > 0$ be a large constant and $\delta_1 > 0$ small enough such that $K\delta_1 < 1$, $\delta_1 [g'(a)]^{2n_1 + 2} < 1$ and

$$0 \leq a + \delta_1 v_n^1(x) - K\delta_1^2 \left[g'(a)\right]^{2n}, \quad a + \delta_1 v_n^1(x) \leq 1, \quad \forall x \in \mathbb{R},\ 0 \leq n \leq n_1, \tag{5.12}$$

which is admissible, since $v_n^1(x)$ is bounded for all $x \in \mathbb{R}$, $0 \leq n \leq n_1$. Define

$$\underline{v}_n^2(x) = a + \delta_1 v_n^1(x) - K\delta_1^2 \left[g'(a)\right]^{2n}, \quad \overline{v}_n^2(x) = 1, \quad x \in \mathbb{R}. \tag{5.13}$$

We now prove that $\overline{v}_n^2$, $\underline{v}_n^2$ are a pair of upper and lower solutions if $x \in \mathbb{R}$, $0 \leq n \leq n_1$. The initial value conditions and $\overline{v}_{n+1}^2 \geq Q[\overline{v}_n^2]$ are clear, so we need to prove that

$$\underline{v}_{n+1}^2(x) \leq Q\left[\underline{v}_n^2\right](x), \quad 0 \leq n \leq n_1,\ x \in \mathbb{R}. \tag{5.14}$$

Namely, we will verify that

$$a+\delta_1 v_{n+1}^1(x)-K\delta_1^2[g'(a)]^{2n+2}\le \frac{1}{\sqrt{4\pi d}}\int_{\mathbb{R}} e^{-(x-y)^2/4d} g\left(a+\delta_1 v_n^1(y)-K\delta_1^2[g'(a)]^{2n}\right)dy. \tag{5.15}$$

By the Taylor's formula, there exists $s=s(g(u))\ge 0$ for $u\in[0,1]$ (see (g2)) such that

$$\begin{aligned} &g\left(a+\delta_1 v_n^1(y)-K\delta_1^2[g'(a)]^{2n}\right)\\ &\quad\ge g(a)+g'(a)\left\{\delta_1 v_n^1(y)-K\delta_1^2[g'(a)]^{2n}\right\}-s\left\{\delta_1 v_n^1(y)-K\delta_1^2[g'(a)]^{2n}\right\}^2. \end{aligned}\tag{5.16}$$

Thus, the definition of $v_n^1(x)$ implies that we only need to prove

$$-(g'(a)-1)K\delta_1^2[g'(a)]^{2n+1}\le \frac{-s}{\sqrt{4\pi d}}\int_{\mathbb{R}} e^{-(x-y)^2/4d}\left\{\delta_1 v_n^1(y)-K\delta_1^2[g'(a)]^{2n}\right\}^2 dy, \tag{5.17}$$

which is true if

$$\begin{aligned} (g'(a)-1)K\delta_1^2[g'(a)]^{2n+1}&\ge 2s\left[\delta_1[g'(a)]^n\right]^2,\\ (g'(a)-1)K\delta_1^2[g'(a)]^{2n+1}&\ge 2s\left[K\delta_1^2[g'(a)]^{2n}\right]^2. \end{aligned}\tag{5.18}$$

Note that $\delta_1>0$ is small enough and $K>0$ is large enough, then the result is clear. This completes the proof of $v_{n_1}^2(x_1)\ge a+2\delta_1$. □

In a similar way, we can prove that $w_{n_1}^2(x_1)\le a-2\delta_1$.

Proof of (iii). Note that $\lim_{N\to\infty}1/\sqrt{4\pi d}\int_{-N}^{N} e^{-y^2/4d}dy=1$ and $a-2\delta_1<a-\delta_1, a+2\delta_1>a+\delta_1$, then the result is clear by letting $h_1>0$ large enough. □

Proof of (iv). By the definition of $u(x)$, it is clear that $u_n(x)$ are strictly monotone in $x\in\mathbb{R}$ for every $n\ge 1$. Thus, the existence and uniqueness of $\xi_-(n)$, $\xi_+(n)$ are obvious. By Lemma 3.7, there exist $\epsilon(\delta_1)$ and $C(\delta_1)$ such that

$$\xi_+(n)-\xi_-(n)\le \xi_+(0)-\xi_-(0)+8\epsilon^{-1}(\delta_1)+2C(\delta_1)n. \tag{5.19}$$

In particular, (5.19) is true if $n=n_1$. We now prove that

$$\xi_+(n+n_1)-\xi_-(n+n_1)\le \max\{\xi_+(n_1)-\xi_-(n_1),2h_1\}\ \text{for any}\ n\ge 0. \tag{5.20}$$

By translation and symmetry, we can assume that for any given $n\ge 0$,

$$\xi_-(n)<0<\xi_+(n),\qquad \xi_+(n)\ge-\xi_-(n). \tag{5.21}$$

Set $h_+ = \max\{\xi_+(n_1), h_1\}$, then $u_n(x + h_+) \geq v_0^3(x)$ for all $x \in \mathbb{R}$. By the comparison principle, $u_{n+n_1}(x+h_+) \geq v_{n_1}^3(x)$. Thus, $u_{n+n_1}(x_1+h_+) \geq a+\delta_1$, which implies that $\xi_+(n+n_1) \leq x_1 + h_+$.

Set $h_- = \max\{\xi_+(n_1) - \xi_-(n_1), h_1\}$. Similar to the above discussion, we can verify that $\xi_-(n+n_1) \geq -h_- + \xi_+(n_1) + x_1$, which further implies that

$$\begin{aligned}\xi_+(n+n_1) - \xi_-(n+n_1) &\leq h_+ + h_- - \xi_-(n_1) \leq \max\{\xi_+(n_1) - \xi_-(n_1), 2h_1\}\\ &\leq \max\left\{\xi_+(0) - \xi_-(0) + 8\epsilon^{-1}(\delta_1) + 2C(\delta_1)n_1, 2h_1\right\}.\end{aligned} \tag{5.22}$$

The proof is complete. □

□

Lemma 5.4. *Assume that $z(\alpha, n)$ is defined by Lemma 5.2. Then, there exist a small positive constant δ_1 and a large positive constant $h_2 \geq 1$ such that*

$$z(a+\delta_1, n) - z(a-\delta_1, n) \leq h_2, \quad \forall n \geq 0. \tag{5.23}$$

Moreover, for any $\delta \in (0, \delta_1/2]$, there exists $m_1(\delta) > 0$ such that

$$z(1-\delta, n) - z(\delta, n) \leq m_1(\delta), \quad n \geq 0. \tag{5.24}$$

Proof. The former is clear by Lemma 5.3. We now prove the latter of the lemma. By Lemma 3.7, there exist $\epsilon = \epsilon(\delta), C = C(\delta)$ such that

$$\begin{aligned}&z(1-2\delta, n) \leq z(a+\delta, n_1) + 4\epsilon^{-1} + C(n-n_1) \text{ for any } n - n_1 > \epsilon^{-1}|\ln\delta|,\\ &z(2\delta, n) \geq z(a-\delta, n_1) - 4\epsilon^{-1} - C(n-n_1) \text{ for any } n - n_1 \geq \epsilon^{-1}|\ln\delta|.\end{aligned} \tag{5.25}$$

Thus, $n > \epsilon^{-1}|\ln\delta| =: \Delta$ implies that

$$z(1-2\delta, n) - z(2\delta, n) \leq z(a+\delta, n - \lfloor\Delta\rfloor + 1) - z(a-\delta, n - \lfloor\Delta\rfloor + 1) + 8\epsilon^{-1} + 2C(\lfloor\Delta\rfloor + 1), \tag{5.26}$$

in which $\lfloor x\rfloor$ denotes the maximal integer less than $x+1$. Note that

$$z(a+\delta_1, n - \lfloor\Delta\rfloor + 1) - z(a-\delta_1, n - \lfloor\Delta\rfloor + 1) \leq h_2, \quad \forall n - \lfloor\Delta\rfloor + 1 \geq 0. \tag{5.27}$$

Then, the result is clear, and the proof is complete. □

Remark 5.5. Lemma 5.4 gives the estimate of variations of $z(a,n)$, which will ensure that the solution of (5.1) converges to a nontrivial traveling wave solution in the following discussion.

Lemma 5.6. *For every $M > 0$, there exists a constant $\hat{\eta}(M) > 0$ such that*

$$\left.\frac{dv_{n+1}(t)}{dt}\right|_{t=x+z(a,n)} > \hat{\eta}(M) > 0, \quad \text{for any } n \geq 1,\ x \in [-M, M]. \tag{5.28}$$

Proof. By the smoothness of the Gaussian, the existence of $(dv_{n+1}(t))/dt,\ t \in \mathbb{R}$ is clear. Let $\eta(x) = (1/\sqrt{4\pi d})e^{-x^2/4d}$. By the definition of h_2, it is clear that

$$v_n(\xi + 1 + z(a,n)) - v_n(\xi + z(a,n)) \geq \frac{\delta_1}{h_2}, \tag{5.29}$$

for any $\xi \in [-h_2, h_2 - 1]$ and $n \geq 0$, which further implies that

$$\min_{x\in\left[-\widehat{M},\widehat{M}\right]} \left.\frac{dv_{n+1}(t)}{dt}\right|_{t=x+z(a,n)} \geq \min_{x\in\left[-\widehat{M}-h_2,\widehat{M}+h_2\right]} \frac{\eta\left(\widehat{M} + h_2\right)\delta_1}{h_2}, \tag{5.30}$$

for any $n \geq 0$. Since $|z(a,n+1) - z(a,n)| \leq h_2 + 8\epsilon^{-1}(\delta_1) + 2C(\delta_1)$, taking $\widehat{M} = M + h_2 + 8\epsilon^{-1}(\delta_1) + 2C(\delta_1)$ and $\hat{\eta}(M) = \eta(\widehat{M} + h_2)$, then the result is clear. The proof is complete. □

Lemma 5.7. *Let $M_3 = m_1(\delta_0) > 0,\ \sigma_2 > 0$ be large enough and $\beta > 0$ small enough. Then, for every $\delta \in (0, \delta_0]$ and $\xi \in \mathbb{R}$, the continuous functions*

$$\begin{aligned} W_n^+(x) &:= \min\left\{v_{n+1}\left(x + \xi + \sigma_2\delta\left(1 - e^{-\beta n}\right)\right) + \delta e^{-\beta n}, 1\right\}, \\ W_n^-(x) &:= \max\left\{v_{n+1}\left(x + \xi - \sigma_2\delta\left(1 - e^{-\beta n}\right)\right) - \delta e^{-\beta n}, 0\right\} \end{aligned} \tag{5.31}$$

are the upper and lower solutions of (1.5) *if $W_0^+(x) \geq v_0(x) \geq W_0^-(x)$.*

By the estimate on the derivative of v_n formulated by Lemma 5.6, namely, the strict monotonicity of v_n, the proof is similar to that of Lemma 3.5, so we omit it here. In particular, these parameters have a property similar to Remark 3.6.

Lemma 5.8. *There exists a nondecreasing function $\phi(\xi)$ such that*

$$v_n(\xi + z(a,n)) \longrightarrow \phi(\xi), \qquad n \longrightarrow \infty,\ \forall \xi \in \mathbb{R}, \tag{5.32}$$

and $\lim_{\xi\to-\infty}\phi(\xi) = 0,\ \lim_{\xi\to\infty}\phi(\xi) = 1.$

Proof. The existence of $\phi(\xi)$ is clear by the boundedness, uniform continuity, and monotonicity of the sequences, and $\phi(0) = a$ is also obvious. Note that $\phi(\xi)$ is monotone, then $\lim_{\xi\to\pm\infty}\phi(\xi) = \phi_\pm$ exist. Lemmas 5.4 and 5.6 further imply that $\phi_- < a$ and $\phi_+ > a$. Then, $\phi_- = 0,\ \phi_+ = 1$ by (g1). The proof is complete. □

Lemma 5.9. *ϕ is the profile of a traveling wave front.*

Proof. Let $\widetilde{\phi}_n(x)$ be the solution of (1.5) with the initial value $\phi(x)$, and we will prove that

$$v_{m+n}(x + z(a,m)) \longrightarrow \widetilde{\phi}_n(x), \quad m \longrightarrow \infty. \tag{5.33}$$

In fact, from the monotonicity of $v_n(x)$ and $\phi(\xi)$, for any $\widehat{\varepsilon} > 0$, there exists $J > 0$ such that for any $m > J$ (see Lemma 5.8),

$$v_m(x - \widehat{\varepsilon} + z(a,m)) - \widehat{\varepsilon} \le \phi(x) \le v_m(x + \widehat{\varepsilon} + z(a,m)) + \widehat{\varepsilon}. \tag{5.34}$$

We further have

$$\begin{aligned} &v_{m+n}\Big(x - \widehat{\varepsilon} + z(a,n) - \sigma_2\widehat{\varepsilon}\Big(1 - e^{-\beta n}\Big)\Big) - \widehat{\varepsilon}e^{-\beta n} \\ &\quad \le \widetilde{\phi}_n(x) \le v_{m+n}\Big(x + \widehat{\varepsilon} + z(a,n) + \sigma_2\widehat{\varepsilon}\Big(1 - e^{-\beta n}\Big)\Big) + \widehat{\varepsilon}e^{-\beta n}, \end{aligned} \tag{5.35}$$

for all $n \ge 0$ and $m > J$. Sending $m \to \infty$ first and then $\widehat{\varepsilon} \to 0$ (see Remark 3.6), we immediately get

$$\lim_{m \to \infty} v_{m+n}(x + z(a,m)) = \widetilde{\phi}_n(x), \quad x \in \mathbb{R}. \tag{5.36}$$

Let $m_0 > 0$ be large enough such that

$$v_1(x - m_0) - \delta_0 \le v_0(x) \le v_1(x + m_0) + \delta_0. \tag{5.37}$$

Then, Lemma 5.7 and comparison principle imply that

$$\widetilde{\phi}_1(x - m_0 - \sigma_2\delta_0) \le \phi(x) \le \widetilde{\phi}_1(x + m_0 + \sigma_2\delta_0). \tag{5.38}$$

Define constants

$$\xi_* = \sup\Big\{\xi : \widetilde{\phi}_1(x + \xi) \le \phi(x)\Big\}, \qquad \xi^* = \inf\Big\{\xi : \widetilde{\phi}_1(x + \xi) \ge \phi(x)\Big\}. \tag{5.39}$$

It is clear that $\xi^* \ge \xi_*$ holds and ξ_*, ξ^* are well defined. In particular, if $\xi_* = \xi^*$, then we complete the proof.

Due to a prior estimate formulated by Lemma 5.6 which is similar to that of Lemma 3.4 established for the bistable wave front, the proof of $\xi_* = \xi^*$ is similar to that of Theorem 4.2 in estimating the convergence of phase shift, and we omit it here. We also refer to Chen [22, Lemma 4.2] for some details. This implies that $\phi(t)$ satisfies

$$\phi(t + \xi^*) = \frac{1}{\sqrt{4\pi d}} \int_{\mathbb{R}} e^{-(t-y)^2/4d} g(\phi(y))\,dy, \quad t \in \mathbb{R}. \tag{5.40}$$

The proof is complete. □

Lemma 5.10. $\lim_{n\to\infty}[z(a,n) - z(a,n+1)]$ *exists. Let* $c = \lim_{n\to\infty}[z(a,n) - z(a,n+1)]$, *then* $\phi(x + cn)$ *is a bistable wave front of* (1.5).

Note that the convergence in Lemma 5.8 is also locally uniform by Ascoli-Arzela lemma, then the proof of Lemma 5.10 is clear by Lemma 5.9 and the Lebesgue's dominated convergence theorem, so we omit it here.

6. Discussion

In this paper, we consider the bistable wave fronts of (1.5), the existence, uniqueness and asymptotic stability of traveling wave fronts as well as the uniqueness of wave speed are established. Our results answer the conjecture of the stability of the bistable wave fronts in Lui [26], and we give more weaker assumptions on g than that in [26–28] for the Gaussian kernel function. Very likely, our methods and results can be extended to more general kernel functions.

In particular, for such a bistable wave front, it is proved that both the wave speed and the wave profile are unique. By Theorem 3.3 and Lemma 3.5, it is clear that the wave speed can characterize the invasion speed (or extinction speed, and this depends on the sign of the wave speed [15]) when a single species with Allen effect is involved in population dynamics. Especially, such a speed formulates the evolution of a single species that admits the Allen effect, and the initial population density on a half space is large (e.g., let the initial value of (3.1) be the Heaviside function), which is different from the spreading speed of monostable case (see [1–3, 5]) that describes the population invasion process from finite domain to the whole space. Moreover, the stability of the bistable wave fronts also implies that the wave profile can further formulate the invasion process. Thus, our results provide further theory illustration for part numerical results (e.g., Wang et al. [15]). In Weinberger [1], the author derived a population genetics model (1.2) with $g(u) = (su^2+u)/(1+su^2+\sigma(1-u)^2$. If $0 < \sigma \le s$ holds, then it also satisfies (g1)–(g3). Thus, we also give the theory illustration for such a population genetics model with gaussian kernel.

Note that Lemmas 5.8–5.10 indicate the estimates of the wave profile and the wave speed, then it is possible to study the wave speed and the wave profile numerically. Since there are several results on such a numerical simulation (see Wang et al. [15]), we omit it here.

Acknowledgment

This work is supported by the Development Program for Outstanding Young Teachers in Lanzhou University of Technology (1010ZCX019).

References

[1] H. F. Weinberger, "Long-time behavior of a class of biological models," *SIAM Journal on Mathematical Analysis*, vol. 13, no. 3, pp. 353–396, 1982.

[2] X. Liang and X.-Q. Zhao, "Asymptotic speeds of spread and traveling waves for monotone semiflows with applications," *Communications on Pure and Applied Mathematics*, vol. 60, no. 1, pp. 1–40, 2007.

[3] H. F. Weinberger, M. A. Lewis, and B. Li, "Analysis of linear determinacy for spread in cooperative models," *Journal of Mathematical Biology*, vol. 45, no. 3, pp. 183–218, 2002.

[4] O. Diekmann, “Thresholds and travelling waves for the geographical spread of infection,” *Journal of Mathematical Biology*, vol. 6, no. 2, pp. 109–130, 1978.

[5] S.-B. Hsu and X.-Q. Zhao, “Spreading speeds and traveling waves for nonmonotone integrodifference equations,” *SIAM Journal on Mathematical Analysis*, vol. 40, no. 2, pp. 776–789, 2008.

[6] M. A. Lewis, B. Li, and H. F. Weinberger, “Spreading speed and linear determinacy for two-species competition models,” *Journal of Mathematical Biology*, vol. 45, no. 3, pp. 219–233, 2002.

[7] B. Li, M. A. Lewis, and H. F. Weinberger, “Existence of traveling waves for integral recursions with nonmonotone growth functions,” *Journal of Mathematical Biology*, vol. 58, no. 3, pp. 323–338, 2009.

[8] G. Lin and W.-T. Li, “Spreading speeds and traveling wavefronts for second order integrodifference equations,” *Journal of Mathematical Analysis and Applications*, vol. 361, no. 2, pp. 520–532, 2010.

[9] G. Lin, W. T. Li, and S. Ruan, “Asymptotic stability of monostable wavefronts in discrete-time integral recursions,” *Science China Mathematics*, vol. 53, no. 5, pp. 1185–1194, 2010.

[10] R. Lui, “Biological growth and spread modeled by systems of recursions: I. Mathematical theory,” *Mathematical Biosciences*, vol. 93, no. 2, pp. 269–295, 1989.

[11] R. Lui, “Biological growth and spread modeled by systems of recursions: II. Biological theory,” *Mathematical Biosciences*, vol. 107, pp. 255–287, 1991.

[12] H. F. Weinberger, “On spreading speeds and traveling waves for growth and migration models in a periodic habitat,” *Journal of Mathematical Biology*, vol. 45, no. 6, pp. 511–548, 2002.

[13] H. F. Weinberger, K. Kawasaki, and N. Shigesada, “Spreading speeds of spatially periodic integro-difference models for populations with nonmonotone recruitment functions,” *Journal of Mathematical Biology*, vol. 57, no. 3, pp. 387–411, 2008.

[14] M. Kot, M. A. Lewis, and P. van den Driessche, “Dispersal data and the spread of invading organisms,” *Ecology*, vol. 77, no. 7, pp. 2027–2042, 1996.

[15] M.-H. Wang, M. Kot, and M. G. Neubert, “Integrodifference equations, Allee effects, and invasions,” *Journal of Mathematical Biology*, vol. 44, no. 2, pp. 150–168, 2002.

[16] M. Kot, “Discrete-time travelling waves: ecological examples,” *Journal of Mathematical Biology*, vol. 30, no. 4, pp. 413–436, 1992.

[17] K. Mischaikow and V. Hutson, “Travelling waves for mutualist species,” *SIAM Journal on Mathematical Analysis*, vol. 24, no. 4, pp. 987–1008, 1993.

[18] H. L. Smith and X.-Q. Zhao, “Global asymptotic stability of traveling waves in delayed reaction-diffusion equations,” *SIAM Journal on Mathematical Analysis*, vol. 31, no. 3, pp. 514–534, 2000.

[19] A. I. Volpert, V. A. Volpert, and V. A. Volpert, *Traveling Wave Solutions of Parabolic Systems*, vol. 140 of *Translations of Mathematical Monographs*, American Mathematical Society, Providence, RI, USA, 1994.

[20] Z.-C. Wang, W.-T. Li, and S. Ruan, “Existence and stability of traveling wave fronts in reaction advection diffusion equations with nonlocal delay,” *Journal of Differential Equations*, vol. 238, no. 1, pp. 153–200, 2007.

[21] Z.-C. Wang, W.-T. Li, and S. Ruan, “Traveling fronts in monostable equations with nonlocal delayed effects,” *Journal of Dynamics and Differential Equations*, vol. 20, no. 3, pp. 573–607, 2008.

[22] X. Chen, “Existence, uniqueness, and asymptotic stability of traveling waves in nonlocal evolution equations,” *Advances in Differential Equations*, vol. 2, no. 1, pp. 125–160, 1997.

[23] X. Chen and J.-S. Guo, “Existence and asymptotic stability of traveling waves of discrete quasilinear monostable equations,” *Journal of Differential Equations*, vol. 184, no. 2, pp. 549–569, 2002.

[24] S. Ma and X. Zou, “Propagation and its failure in a lattice delayed differential equation with global interaction,” *Journal of Differential Equations*, vol. 212, no. 1, pp. 129–190, 2005.

[25] S. Ma and X. Zou, “Existence, uniqueness and stability of travelling waves in a discrete reaction-diffusion monostable equation with delay,” *Journal of Differential Equations*, vol. 217, no. 1, pp. 54–87, 2005.

[26] R. Lui, “Existence and stability of travelling wave solutions of a nonlinear integral operator,” *Journal of Mathematical Biology*, vol. 16, no. 3, pp. 199–220, 1983.

[27] R. Lui, “A nonlinear integral operator arising from a model in population genetics: III. Heterozygote inferior case,” *SIAM Journal on Mathematical Analysis*, vol. 16, no. 6, pp. 1180–1206, 1985.

[28] R. Lui, “A nonlinear integral operator arising from a model in population genetics: IV. Clines,” *SIAM Journal on Mathematical Analysis*, vol. 17, no. 1, pp. 152–168, 1986.

[29] P. Creegan and R. Lui, “Some remarks about the wave speed and travelling wave solutions of a nonlinear integral operator,” *Journal of Mathematical Biology*, vol. 20, no. 1, pp. 59–68, 1984.

[30] P. C. Fife and J. B. McLeod, “The approach of solutions of nonlinear diffusion equations to travelling front solutions,” *Archive for Rational Mechanics and Analysis*, vol. 65, no. 4, pp. 335–361, 1977.

[31] S. Ma and J. Wu, "Existence, uniqueness and asymptotic stability of traveling wavefronts in a non-local delayed diffusion equation," *Journal of Dynamics and Differential Equations*, vol. 19, no. 2, pp. 391–436, 2007.

[32] H. Yagisita, "Existence of traveling wave solutions for a nonlocal bistable equation: an abstract approach," *Kyoto University, Research Institute for Mathematical Sciences*, vol. 45, no. 4, pp. 955–979, 2009.

[33] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, vol. 258 of *Fundamental Principles of Mathematical Sciences*, Springer, New York, NY, USA, 2nd edition, 1994.