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Research Article

$H(\cdot,\cdot)$-Cocoercive Operator and an Application for Solving Generalized Variational Inclusions

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The purpose of this paper is to introduce a new $H(\cdot,\cdot)$-cocoercive operator, which generalizes many existing monotone operators. The resolvent operator associated with $H(\cdot,\cdot)$-cocoercive operator is defined, and its Lipschitz continuity is presented. By using techniques of resolvent operator, a new iterative algorithm for solving generalized variational inclusions is constructed. Under some suitable conditions, we prove the convergence of iterative sequences generated by the algorithm. For illustration, some examples are given.

1. Introduction

Various concepts of generalized monotone mappings have been introduced in the literature. Cocoercive mappings which are generalized form of monotone mappings are defined by Tseng [1], Magnanti and Perakis [2], and Zhu and Marcotte [3]. The resolvent operator techniques are important to study the existence of solutions and to develop iterative schemes for different kinds of variational inequalities and their generalizations, which are providing mathematical models to some problems arising in optimization and control, economics, and engineering sciences. In order to study various variational inequalities and variational inclusions, Fang and Huang, Lan, Cho, and Verma investigated many generalized operators such as H-monotone [4], H-accretive [5], (H,η)-accretive [6], (H,η)-monotone[7, 8], (A,η)-accretive mappings [9]. Recently, Zou and Huang [10] introduced and studied $H(\cdot,\cdot)$-accretive operators and Xu and Wang [11] introduced and studied $(H(\cdot,\cdot),\eta)$-monotone operators.

Motivated and inspired by the excellent work mentioned above, in this paper, we introduce and discuss new type of operators called $H(\cdot,\cdot)$-cocoercive operators. We define resolvent operator associated with $H(\cdot,\cdot)$-cocoercive operators and prove the Lipschitz continuity of the resolvent operator. We apply $H(\cdot,\cdot)$-cocoercive operators to solve a generalized variational inclusion problem. Some examples are constructed for illustration.

2. Preliminaries

Throughout the paper, we suppose that X is a real Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$, d is the metric induced by the norm $\|\cdot\|$, 2^X (resp., $CB(X)$) is the family of all nonempty (resp., closed and bounded) subsets of X, and $\mathfrak{D}(\cdot,\cdot)$ is the Hausdorff metric on $CB(X)$ defined by

$$\mathfrak{D}(P,Q)=\max\left\{\sup_{x\in P} d(x,Q),\sup_{y\in Q} d(P,y)\right\}, \tag{2.1}$$

where $d(x,Q)=\inf_{y\in Q} d(x,y)$ and $d(P,y)=\inf_{x\in P} d(x,y)$.

Definition 2.1. A mapping $g: X\to X$ is said to be

(i) Lipschitz continuous if there exists a constant $\lambda_g>0$ such that

$$\|g(x)-g(y)\|\le\lambda_g\|x-y\|,\quad \forall x,y\in X; \tag{2.2}$$

(ii) monotone if

$$\langle g(x)-g(y),x-y\rangle\ge 0,\quad \forall x,y\in X; \tag{2.3}$$

(iii) strongly monotone if there exists a constant $\xi>0$ such that

$$\langle g(x)-g(y),x-y\rangle\ge\xi\|x-y\|^2,\quad \forall x,y\in X; \tag{2.4}$$

(iv) α-expansive if there exists a constant $\alpha>0$ such that

$$\|g(x)-g(y)\|\ge\alpha\|x-y\|,\quad \forall x,y\in X, \tag{2.5}$$

if $\alpha=1$, then it is expansive.

Definition 2.2. A mapping $T: X\to X$ is said to be cocoercive if there exists a constant $\mu'>0$ such that

$$\langle T(x)-T(y),x-y\rangle\ge\mu'\|T(x)-T(y)\|^2,\quad \forall x,y\in X. \tag{2.6}$$

Note 1. Clearly T is $1/\mu'$-Lipschitz continuous and also monotone but not necessarily strongly monotone and Lipschitz continuous (consider a constant mapping). Conversely, strongly monotone and Lipschitz continuous mappings are cocoercive, and it follows that cocoercivity is an intermediate concept that lies between simple and strong monotonicity.

Definition 2.3. A multivalued mapping $M : X \to 2^X$ is said to be cocoercive if there exists a constant $\mu'' > 0$ such that

$$\langle u - v, x - y \rangle \geq \mu'' \|u - v\|^2, \quad \forall x, y \in X,\ u \in M(x),\ v \in M(y). \tag{2.7}$$

Definition 2.4. A mapping $T : X \to X$ is said to be relaxed cocoercive if there exists a constant $\gamma' > 0$ such that

$$\langle T(x) - T(y), x - y \rangle \geq (-\gamma') \|T(x) - T(y)\|^2, \quad \forall x, y \in X. \tag{2.8}$$

Definition 2.5. Let $H : X \times X \to X$ and $A, B : X \to X$ be the mappings.

(i) $H(A, \cdot)$ is said to be cocoercive with respect to A if there exists a constant $\mu > 0$ such that

$$\langle H(Ax, u) - H(Ay, u), x - y \rangle \geq \mu \|Ax - Ay\|^2, \quad \forall x, y \in X; \tag{2.9}$$

(ii) $H(\cdot, B)$ is said to be relaxed cocoercive with respect to B if there exists a constant $\gamma > 0$ such that

$$\langle H(u, Bx) - H(u, By), x - y \rangle \geq (-\gamma) \|Bx - By\|^2, \quad \forall x, y \in X; \tag{2.10}$$

(iii) $H(A, \cdot)$ is said to be r_1-Lipschitz continuous with respect to A if there exists a constant $r_1 > 0$ such that

$$\|H(Ax, \cdot) - H(Ay, \cdot)\| \leq r_1 \|x - y\|, \quad \forall x, y \in X; \tag{2.11}$$

(iv) $H(\cdot, B)$ is said to be r_2-Lipschitz continuous with respect to B if there exists a constant $r_2 > 0$ such that

$$\|H(\cdot, Bx) - H(\cdot, By)\| \leq r_2 \|x - y\|, \quad \forall x, y \in X. \tag{2.12}$$

Example 2.6. Let $X = \mathbb{R}^2$ with usual inner product. Let $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\begin{gathered} Ax = (2x_1 - 2x_2, -2x_1 + 4x_2), \\ By = (-y_1 + y_2, -y_2), \quad \forall (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2. \end{gathered} \tag{2.13}$$

Suppose that $H(A, B) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$H(Ax, By) = Ax + By, \quad \forall x, y \in \mathbb{R}^2. \tag{2.14}$$

Then $H(A,B)$ is $(1/6)$-cocoercive with respect to A and $(1/2)$-relaxed cocoercive with respect to B since

$$\begin{aligned}
\langle H(Ax,u)-H(Ay,u),x-y\rangle &= \langle Ax-Ay,x-y\rangle \\
&= \langle (2x_1-2x_2,-2x_1+4x_2)-(2y_1-2y_2,-2y_1+4y_2), \\
&\qquad (x_1-y_1,x_2-y_2)\rangle \\
&= \langle (2(x_1-y_1)-2(x_2-y_2),-2(x_1-y_1)+4(x_2-y_2)), \\
&\qquad (x_1-y_1,x_2-y_2)\rangle \\
&= 2(x_1-y_1)^2+4(x_2-y_2)^2-4(x_1-y_1)(x_2-y_2), \\
\|Ax-Ay\|^2 &= \langle ((2x_1-2x_2,-2x_1+4x_2)-(2y_1-2y_2,-2y_1+4y_2)), \\
&\qquad ((2x_1-2x_2,-2x_1+4x_2)-(2y_1-2y_2,-2y_1+4y_2))\rangle \\
&= 8(x_1-y_1)^2+20(x_2-y_2)^2-24(x_1-y_1)(x_2-y_2) \\
&\le 12(x_1-y_1)^2+24(x_2-y_2)^2-24(x_1-y_1)(x_2-y_2) \\
&= 6\left\{2(x_1-y_1)^2+4(x_2-y_2)^2-4(x_1-y_1)(x_2-y_2)\right\} \\
&= 6\{\langle H(u,Ax)-H(u,Ay),x-y\rangle\},
\end{aligned} \tag{2.15}$$

which implies that

$$\langle H(Ax,u)-H(Ay,u),x-y\rangle \ge \frac{1}{6}\|Ax-Ay\|^2, \tag{2.16}$$

That is, $H(A,B)$ is $(1/6)$-cocoercive with respect to A.

$$\begin{aligned}
\langle H(u,Bx)-H(u,By),x-y\rangle &= \langle Bx-By,x-y\rangle \\
&= \langle (-x_1+x_2,-x_2)-(-y_1+y_2,-y_2),(x_1-y_1,x_2-y_2)\rangle \\
&= \langle (-(x_1-y_1)+(x_2-y_2),-(x_2-y_2)),(x_1-y_1,x_2-y_2)\rangle \\
&= -(x_1-y_1)^2-(x_2-y_2)^2+(x_1-y_1)(x_2-y_2) \\
&= -\left\{(x_1-y_1)^2+(x_2-y_2)^2-(x_1-y_1)(x_2-y_2)\right\}, \\
\|Bx-By\|^2 &= \langle (-(x_1-y_1)+(x_2-y_2),-(x_2-y_2)), \\
&\qquad (-(x_1-y_1)+(x_2-y_2),-(x_2-y_2))\rangle
\end{aligned}$$

$$
\begin{aligned}
&= (x_1 - y_1)^2 + 2(x_2 - y_2)^2 - 2(x_1 - y_1)(x_2 - y_2) \\
&\le 2\left\{(x_1 - y_1)^2 + (x_2 - y_2)^2 - (x_1 - y_1)(x_2 - y_2)\right\} \\
&= 2(-1)\langle H(Bx, u) - H(By, u), x - y\rangle
\end{aligned}
\tag{2.17}
$$

which implies that

$$
\langle H(u, Bx) - H(u, By), x - y\rangle \ge -\frac{1}{2}\|Bx - By\|^2, \tag{2.18}
$$

that is, $H(A, B)$ is $(1/2)$-relaxed cocoercive with respect to B.

3. $H(\cdot,\cdot)$-Cocoercive Operator

In this section, we define a new $H(\cdot,\cdot)$-cocoercive operator and discuss some of its properties.

Definition 3.1. Let $A, B : X \to X, H : X \times X \to X$ be three single-valued mappings. Let $M : X \to 2^X$ be a set-valued mapping. M is said to be $H(\cdot,\cdot)$-cocoercive with respect to mappings A and B (or simply $H(\cdot,\cdot)$-cocoercive in the sequel) if M is cocoercive and $(H(A, B) + \lambda M)(X) = X$, for every $\lambda > 0$.

Example 3.2. Let X, A, B, and H be the same as in Example 2.6, and let $M : \mathbb{R}^2 \to \mathbb{R}^2$ be define by $M(x_1, x_2) = (0, x_2), \forall (x_1, x_2) \in \mathbb{R}^2$. Then it is easy to check that M is cocoercive and $(H(A, B) + \lambda M)(\mathbb{R}^2) = \mathbb{R}^2, \forall \lambda > 0$, that is, M is $H(\cdot,\cdot)$-cocoercive with respect to A and B.

Remark 3.3. Since cocoercive operators include monotone operators, hence our definition is more general than definition of $H(\cdot,\cdot)$-monotone operator [10]. It is easy to check that $H(\cdot,\cdot)$-cocoercive operators provide a unified framework for the existing $H(\cdot,\cdot)$-monotone, H-monotone operators in Hilbert space and $H(\cdot,\cdot)$-accretive, H-accretive operators in Banach spaces.

Since $H(\cdot,\cdot)$-cocoercive operators are more general than maximal monotone operators, we give the following characterization of $H(\cdot,\cdot)$-cocoercive operators.

Proposition 3.4. *Let $H(A, B)$ be μ-cocoercive with respect to A, γ-relaxed cocoercive with respect to B, A is α-expansive, B is β-Lipschitz continuous, and $\mu > \gamma$, $\alpha > \beta$. Let $M : X \to 2^X$ be $H(\cdot,\cdot)$-cocoercive operator. If the following inequality*

$$
\langle x - y, u - v\rangle \ge 0 \tag{3.1}
$$

holds for all $(v, y) \in \text{Graph}(M)$, then $x \in Mu$, where

$$
\text{Graph}(M) = \{(x, u) \in X \times X : u \in M(x)\}. \tag{3.2}
$$

Proof. Suppose that there exists some (u_0, x_0) such that

$$\langle x_0 - y, u_0 - v \rangle \geq 0, \quad \forall (v, y) \in \operatorname{Graph}(M). \tag{3.3}$$

Since M is $H(\cdot,\cdot)$-cocoercive, we know that $(H(A,B) + \lambda M)(X) = X$ holds for every $\lambda > 0$, and so there exists $(u_1, x_1) \in \operatorname{Graph}(M)$ such that

$$H(Au_1, Bu_1) + \lambda x_1 = H(Au_0, Bu_0) + \lambda x_0 \in X. \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$\begin{aligned}
0 \leq \langle \lambda x_0 + H(Au_0, Bu_0) - \lambda x_1 - H(Au_1, Bu_1), u_0 - u_1 \rangle, \\
0 \leq \lambda \langle x_0 - x_1, u_0 - u_1 \rangle &= -\langle H(Au_0, Bu_0) - H(Au_1, Bu_1), u_0 - u_1 \rangle \\
&= -\langle H(Au_0, Bu_0) - H(Au_1, Bu_0), u_0 - u_1 \rangle \\
&\quad -\langle H(Au_1, Bu_0) - H(Au_1, Bu_1), u_0 - u_1 \rangle \\
&\leq -\mu \|Au_0 - Au_1\|^2 + \gamma \|\mathrm{B}u_0 - Bu_1\|^2 \\
&\leq -\mu\alpha^2 \|u_0 - u_1\|^2 + \gamma\beta^2 \|u_0 - u_1\|^2 \\
&= -(\mu\alpha^2 - \gamma\beta^2) \|u_0 - u_1\|^2 \leq 0,
\end{aligned} \tag{3.5}$$

which gives $u_1 = u_0$ since $\mu > \gamma, \alpha > \beta$. By (3.4), we have $x_1 = x_0$. Hence $(u_0, x_0) = (u_1, x_1) \in \operatorname{Graph}(M)$ and so $x_0 \in Mu_0$. □

Theorem 3.5. *Let X be a Hilbert space and $M : X \to 2^X$ a maximal monotone operator. Suppose that $H : X \times X \to X$ is a bounded cocoercive and semicontinuous with respect to A and B. Let $H : X \times X \to X$ be also μ-cocoercive with respect to A and γ-relaxed cocoercive with respect to B. The mapping A is α-expansive, and B is β-Lipschitz continuous. If $\mu > \gamma$ and $\alpha > \beta$, then M is $H(\cdot,\cdot)$-cocoercive with respect to A and B.*

Proof. For the proof we refer to [10]. □

Theorem 3.6. *Let $H(A,B)$ be a μ-cocoercive with respect to A and γ-relaxed cocoercive with respect to B, A is α-expansive, and B is β-Lipschitz continuous, $\mu > \gamma$ and $\alpha > \beta$. Let M be an $H(\cdot,\cdot)$-cocoercive operator with respect to A and B. Then the operator $(H(A,B) + \lambda M)^{-1}$ is single-valued.*

Proof. For any given $u \in X$, let $x, y \in (H(A,B) + \lambda M)^{-1}(u)$. It follows that

$$\begin{aligned}
-H(Ax, Bx) + u \in \lambda Mx, \\
-H(Ay, By) + u \in \lambda My.
\end{aligned} \tag{3.6}$$

As M is cocoercive (thus monotone), we have

$$\begin{aligned}0 &\le \langle -H(Ax,Bx)+u-(-H(Ay,By)+u)\ ,x-y\rangle\\ &= -\langle H(Ax,Bx)-H(Ay,By),x-y\rangle\\ &= -\langle H(Ax,Bx)-H(Ay,Bx)+H(Ay,Bx)-H(Ay,By),x-y\rangle\\ &= -\langle H(Ax,Bx)-H(Ay,Bx),x-y\rangle-\langle H(Ay,Bx)-H(Ay,By),x-y\rangle.\end{aligned} \tag{3.7}$$

Since H is μ-cocoercive with respect to A and γ-relaxed cocoercive with respect to B, A is α-expansive and B is β-Lipschitz continuous, thus (3.7) becomes

$$0 \le -\mu\alpha^2\|x-y\|^2+\gamma\beta^2\|x-y\|^2=-\left(\mu\alpha^2-\gamma\beta^2\right)\|x-y\|^2 \le 0 \tag{3.8}$$

since $\mu>\gamma,\alpha>\beta$. Thus, we have $x=y$ and so $(H(A,B)+\lambda M)^{-1}$ is single-valued. □

Definition 3.7. Let $H(A,B)$ be μ-cocoercive with respect to A and γ-relaxed cocoercive with respect to B, A is α-expansive, B is β-Lipschitz continuous, and $\mu>\gamma$, $\alpha>\beta$. Let M be an $H(\cdot,\cdot)$-cocoercive operator with respect to A and B. The resolvent operator $R_{\lambda,M}^{H(\cdot,\cdot)}: X \to X$ is defined by

$$R_{\lambda,M}^{H(\cdot,\cdot)}(u)=(H(A,B)+\lambda M)^{-1}(u), \quad \forall u \in X. \tag{3.9}$$

Now, we prove the Lipschitz continuity of resolvent operator defined by (3.9) and estimate its Lipschitz constant.

Theorem 3.8. *Let $H(A,B)$ be μ-cocoercive with respect to A, γ-relaxed cocoercive with respect to B, A is α-expansive, B is β-Lipschitz continuous, and $\mu>\gamma$, $\alpha>\beta$. Let M be an $H(\cdot,\cdot)$-cocoercive operator with respect to A and B. Then the resolvent operator $R_{\lambda,M}^{H(\cdot,\cdot)}: X \to X$ is $1/\mu\alpha^2-\gamma\beta^2$-Lipschitz continuous, that is,*

$$\left\|R_{\lambda,M}^{H(\cdot,\cdot)}(u)-R_{\lambda,M}^{H(\cdot,\cdot)}(v)\right\| \le \frac{1}{\mu\alpha^2-\gamma\beta^2}\|u-v\|, \quad \forall u,v \in X. \tag{3.10}$$

Proof. Let u and v be any given points in X. It follows from (3.9) that

$$\begin{aligned}R_{\lambda,M}^{H(\cdot,\cdot)}(u)&=(H(A,B)+\lambda M)^{-1}(u),\\ R_{\lambda,M}^{H(\cdot,\cdot)}(v)&=(H(A,B)+\lambda M)^{-1}(v).\end{aligned} \tag{3.11}$$

This implies that

$$\begin{aligned}
&\frac{1}{\lambda}\Big(u - H\Big(A\Big(R_{\lambda,M}^{H(\cdot,\cdot)}(u)\Big), B\Big(R_{\lambda,M}^{H(\cdot,\cdot)}(u)\Big)\Big)\Big) \in M\Big(R_{\lambda,M}^{H(\cdot,\cdot)}(u)\Big),\\
&\frac{1}{\lambda}\Big(v - H\Big(A\Big(R_{\lambda,M}^{H(\cdot,\cdot)}(v)\Big), B\Big(R_{\lambda,M}^{H(\cdot,\cdot)}(v)\Big)\Big)\Big) \in M\Big(R_{\lambda,M}^{H(\cdot,\cdot)}(v)\Big).
\end{aligned} \tag{3.12}$$

For the sake of clarity, we take

$$Pu = R_{\lambda,M}^{H(\cdot,\cdot)}(u), \qquad Pv = R_{\lambda,M}^{H(\cdot,\cdot)}(v). \tag{3.13}$$

Since M is cocoercive (hence monotone), we have

$$\begin{aligned}
&\frac{1}{\lambda}\langle u - H(A(Pu), B(Pu)) - (v - H(A(Pv), B(Pv))), Pu - Pv\rangle \geq 0,\\
&\frac{1}{\lambda}\langle u - v - H(A(Pu), B(Pu)) + H(A(Pv), B(Pv)), Pu - Pv\rangle \geq 0,
\end{aligned} \tag{3.14}$$

which implies that

$$\langle u - v, Pu - Pv\rangle \geq \langle H(A(Pu), B(Pu)) - H(A(Pv), B(Pv)), Pu - Pv\rangle. \tag{3.15}$$

Further, we have

$$\begin{aligned}
\|u - v\|\|Pu - Pv\| &\geq \langle u - v, Pu - Pv\rangle\\
&\geq \langle H(A(Pu), B(Pu)) - H(A(Pv), B(Pv)), Pu - Pv\rangle\\
&= \langle H(A(Pu), B(Pu)) - H(A(Pv), B(Pu)) + H(A(Pv), B(Pu))\\
&\quad - H(A(Pv), B(Pv)), Pu - Pv\rangle\\
&= \langle H(A(Pu), B(Pu)) - H(A(Pv), B(Pu)), Pu - Pv\rangle\\
&\quad + \langle H(A(Pv), B(Pu)) - H(A(Pv), B(Pv)), Pu - Pv\rangle\\
&\geq \mu\|A(Pu) - A(Pv)\|^2 - \gamma\|B(Pu) - B(Pv)\|^2\\
&\geq \mu\alpha^2\|Pu - Pv\|^2 - \gamma\beta^2\|Pu - Pv\|^2,
\end{aligned} \tag{3.16}$$

and so

$$\|u - v\|\|Pu - Pv\| \geq \left(\mu\alpha^2 - \gamma\beta^2\right)\|Pu - Pv\|^2, \tag{3.17}$$

thus,

$$\|Pu - Pv\| \leq \frac{1}{\mu\alpha^2 - \gamma\beta^2}\|u - v\|,$$
$$\text{that is} \quad \left\|R_{\lambda,M}^{H(\cdot,\cdot)}(u) - R_{\lambda,M}^{H(\cdot,\cdot)}(v)\right\| \leq \frac{1}{\mu\alpha^2 - \gamma\beta^2}\|u - v\|, \quad \forall u, v \in X. \tag{3.18}$$

This completes the proof. □

4. Application of $H(\cdot,\cdot)$-Cocoercive Operators for Solving Variational Inclusions

We apply $H(\cdot,\cdot)$-cocoercive operators for solving a generalized variational inclusion problem.

We consider the problem of finding $u \in X$ and $w \in T(u)$ such that

$$0 \in w + M(g(u)), \tag{4.1}$$

where $g : X \to X, M : X \to 2^X$, and $T : X \to CB(X)$ are the mappings. Problem (4.1) is introduced and studied by Huang [12] in the setting of Banach spaces.

Lemma 4.1. *The (u, w), where $u \in X, w \in T(u)$, is a solution of the problem* (4.1)*, if and only if (u, w) is a solution of the following:*

$$g(u) = R_{\lambda,M}^{H(\cdot,\cdot)}[H(A(gu), B(gu)) - \lambda w], \tag{4.2}$$

where $\lambda > 0$ is a constant.

Proof. By using the definition of resolvent operator $R_{\lambda,M}^{H(\cdot,\cdot)}$, the conclusion follows directly. □

Based on (4.2), we construct the following algorithm.

Algorithm 4.2. For any $u_0 \in X,\ w_0 \in T(u_0)$, compute the sequences $\{u_n\}$ and $\{w_n\}$ by iterative schemes such that

$$g(u_{n+1}) = R_{\lambda,M}^{H(\cdot,\cdot)}[H(A(gu_n), B(gu_n)) - \lambda w_n],$$
$$w_n \in T(u_n), \|w_n - w_{n+1}\| \leq \left(1 + \frac{1}{n+1}\right)\mathfrak{D}(T(u_n), T(u_{n+1})), \tag{4.3}$$

for all $n = 0, 1, 2, \ldots$, and $\lambda > 0$ is a constant.

Theorem 4.3. *Let X be a real Hilbert space and $A, B, g : X \to X, H : X \times X \to X$ the single-valued mappings. Let $T : X \to CB(X)$ be a multi-valued mapping and $M : X \to 2^X$ the multi-valued $H(\cdot,\cdot)$-cocoercive operator. Assume that*

(i) *T is δ-Lipschitz continuous in the Hausdorff metric $\mathfrak{D}(\cdot,\cdot)$;*

(ii) *$H(A, B)$ is μ-cocoercive with respect to A and γ-relaxed cocoercive with respect to B;*

(iii) *A is α-expansive;*

(iv) *B is β-Lipschitz continuous;*

(v) *g is λ_g-Lipschitz continuous and ξ-strongly monotone;*

(vi) *$H(A,B)$ is r_1-Lipschitz continuous with respect to A and r_2-Lipschitz continuous with respect to B;*

(vii) $(r_1+r_2)\lambda_g < [(\mu\alpha^2-\gamma\beta^2)\xi - \lambda\delta];\ \mu>\gamma,\ \alpha>\beta.$

Then the generalized variational inclusion problem (4.1) has a solution (u,w) with $u\in X$, $w\in T(u)$, and the iterative sequences $\{u_n\}$ and $\{w_n\}$ generated by Algorithm 4.2 converge strongly to u and w, respectively.

Proof. Since T is δ-Lipschitz continuous, it follows from Algorithm 4.2 that

$$\|w_n - w_{n+1}\| \le \left(1+\frac{1}{n+1}\right)\mathfrak{D}(T(u_n),T(u_{n+1})) \le \left(1+\frac{1}{n+1}\right)\delta\|u_n-u_{n+1}\|, \tag{4.4}$$

for $n=0,1,2,\ldots$.

Using the ξ-strong monotonicity of g, we have

$$\begin{aligned}\|g(u_{n+1})-g(u_n)\|\|u_{n+1}-u_n\| &\ge \langle g(u_{n+1})-g(u_n), u_{n+1}-u_n\rangle \\ &\ge \xi\|u_{n+1}-u_n\|^2\end{aligned} \tag{4.5}$$

which implies that

$$\|u_{n+1}-u_n\| \le \frac{1}{\xi}\|g(u_{n+1})-g(u_n)\|. \tag{4.6}$$

Now we estimate $\|g(u_{n+1})-g(u_n)\|$ by using the Lipschitz continuity of $R^{H(\cdot,\cdot)}_{\lambda,M}$,

$$\begin{aligned}\|g(u_{n+1})-g(u_n)\| &= \Big\|R^{H(\cdot,\cdot)}_{\lambda,M}[H(A(gu_n),B(gu_n))-\lambda w_n] \\ &\quad - R^{H(\cdot,\cdot)}_{\lambda,M}[H(A(gu_{n-1}),B(gu_{n-1}))-\lambda w_{n-1}]\Big\| \\ &\le \frac{1}{\mu\alpha^2-\gamma\beta^2}\|H(A(gu_n),B(gu_n))-H(A(gu_{n-1}),B(gu_{n-1}))\| \\ &\quad + \frac{\lambda}{\mu\alpha^2-\gamma\beta^2}\|w_n-w_{n-1}\| \\ &\le \frac{1}{\mu\alpha^2-\gamma\beta^2}\|H(A(gu_n),B(gu_n))-H(A(gu_{n-1}),B(gu_n))\| \\ &\quad + \frac{1}{\mu\alpha^2-\gamma\beta^2}\|H(A(gu_{n-1}),B(gu_n))-H(A(gu_{n-1}),B(gu_{n-1}))\| \\ &\quad + \frac{\lambda}{\mu\alpha^2-\gamma\beta^2}\|w_n-w_{n-1}\|.\end{aligned} \tag{4.7}$$

Since $H(A,B)$ is r_1-Lipschitz continuous with respect to A and r_2-Lipschitz continuous with respect to B, g is λ_g-Lipschitz continuous and using (4.4), (4.7) becomes

$$\begin{aligned}\|g(u_{n+1})-g(u_n)\| \leq{}& \frac{r_1\lambda_g}{\mu\alpha^2-\gamma\beta^2}\|u_n-u_{n-1}\| + \frac{r_2\lambda_g}{\mu\alpha^2-\gamma\beta^2}\|u_n-u_{n-1}\| \\ &+ \frac{\lambda}{\mu\alpha^2-\gamma\beta^2}\left(1+\frac{1}{n}\right)\delta\|u_n-u_{n-1}\|\end{aligned} \tag{4.8}$$

or

$$\|g(\mathrm{u}_{\mathrm{n+1}}-g(u_n)\| \leq \left[\frac{r_1\lambda_g}{\mu\alpha^2-\gamma\beta^2}+\frac{r_2\lambda_g}{\mu\alpha^2-\gamma\beta^2}+\frac{\lambda}{\mu\alpha^2-\gamma\beta^2}\left(1+\frac{1}{n}\right)\delta\right]\|u_n-u_{n-1}\|. \tag{4.9}$$

Using (4.9), (4.6) becomes

$$\|u_{n+1}-u_n\| \leq \theta_n\|u_n-u_{n-1}\|, \tag{4.10}$$

where

$$\theta_n = \frac{(r_1+r_2)\lambda_g+\lambda\delta(1+1/n)}{(\mu\alpha^2-\gamma\beta^2)\xi}. \tag{4.11}$$

Let

$$\theta = \frac{(r_1+r_2)\lambda_g+\lambda\delta}{(\mu\alpha^2-\gamma\beta^2)\xi}. \tag{4.12}$$

We know that $\theta_n \to \theta$ and $n \to \infty$. From assumption (vii), it is easy to see that $\theta < 1$. Therefore, it follows from (4.10) that $\{u_n\}$ is a Cauchy sequence in X. Since X is a Hilbert space, there exists $u \in \mathrm{X}$ such that $u_n \to u$ as $n \to \infty$. From (4.4), we know that $\{w_n\}$ is also a Cauchy sequence in X, thus there exists $w \in \mathrm{X}$ such that $w_n \to w$ and $n \to \infty$. By the continuity of $g, R_{\lambda,M}^{H(\cdot,\cdot)} H, A, B$, and T and Algorithm 4.2, we have

$$g(u) = R_{\lambda,M}^{H(\cdot,\cdot)}[H(A(gu),B(gu))-\lambda w]. \tag{4.13}$$

Now, we prove that $w \in T(u)$. In fact, since $w_n \in T(u_n)$, we have

$$\begin{aligned}d(w,T(u)) &\leq \|w-w_n\| + d(w_n,T(u)) \\ &\leq \|w-w_n\| + \mathfrak{D}(T(u_n),T(u)) \\ &\leq \|w-w_n\| + \delta\|u_n-u\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty,\end{aligned} \tag{4.14}$$

which implies that $d(w,T(u)) = 0$. Since $T(u) \in CB(X)$, it follows that $w \in T(u)$. By Lemma 4.1, we know that (u,w) is a solution of problem (4.1). This completes the proof. □

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