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Research Article

On the Reducibility for a Class of Quasi-Periodic Hamiltonian Systems with Small Perturbation Parameter

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We consider the following real two-dimensional nonlinear analytic quasi-periodic Hamiltonian system $\dot{x} = J\nabla_x H$, where $H(x,t,\varepsilon) = (1/2)\beta(x_1^2 + x_2^2) + F(x,t,\varepsilon)$ with $\beta \neq 0, \partial_x F(0,t,\varepsilon) = O(\varepsilon)$ and $\partial_{xx} F(0,t,\varepsilon) = O(\varepsilon)$ as $\varepsilon \to 0$. Without any nondegeneracy condition with respect to ε, we prove that for most of the sufficiently small ε, by a quasi-periodic symplectic transformation, it can be reduced to a quasi-periodic Hamiltonian system with an equilibrium.

1. Introduction

We first give some definitions and notations for our problem. A function $f(t)$ is called a quasi-periodic function with frequencies $\omega = (\omega_1, \omega_2, \ldots, \omega_l)$ if $f(t) = F(\omega_1 t, \omega_2 t, \ldots, \omega_l t)$ with $\theta_i = \omega_i t$, where $F(\theta_1, \theta_2, \ldots, \theta_l)$ is 2π periodic in all the arguments $\theta_j,\ j = 1, 2, \ldots, l$. If $F(\theta)$ $(\theta = (\theta_1, \theta_2, \ldots, \theta_l))$ is analytic on $D_\rho = \{\theta \in C^l/2\pi Z^l \mid |\operatorname{Im}\theta_i| \leq \rho,\ i = 1, 2, \ldots, l\}$, we call $f(t)$ analytic quasi-periodic on D_ρ. If all $q_{ij}(t)$ $(i, j = 1, 2 \ldots, n)$ are analytic quasi-periodic on D_ρ, then the matrix function $Q(t) = (q_{ij}(t))_{1\leq i,j\leq n}$ is called analytic quasi-periodic on D_ρ.

If $f(t)$ is analytic quasi-periodic on D_ρ, we can write it as Fourier series:

$$f(t) = \sum_{k\in Z^l} f_k e^{\mathrm{i}\langle k,\omega\rangle t}. \tag{1.1}$$

Define a norm of f by $\|f\|_\rho = \sum_{k\in Z^l} |f_k| e^{|k|\rho}$. It follows that $|f_k| \leq \|f\|_\rho e^{-|k|\rho}$. If the matrix function $Q(t)$ is analytic quasi-periodic on D_ρ, we define the norm of Q by $\|Q\|_\rho = n \times \max_{1\leq i,j\leq n} \|q_{ij}\|_\rho$. It is easy to verify $\|Q_1 Q_2\|_\rho \leq \|Q_1\|_\rho \|Q_2\|_\rho$. The average of $Q(t)$ is denoted by $[Q] = ([q_{ij}])_{1\le i,j\le n}$, where

$$[q_{ij}] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} q_{ij}(t)dt. \tag{1.2}$$

For the existence of the above limit, see [1].

Denote

$$D(r,\rho,\varepsilon_0) = \left\{(x,\theta,\varepsilon) \in C^n \times \left(\frac{C^l}{2\pi Z^l}\right) \times C \mid |x| \le r,\ \theta \in D_\rho,\ |\varepsilon| \le \varepsilon_0\right\}, \tag{1.3}$$

where $x = (x_1, x_2, \dots, x_n)$ and $|x| = |x_1| + |x_2| + \cdots + |x_n|$.

Let $f(x,t,\varepsilon)$ be analytic quasi-periodic of t and analytic in x and ε on $D(r,\rho,\varepsilon_0)$. Then $f(x,t,\varepsilon)$ can be expanded as

$$f(x,t,\varepsilon) = \sum_{m=0}^{\infty}\sum_{k\in Z^l} f_{mk}(x)\varepsilon^m\, e^{\mathrm{i}\langle k,\omega\rangle t}. \tag{1.4}$$

Define a norm by

$$\|f\|_{D(r,\rho,\varepsilon_0)} = \sum_{m=0}^{\infty}\sum_{k\in Z^l} \left|f_{mk}\right|_r \varepsilon_0^m\, e^{\rho|k|}, \tag{1.5}$$

where $|f_{mk}|_r = \sup_{|x|\le r}|f_{mk}(x)|$. Note that

$$\|f_1\cdot f_2\|_{D(r,\rho,\varepsilon_0)} \le \|f_1\|_{D(r,\rho,\varepsilon_0)} \cdot \|f_2\|_{D(r,\rho,\varepsilon_0)}. \tag{1.6}$$

Problems

The reducibility on the linear differential system has been studied for a long time. The well-known Floquet theorem tells us that if $A(t)$ is a T-periodic matrix, then the linear system $\dot{x} = A(t)x$ is always reducible to the constant coefficient one by a T-periodic change of variables. However, this cannot be generalized to the quasi-periodic system. In [2], Johnson and Sell considered the quasi-periodic system $\dot{x} = A(t)x$, where $A(t)$ is a quasi-periodic matrix. Under some “full spectrum” conditions, they proved that $\dot{x} = A(t)x$ is reducible. That is, there exists a quasi-periodic nonsingular transformation $x = \phi(t)y$, where $\phi(t)$ and $\phi(t)^{-1}$ are quasi-periodic and bounded, such that $\dot{x} = A(t)x$ is transformed to $\dot{y} = By$, where B is a constant matrix.

In [3], Jorba and Simó considered the reducibility of the following linear system:

$$\dot{x} = (A + \varepsilon Q(t))x, \quad x \in R^n, \tag{1.7}$$

where A is an $n \times n$ constant matrix with n different eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $Q(t)$ is analytic quasi-periodic with respect to t with frequencies $\omega = (\omega_1, \omega_2, \ldots, \omega_l)$. Here ε is a small perturbation parameter. Suppose that the following nonresonance conditions hold:

$$\left| \langle k, \omega \rangle \sqrt{-1} + \lambda_i - \lambda_j \right| \geq \frac{\alpha}{|k|^\tau}, \tag{1.8}$$

for all $k \in Z^l \setminus \{0\}$, where $\alpha > 0$ is a small constant and $\tau > l - 1$. Assume that $\lambda_j^0(\varepsilon)$ $(j = 1, 2, \ldots, n)$ are eigenvalues of $A + \varepsilon[Q]$. If the following non-degeneracy conditions hold:

$$\frac{d}{d\varepsilon}\left(\lambda_i^0(\varepsilon) - \lambda_j^0(\varepsilon)\right)\Big|_{\varepsilon=0} \neq 0, \quad \forall i \neq j, \tag{1.9}$$

then authors proved that for sufficiently small $\varepsilon_0 > 0$, there exists a nonempty Cantor subset $E \subset (0, \varepsilon_0)$, such that for $\varepsilon \in E$, the system (1.7) is reducible. Moreover, $\text{meas}((0, \varepsilon_0) \setminus E) = o(\varepsilon_0)$.

Some related problems were considered by Eliasson in [4, 5]. In the paper [4], to study one-dimensional linear Schrödinger equation

$$\frac{d^2 q}{dt^2} + Q(\omega t) q = Eq, \tag{1.10}$$

Eliasson considered the following equivalent two-dimensional quasi-periodic Hamiltonian system:

$$\dot{p} = (E - Q(\omega t))q, \quad \dot{q} = p, \tag{1.11}$$

where Q is an analytic quasi-periodic function and E is an energy parameter. The result in [4] implies that for almost every sufficiently large E, the quasi-periodic system (1.11) is reducible. Later, in [5] the author considered the almost reducibility of linear quasi-periodic systems. Recently, the similar problem was considered by Her and You [6]. Let $C^\omega(\Lambda, gl(m, C))$ be the set of $m \times m$ matrices $A(\lambda)$ depending analytically on a parameter λ in a closed interval $\Lambda \subset R$. In [6], Her and You considered one-parameter families of quasi-periodic linear equations

$$\dot{x} = \left(A(\lambda) + g(\omega_1 t, \ldots, \omega_l t, \lambda)\right)x, \tag{1.12}$$

where $A \in C^\omega(\Lambda, gl(m, C))$, and g is analytic and sufficiently small. They proved that under some nonresonance conditions and some non-degeneracy conditions, there exists an open and dense set $\mathcal{A}$ in $C^\omega(\Lambda, gl(m, C))$, such that for each $A \in \mathcal{A}$, the system (1.12) is reducible for almost all $\lambda \in \Lambda$.

In 1996, Jorba and Simó extended the conclusion of the linear system to the nonlinear case. In [7], Jorba and Simó considered the quasi-periodic system

$$\dot{x} = (A + \varepsilon Q(t))x + \varepsilon g(t) + h(x, t), \quad x \in R^n, \tag{1.13}$$

where A has n different nonzero eigenvalues λ_i. They proved that under some nonresonance conditions and some non-degeneracy conditions, there exists a nonempty Cantor subset $E \subset (0, \varepsilon_0)$, such that the system (1.13) is reducible for $\varepsilon \in E$.

In [8], the authors found that the non-degeneracy condition is not necessary for the two-dimensional quasi-periodic system. They considered the two-dimensional nonlinear quasi-periodic system:

$$\dot{x} = Ax + f(x, t, \varepsilon), \quad x \in R^2, \tag{1.14}$$

where A has a pair of pure imaginary eigenvalues $\pm\sqrt{-1}\omega_0$ with $\omega_0 \neq 0$ satisfying the nonresonance conditions

$$|\langle k, \omega \rangle| \geq \frac{\alpha}{|k|^\tau}, \qquad |\langle k, \omega \rangle - 2\omega_0| \geq \frac{\alpha}{|k|^\tau} \tag{1.15}$$

for all $k \in Z^l \setminus \{0\}$, where $\alpha > 0$ is a small constant and $\tau > l - 1$. Assume that $f(0, t, \varepsilon) = O(\varepsilon)$ and $\partial_x f(0, t, \varepsilon) = O(\varepsilon)$ as $\varepsilon \to 0$. They proved that either of the following two results holds:

(1) for $\forall \varepsilon \in (0, \varepsilon_0)$, the system (1.14) is reducible to $\dot{y} = By + O(y)$ as $y \to 0$;

(2) there exists a nonempty Cantor subset $E \subset (0, \varepsilon_0)$, such that for $\varepsilon \in E$ the system (1.14) is reducible to $\dot{y} = By + O(y^2)$ as $y \to 0$.

Note that the result (1) happens when the eigenvalue of the perturbed matrix of A in KAM steps has nonzero real part. But the authors were interested in the equilibrium of the transformed system and obtained a small quasi-periodic solution for the original system.

Motivated by [8], in this paper we consider the Hamiltonian system and we have a better result.

2. Main Results

Theorem 2.1. *Consider the following real two-dimensional Hamiltonian system*

$$\dot{x} = J\nabla_x H, \quad x \in R^2, \tag{2.1}$$

where $H(x, t, \varepsilon) = (1/2)\beta(x_1^2 + x_2^2) + F(x, t, \varepsilon)$ *with* $\beta \neq 0$, $F(x, t, \varepsilon)$ *is analytic quasi-periodic with respect to* t *with frequencies* $\omega = (\omega_1, \omega_2, \ldots, \omega_l)$ *and real analytic with respect to* x *and* ε *on* $D(r, \rho, \varepsilon_0)$*, and*

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.2}$$

Here $\varepsilon \in (0, \varepsilon_0)$ is a small parameter. Suppose that $\partial_x F(0, t, \varepsilon) = O(\varepsilon)$ and $\partial_{xx} F(0, t, \varepsilon) = O(\varepsilon)$ as $\varepsilon \to 0$. Moreover, assume that β and ω satisfy

$$|\langle k, \omega \rangle| \geq \frac{\alpha_0}{|k|^\tau}, \tag{2.3}$$

$$\left|\langle k, \omega \rangle - 2\beta\right| \geq \frac{\alpha_0}{|k|^\tau} \tag{2.4}$$

for all $k \in Z^l \setminus \{0\}$, where $\alpha_0 > 0$ is a small constant and $\tau > l - 1$.

Then there exist a sufficiently small $\varepsilon_* \in (0, \varepsilon_0]$ and a nonempty Cantor subset $E_* \subset (0, \varepsilon_*)$, such that for $\varepsilon \in E_*$, there exists an analytic quasi-periodic symplectic transformation $x = \phi_*(t) y + \psi_*(t)$ on $D_{\rho/2}$ with the frequencies ω, which changes (2.1) *into the Hamiltonian system $\dot{y} = J \nabla_y H_*$, where $H_*(y, t, \varepsilon) = 1/2\beta_*(\varepsilon)(y_1^2 + y_2^2) + F_*(y, t, \varepsilon)$, where $F_*(y, t, \varepsilon) = O(y^3)$ as $y \to 0$. Moreover, $\operatorname{meas}((0, \varepsilon_*) \setminus E_*) = o(\varepsilon_*)$ as $\varepsilon_* \to 0$. Furthermore, $\beta_*(\varepsilon) = \beta + O(\varepsilon)$ and $\|\phi_* - Id\|_{\rho/2} + \|\psi_*\|_{\rho/2} = O(\varepsilon)$, where Id is the 2-order unit matrix.*

3. The Lemmas

The proof of Theorem 2.1 is based on KAM-iteration. The idea is the same as [7, 8]. When the non-degeneracy conditions do not happen, the small parameter ε is not involved in the nonresonance conditions. So without deleting any parameter, the KAM step will be valid. Once the non-degeneracy conditions occur at some step, they will be kept for ever and we can apply the results with the non-degeneracy conditions. Thus, after infinite KAM steps, the transformed system is convergent to a desired form.

We first give some lemmas. Let $R = (r_{ij})_{1 \leq i, j \leq 2}$ be a Hamiltonian matrix. Then we have $r_{11} + r_{22} = 0$. Define a matrix $R_A = (1/2) d J$ with $d = r_{12} - r_{21}$. Let

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ \sqrt{-1} & -\sqrt{-1} \end{pmatrix}. \tag{3.1}$$

It is easy to verify

$$\begin{gathered} B^{-1} R_A B = \frac{1}{2} \operatorname{diag}\left(\sqrt{-1} d, -\sqrt{-1} d\right), \\ B^{-1}(R - R_A) B = \frac{1}{2} \begin{pmatrix} 0 & \sigma' - \sqrt{-1}\kappa' \\ \sigma' + \sqrt{-1}\kappa' & 0 \end{pmatrix}, \end{gathered} \tag{3.2}$$

where $\sigma' = 2 r_{11}$ and $\kappa' = r_{21} + r_{12}$.

In the same way as in [7, 8], in KAM steps we need to solve linear homological equations. For this purpose we need the following lemma.

Lemma 3.1. *Consider the following equation of the matrix:*

$$\dot{P} = AP - PA + R(t), \tag{3.3}$$

where $A = \beta(\varepsilon)J$ with $|\beta(\varepsilon)| > \mu$, $\mu > 0$ is a constant, and $R(t) = (r_{ij}(t))_{1\le i,j\le 2}$ is a real analytic quasi-periodic Hamiltonian matrix on D_ρ with frequencies ω. Suppose $\beta(\varepsilon)$ and R are smooth with respect to ε and $|\varepsilon\beta'(\varepsilon)| \le c_0$ for $\varepsilon \in E \subset (0,\varepsilon_*)$, where c_0 is a constant. Note that here and below the dependence of ε is usually implied and one does not write it explicitly for simplicity. Assume $[R]_A = 0$, where $[R]$ is the average of R. Suppose that for $\varepsilon \in E$, the small divisors conditions (2.3) *and the following small divisors conditions hold:*

$$\left|\langle k,\omega\rangle - 2\beta(\varepsilon)\right| \ge \frac{\alpha}{|k|^{\tau'}}, \tag{3.4}$$

where $\tau' > 2\tau + l$. Let $0 < s < \rho$ and $\rho_1 = \rho - s$. Then there exists a unique real analytic quasi-periodic Hamiltonian matrix P(t) with frequencies ω, which solves the homological linear equation (3.3) *and satisfies*

$$\|P\|_{\rho_1} \le \frac{c}{\alpha s^{v}}\|R\|_\rho, \qquad \|\varepsilon\partial_\varepsilon P\|_{\rho_1} \le \frac{c}{\alpha^2 s^{v'}}\left(\|R\|_\rho + \|\varepsilon\partial_\varepsilon R\|_\rho\right), \tag{3.5}$$

where $v = \tau' + l$, $v' = 2\tau' + l$ and $c > 0$ is a constant.

Remark 3.2. The subset E of $(0,\varepsilon_*)$ is usually a Cantor set and so the derivative with respect to ε should be understood in the sense of Whitney [9].

Proof. Let $\overline{P} = B^{-1}PB$, where B is defined by (3.1). Similarly, define $\overline{A}$, $\overline{R}$, $\overline{R}_A$. Then (3.3) becomes

$$\dot{\overline{P}} = \overline{A}\,\overline{P} - \overline{P}\,\overline{A} + \overline{R}(t), \tag{3.6}$$

where

$$\overline{A} = \operatorname{diag}\left(\sqrt{-1}\beta, -\sqrt{-1}\beta\right). \tag{3.7}$$

Moreover, $\overline{R}_A$ and $\overline{R} - \overline{R}_A$ have the same forms as (3.2) and (3.2), respectively

Noting that $[R]_A = 0$, we have $[\overline{R}]_A = 0$. Write $\overline{P} = (\overline{p}_{ij})_{i,j}$ and $\overline{R} = (\overline{r}_{ij})_{i,j}$. Obviously, we have $\overline{r}_{11} = -\overline{r}_{22}$ with $[\overline{r}_{ii}] = 0$.

Insert the Fourier series of $\overline{P}$ and $\overline{R}$ into (3.6). Then it follows that $\overline{p}_{ii}^0 = 0$, $\overline{p}_{ii}^k = \overline{r}_{ii}^k/(\langle k,\omega\rangle\sqrt{-1})$ for $k \ne 0$, and

$$\overline{p}_{ij}^k = \frac{\overline{r}_{ij}^k}{\sqrt{-1}\left(\langle k,\omega\rangle \pm 2\beta\right)} \quad \text{for } i \ne j. \tag{3.8}$$

Since $\overline{R}$ is analytic on D_ρ, we have $|\overline{R}_k| \le \|\overline{R}\|_\rho e^{-|k|\rho}$. So it follows

$$\left\|\overline{P}\right\|_{\rho-s} \le \sum_{k\in Z^l}\left|\overline{P}_k\right| e^{|k|\,(\rho-s)} \le \frac{c}{\alpha s^{v}}\|R\|_\rho. \tag{3.9}$$

Note that here and below we always use c to indicate constants, which are independent of KAM steps.

Since A and $R(t)$ are real matrices, it is easy to obtain that $P(t)$ is also a real matrix. Obviously, it follows that $\overline{p}_{11} = -\overline{p}_{22}$ and the trace of the matrix $\overline{P}$ is zero. So is the trace of P. Thus, P is a Hamiltonian matrix.

Now we estimate $\|\varepsilon \partial P/\partial \varepsilon\|_{\rho_1}$. We only consider $\overline{p}_{12}$ and $\overline{p}_{21}$ since $\overline{p}_{11}$ and $\overline{p}_{22}$ are easy.

For $i \neq j$ we have

$$\frac{d\overline{p}_{ij}^{k}(\varepsilon)}{d\varepsilon} = \frac{\pm 2\beta'(\varepsilon)\overline{r}_{ij}^{k} - (\langle k,\omega\rangle \pm 2\beta)\overline{r}_{ij}^{k\,\prime}(\varepsilon)}{-\sqrt{-1}(\langle k,\omega\rangle \pm 2\beta)^{2}}. \tag{3.10}$$

Then, in the same way as above we obtain the estimate for $\|\varepsilon(\partial P/\partial \varepsilon)\|_{\rho_1}$. □

The following lemma will be used for the zero order term in KAM steps.

Lemma 3.3. *Consider the equation*

$$\dot{x} = Ax + g(t), \tag{3.11}$$

where A is the same as in Lemma 3.1, and g is real analytic quasi-periodic in t on D_ρ with frequencies ω and smooth with respect to ε. Suppose that the small divisors conditions (3.4) *hold. Then there exists a unique real analytic quasi-periodic solution $x(t)$ with frequencies ω, which satisfies*

$$\|x\|_{\rho_1} \leq \frac{c}{\alpha s^{v}}\|g\|_{\rho}, \qquad \left\|\varepsilon\frac{\partial x}{\partial \varepsilon}\right\|_{\rho_1} \leq \frac{c}{\alpha^2 s^{v'}}\left(\|g\|_{\rho} + \left\|\varepsilon\frac{\partial g}{\partial \varepsilon}\right\|_{\rho}\right), \tag{3.12}$$

where s,ρ_1,v,v' are defined in Lemma 3.1.

Proof. Similarly, let $\overline{x} = B^{-1}x$, $\overline{A} = B^{-1}AB$ and $\overline{g}(t) = B^{-1}g(t)$. Then (3.11) becomes

$$\dot{\overline{x}} = \overline{A}\,\overline{x} + \overline{g}(t), \tag{3.13}$$

where $\overline{A} = \operatorname{diag}(\sqrt{-1}\beta, -\sqrt{-1}\beta)$. Expanding $\overline{x} = (\overline{x}_1, \overline{x}_2)$ and $\overline{g} = (\overline{g}_1, \overline{g}_2)$ into Fourier series and using (3.13), we have

$$\overline{x}_i^{k} = \frac{\overline{g}_i^{k}}{\sqrt{-1}\left(\langle k,\omega\rangle + (-1)^{i}\beta\right)}. \tag{3.14}$$

Using $2k$ in place of k in (3.4), we have

$$\left|\langle k,\omega\rangle-\beta(\varepsilon)\right|\geq\frac{\alpha}{2|k|^{\tau'}}. \tag{3.15}$$

Thus, in the same way as the proof of Lemma 3.1, we can estimate $\|x\|_{\rho_1}$ and $\|\varepsilon\partial_\varepsilon x\|_{\rho_1}$. We omit the details. □

The following lemma is used in the estimate of Lebesgue measure for the parameter ε in the case of non-degeneracy.

Lemma 3.4. *Let $\psi(\varepsilon)=\sigma\varepsilon^N+\varepsilon^N f(\varepsilon)$, where N is a positive integer and f satisfies that $f(\varepsilon)\to 0$ as $\varepsilon\to 0$ and $|f'(\varepsilon)|\leq c$ for $\varepsilon\in(0,\varepsilon_*)$. Let $\phi(\varepsilon)=\langle k,\omega\rangle-2\beta-\psi(\varepsilon)$. Let*

$$O=\left\{\varepsilon\in(0,\varepsilon_*)\mid\left|\phi(\varepsilon)\right|\geq\frac{\alpha}{|k|^{\tau'}},\ \forall k\neq 0\right\}, \tag{3.16}$$

where $\tau'\geq 2\tau+l$, $\alpha\leq(1/2)\alpha_0$,$\sigma\neq 0$. Suppose that the small condition (2.4) *holds. Then when ε_* is sufficiently small, one has*

$$\operatorname{meas}(0,\varepsilon_*)\setminus O\leq c\frac{\alpha}{\alpha_0^2}\varepsilon_*^{N+1}, \tag{3.17}$$

where c is a constant independent of $\alpha_0,\alpha,\varepsilon_*$

Proof. Let

$$O_k=\left\{\varepsilon\in(0,\varepsilon_*)\mid\left|\phi(\varepsilon)\right|<\frac{\alpha}{|k|^{\tau'}}\right\}. \tag{3.18}$$

By assumption, if ε_* is sufficient small, we have that $|\psi(\varepsilon)|\leq 2\sigma\varepsilon^N$ and $|\psi'(\varepsilon)|\geq(\sigma/2)\varepsilon^{N-1}$ for $\varepsilon\in(0,\varepsilon_*)$. If $\varepsilon^N\leq\alpha_0/(4\sigma|k|^\tau)$, by (2.4) we have

$$\left|\phi(\varepsilon)\right|\geq\left|\langle k,\omega\rangle-2\beta\right|-\left|\psi(\varepsilon)\right|\geq\frac{\alpha}{|k|^{\tau'}}. \tag{3.19}$$

Thus, we only consider the case that $\varepsilon_*^N\geq\varepsilon^N\geq(\alpha_0/(4\sigma|k|^\tau))$. We have $|k|\geq(\alpha_0/(4\sigma\varepsilon_*^N))^{1/\tau}=K$. Since

$$\left|\phi'(\varepsilon)\right|=\left|\psi'(\varepsilon)\right|\geq\frac{\sigma}{2}\varepsilon^{N-1}\geq\frac{\alpha_0}{8|k|^\tau\varepsilon_*}, \tag{3.20}$$

we have $\text{meas}(O_k) \le ((2\alpha)/|k|^{\tau'}) \times ((8|k|^{\tau}\varepsilon_*)/\alpha_0) = (16\alpha\varepsilon_*)/(|k|^{\tau'-\tau}\alpha_0)$. So

$$\begin{aligned} \text{meas}((0,\varepsilon_*) \setminus 0) &\le \sum_{|k|\ge K} \text{meas}(O_k) \le \frac{16\alpha}{\alpha_0}\varepsilon_* \sum_{|k|\ge K} \frac{1}{|k|^{\tau'-\tau}} \\ &\le \frac{c\alpha}{\alpha_0}\varepsilon_* K^{l-\tau'+\tau} \le \frac{c\alpha}{\alpha_0^2}\varepsilon_*^{N+1}, \end{aligned} \tag{3.21}$$

where c is a constant independent of α_0, α, and ε_*. □

Below we give a lemma with the non-degeneracy conditions.

Lemma 3.5. *Consider the real nonlinear Hamiltonian system $\dot{x} = J\nabla_x H$, where*

$$H(x,t,\varepsilon) = \frac{1}{2}\beta\left(x_1^2 + x_2^2\right) + F(x,t,\varepsilon) \quad \text{with } \beta \ne 0. \tag{3.22}$$

Suppose that $F(x,t,\varepsilon)$ is analytic quasi-periodic with respect to t with frequencies ω and real analytic with respect to x and ε on $D(r,\rho,\varepsilon_0)$. Let $f(x,t,\varepsilon) = J\nabla_x F(x,t,\varepsilon)$. Assume that $f(0,t,\varepsilon) = O(\varepsilon^{2m_0})$ and $\partial_x f(0,t,\varepsilon) = O(\varepsilon^{m_0})$ as $\varepsilon \to 0$, where m_0 is a positive integer. Let $Q(t,\varepsilon) = \partial_x f(0,t,\varepsilon) = \sum_{k\ge m_0} Q_k(t)\varepsilon^k$. Suppose there exists $m_0 \le k \le 2m_0 - 1$ such that $[Q_k]_A \ne 0$ and the nonresonance conditions (2.3) *and* (2.4) *hold. Then, for sufficiently small $\varepsilon_* > 0$, there exists a nonempty Cantor subset $E_* \subset (0,\varepsilon_*)$, such that for $\varepsilon \in E_*$, there exists a quasi-periodic symplectic transformation $x = \phi_*(t)y + \psi_*(t)$ with the frequencies ω, which changes the Hamiltonian system to $\dot{y} = J\nabla_y H_*$, where*

$$H_*(y,t,\varepsilon) = \frac{1}{2}\beta_*(\varepsilon)\left(y_1^2 + y_2^2\right) + F_*(y,t,\varepsilon), \tag{3.23}$$

where $F_*(y,t,\varepsilon) = O(y^3)$ as $y \to 0$. Moreover, $\text{meas}((0,\varepsilon_*) \setminus E_*) = O(\varepsilon_*^{m_0+1})$ as $\varepsilon_* \to 0$. Furthermore, $\beta_*(\varepsilon) = \beta + O(\varepsilon^{m_0})$ and $\|\phi_* - Id\|_{\rho/2} + \|\psi_*\|_{\rho/2} = O(\varepsilon^{m_0})$.

Proof

KAM Step

The proof is based on a modified KAM iteration. In spirit, it is very similar to [7, 8]. The important thing is to make symplectic transformations so that the Hamiltonian structure can be preserved. Note that $[Q_k]_A \ne 0$ for some $m_0 \le k \le 2m_0 - 1$ is a non-degeneracy condition.

Consider the following Hamiltonian system

$$\dot{x} = Ax + f(x,t,\varepsilon), \tag{3.24}$$

where $A = \beta(\varepsilon)J$ and f is analytic quasi-periodic with respect to t with frequencies ω and real analytic with respect to x and ε on $D = D(r,\rho,\varepsilon_*)$.

Let $\|f\|_D \le \alpha r\widetilde{\varepsilon}$ and $\|\varepsilon\partial_\varepsilon f\|_D \le \alpha r\widetilde{\varepsilon}$. Let $Q(t,\varepsilon) = \partial_x f(0,t,\varepsilon)$, $g(t,\varepsilon) = f(0,t,\varepsilon)$ and

$$h(x,t,\varepsilon) = f(x,t,\varepsilon) - g(t,\varepsilon) - Q(t,\varepsilon)x. \tag{3.25}$$

Then h is the higher-order term of f. Moreover, the matrix $Q(t,\varepsilon)$ is Hamiltonian. Let $[Q]_A = \widehat{\beta}(\varepsilon)J$.

The system (3.24) is written as

$$\dot{x} = (A_+ + R(t,\varepsilon))x + g(t,\varepsilon) + h(x,t,\varepsilon), \tag{3.26}$$

where $A_+ = A + [Q]_A = \beta_+(\varepsilon)J$ and $R = Q - [Q]_A$. By assumption we have

$$\|g\|_\rho \le \alpha r\widetilde{\varepsilon}, \qquad \|Q\|_\rho \le \alpha\widetilde{\varepsilon}, \qquad \|h\|_D \le 3\alpha r\widetilde{\varepsilon}. \tag{3.27}$$

Moreover, we have

$$\|\varepsilon\partial_\varepsilon g\|_\rho \le \alpha r\widetilde{\varepsilon}, \qquad \|\varepsilon\partial_\varepsilon Q\|_\rho \le \alpha\widetilde{\varepsilon}, \qquad \|\varepsilon\partial_\varepsilon h\|_D \le 3\alpha r\widetilde{\varepsilon}. \tag{3.28}$$

Now we want to construct the symplectic change of variables $x = T'y = e^{P(t)}y$ to (3.26), where P is a Hamiltonian matrix to be defined later. Then we have

$$\begin{aligned}\dot{y} =& \left(e^{-P}(A_+ + R - \dot{P})e^P + e^{-P}\left(\dot{P}e^P - \frac{d}{dt}e^{P(t)}\right)\right)y \\ &+ e^{-P}g(t,\varepsilon) + e^{-P}h\left(e^P y,t,\varepsilon\right).\end{aligned} \tag{3.29}$$

Let $W = e^P - I - P$ and $\widetilde{W} = e^{-P} - I - P$. Then the system (3.29) becomes

$$\dot{y} = (A_+ + R - \dot{P} + A_+P - PA_+)y + Q'y + e^{-P}g(t,\varepsilon) + e^{-P}h\left(e^P y,t,\varepsilon\right), \tag{3.30}$$

where

$$\begin{aligned}Q' =& - P(R - \dot{P}) + (R - \dot{P})P - P(A_+ + R - \dot{P})P \\ &- P(A_+ + R - \dot{P})W + (A_+ + R - \dot{P})W \\ &+ \widetilde{W}(A_+ + R - \dot{P})e^P + e^{-P}\left(\dot{P}e^P - \frac{d}{dt}e^P\right).\end{aligned} \tag{3.31}$$

We would like to have

$$\dot{P} - A_+P + PA_+ = R, \tag{3.32}$$

where $R = Q - [Q]_A$. Suppose the small divisors conditions (2.3) hold. Let $E_+ \subset (0, \varepsilon_*)$ be a subset such that for $\varepsilon \in E_+$ the small divisors conditions hold:

$$\left|\langle k, \omega\rangle - 2\beta_+(\varepsilon)\right| \geq \frac{\alpha_+}{|k|^{\tau'}}, \quad \forall k \in Z^l \setminus \{0\}, \tag{3.33}$$

where $\tau' > 2\tau + l$. By Lemma 3.1, we have a quasi-periodic Hamiltonian matrix $P(t)$ with frequencies ω to solve the above equation with the following estimates:

$$\begin{gathered} \|P\|_{\rho-s} \leq \frac{c\|Q\|_\rho}{\alpha_+ s^v} \leq \frac{c\tilde{\varepsilon}}{s^v}, \\ \left\|\varepsilon\frac{\partial P}{\partial \varepsilon}\right\|_{\rho-s} \leq \frac{c}{\alpha_+^2 s^{v'}}\left(\|Q\|_\rho + \left\|\varepsilon\frac{\partial Q}{\partial \varepsilon}\right\|_\rho\right) \leq \frac{c\tilde{\varepsilon}}{\alpha_+ s^{v'}}, \end{gathered} \tag{3.34}$$

where $v = \tau' + l,\ v' = 2\tau' + l$ and $c > 0$ is a constant. Then the system (3.30) becomes

$$\dot{y} = A_+ y + f'(y, t, \varepsilon), \tag{3.35}$$

where $f' = Q'y + e^{-P}g(t, \varepsilon) + e^{-P}h(e^P y, t, \varepsilon)$.

By Lemma 3.3, let us denote by $\underline{x}$ the solution of $\dot{x} = A_+ x + g'(t, \varepsilon)$ on $D_{\rho-2s}$, where $g' = e^{-P}g(t, \varepsilon)$. Then, by Lemma 3.3 we have

$$\begin{gathered} \|\underline{x}\|_{\rho-2s} \leq \frac{c\|g\|_{\rho-s}}{\alpha_+ s^v} \leq \frac{cr\tilde{\varepsilon}}{s^v}, \\ \left\|\varepsilon\frac{\partial \underline{x}}{\partial \varepsilon}\right\|_{\rho-2s} \leq \frac{c}{\alpha_+^2 s^{v'}}\left(\|g\|_{\rho-s} + \left\|\varepsilon\frac{\partial g}{\partial \varepsilon}\right\|_{\rho-s}\right) \leq \frac{cr\tilde{\varepsilon}}{\alpha_+ s^{v'}}. \end{gathered} \tag{3.36}$$

Under the symplectic change of variables $y = T''x_+ = \underline{x} + x_+$, the Hamiltonian system (3.35) is changed to

$$\dot{x}_+ = A_+ x_+ + f_+(x_+, t, \varepsilon), \tag{3.37}$$

where $A_+ = \beta_+ J$ and

$$f_+ = Q' \cdot T'' + e^{-P} h \circ T' \circ T''. \tag{3.38}$$

Let the symplectic transformation $T = T' \circ T''$. Then $x = Tx_+ = \phi(t)x_+ + \psi(t)$, where $\phi(t) = e^{P(t)}$ and $\psi(t) = e^{P(t)}\underline{x}(t)$. It is easy to obtain that if $\|P\|_{\rho-2s} \leq 1/2$, then

$$\|\phi - I\|_{\rho-2s} \leq \frac{c\widetilde{\varepsilon}}{s^{v}}, \qquad \|\varepsilon\partial_\varepsilon\phi\|_{\rho-2s} \leq \frac{c\widetilde{\varepsilon}}{\alpha_+ s^{v'}},$$
$$\|\psi\|_{\rho-2s} \leq \frac{cr\widetilde{\varepsilon}}{s^{v}}, \qquad \|\varepsilon\partial_\varepsilon\psi\|_{\rho-2s} \leq \frac{cr\widetilde{\varepsilon}}{\alpha_+ s^{v'}}. \tag{3.39}$$

Under the symplectic change of variables $x = Tx_+$, the Hamiltonian system (3.24) becomes (3.37).

Below we give the estimates for A_+ and f_+. Obviously, it follows that $A_+(\varepsilon) - A = [Q]_A = \widehat{\beta}(\varepsilon)J$ and

$$\left|\beta_+(\varepsilon) - \beta(\varepsilon)\right| = \left|\widehat{\beta}(\varepsilon)\right| \leq c\alpha\widetilde{\varepsilon}, \qquad \left|\varepsilon\left(\beta'_+(\varepsilon) - \beta'(\varepsilon)\right)\right| = \left|\varepsilon\widehat{\beta}'(\varepsilon)\right| \leq c\alpha\widetilde{\varepsilon}. \tag{3.40}$$

By (3.38) we have

$$f_+(x_+, t, \varepsilon) = Q'(t)\left(x_+ + \underline{x}(t)\right) + e^{-P(t)}h\left(e^{P(t)}\left(x_+ + \underline{x}(t)\right), t, \varepsilon\right). \tag{3.41}$$

Let $\rho_+ = \rho - 2s$, and $r_+ = \eta r$ with $\eta \leq 1/8$. If $c\widetilde{\varepsilon}/\alpha_+ s^{v+v'} \leq \eta$, it follows that $\|\underline{x}\|_{\rho-2s} \leq (1/8)r$. Let $D_+ = D(r_+, s_+, \varepsilon_*)$. Note that Q' and h only consist of high-order terms of P and x, respectively. It is easy to see $|e^{P(t)}(x_+ + \underline{x}(t))| \leq 4\eta r \leq r$. By all the estimates (3.27), (3.28), (3.34), and (3.36), and using usual technique of KAM estimate, we have

$$\|f_+\|_{D_+} \leq \frac{c\widetilde{\varepsilon}^2}{s^{2v}}\eta r + c\alpha r\widetilde{\varepsilon}\eta^2 \leq \left(\frac{c\widetilde{\varepsilon}}{s^{2v}} + c\alpha\eta\right)r_+\widetilde{\varepsilon},$$
$$\|\varepsilon\partial_\varepsilon f_+\|_{D_+} \leq \frac{c\widetilde{\varepsilon}^2}{\alpha_+ s^{v+v'}}\eta r + c\alpha r\widetilde{\varepsilon}\eta^2 \leq \left(\frac{c\widetilde{\varepsilon}}{\alpha s^{v+v'}} + c\alpha\eta\right)r_+\widetilde{\varepsilon}. \tag{3.42}$$

Let $\alpha_+ = \alpha/2$ and $\eta = c\widetilde{\varepsilon}/(\alpha^2 s^{v+v'})$. Then we have

$$\|f_+\|_{D_+} \leq c\alpha_+ r_+\eta\widetilde{\varepsilon} = \alpha_+ r_+\widetilde{\varepsilon}_+, \qquad \widetilde{\varepsilon}_+ = c\eta\widetilde{\varepsilon}. \tag{3.43}$$

Similarly, we have

$$\|\varepsilon\partial_\varepsilon f_+\|_{D_+} \leq \alpha_+ r_+\widetilde{\varepsilon}_+. \tag{3.44}$$

Note that KAM steps only make sense for the small parameter ε satisfying small divisors conditions. However, by Whitney's extension theorem, for convenience all the functions are supposed to be defined for ε on $[0, \varepsilon_*]$.

KAM Iteration

Now we can give the iteration procedure in the same way as in [7] and prove its convergence.

At the initial step, let $f_0 = f$. Let $f(x,t,\varepsilon) = f(0,t,\varepsilon) + \partial_x f(0,t,\varepsilon)x + h(x,t,\varepsilon)$. By assumption, if ε_* is sufficiently small, we have that for all $\varepsilon \in [0,\varepsilon_*]$

$$\begin{gathered} \left|f(0,t,\varepsilon)\right| \le c\varepsilon^{2m_0}, \qquad \left|\partial_x f(0,t,\varepsilon)\right| \le c\varepsilon^{m_0}, \\ \left|\varepsilon\partial_\varepsilon f(0,t,\varepsilon)\right| \le c\varepsilon^{2m_0}, \qquad \left|\varepsilon\partial_\varepsilon\partial_x f(0,t,\varepsilon)\right| \le c\varepsilon^{m_0}. \end{gathered} \tag{3.45}$$

Moreover,

$$|h(x,t,\varepsilon)| \le c|x|^2, \quad |\varepsilon\partial_\varepsilon h(x,t,\varepsilon)| \le c|x|^2, \quad \forall |x| \le \varepsilon^{m_0}, \quad \forall \varepsilon \in [0,\varepsilon_*]. \tag{3.46}$$

Let $r_0 = \varepsilon^{m_0}$, $\rho_0 = \rho$, $s_0 = \rho_0/8$, $D_0 = D(r_0,\rho_0,\varepsilon_*)$, and $\tilde{\varepsilon}_0 = c\varepsilon^{m_0}/\alpha_0$. Then we have

$$\left|f_0\right|_{D_0} \le \alpha_0 r_0 \tilde{\varepsilon}_0, \qquad \left|\varepsilon\partial_\varepsilon f_0\right|_{D_0} \le \alpha_0 r_0 \tilde{\varepsilon}_0. \tag{3.47}$$

For $n \ge 1$, let

$$\begin{gathered} \alpha_n = \frac{\alpha_{n-1}}{2}, \qquad s_n = \frac{s_{n-1}}{2}, \qquad \rho_n = \rho_{n-1} - 2s_{n-1}, \\ \eta_{n-1} = \frac{c\tilde{\varepsilon}_{n-1}}{\alpha_{n-1}^2 s_{n-1}^{v+v'}}, \qquad r_n = \eta_{n-1} r_{n-1}, \qquad \tilde{\varepsilon}_n = c\eta_{n-1}\tilde{\varepsilon}_{n-1}. \end{gathered} \tag{3.48}$$

Then we have a sequence of quasi-periodic symplectic transformations $\{T_n\}$ satisfying $T_n x = \phi_n(t)x + \varphi_n(t)$ with

$$\left\|\phi_n - I\right\|_{\rho_{n+1}} \le \frac{c\tilde{\varepsilon}_n}{s_n^v}, \qquad \left\|\varphi_n\right\|_{\rho_{n+1}} \le \frac{c r_n \tilde{\varepsilon}_n}{s_n^v}. \tag{3.49}$$

Let $T^n = T_0 \circ T_1 \cdots \circ T_{n-1}$. Then under the transformation $x = T^n y$ the Hamiltonian system $\dot{x} = A_0 x + f_0(x,t,\varepsilon)$ is changed to $\dot{y} = A_n y + f_n(y,t,\varepsilon)$.

Moreover, $A_n(\varepsilon) = \beta_n(\varepsilon)J$ satisfies $A_{n+1} - A_n = [Q_n]_A$ and

$$\left|\beta_{n+1}(\varepsilon) - \beta_n(\varepsilon)\right| \le c\alpha_n\tilde{\varepsilon}_n, \qquad \left|\varepsilon\left(\beta'_{n+1}(\varepsilon) - \beta'_n(\varepsilon)\right)\right| \le c\alpha_n\tilde{\varepsilon}_n, \tag{3.50}$$

$$\left\|f_n\right\|_{D_n} \le \alpha_n r_n \tilde{\varepsilon}_n. \tag{3.51}$$

Convergence

By the above definitions we have $\eta_n/\eta_{n-1} = c\tilde{\varepsilon}_n/\tilde{\varepsilon}_{n-1} = c\eta_{n-1}$. Thus, we have $\eta_n \le c\eta_{n-1}^2$ and so $c\eta_n \le (c\eta_{n-1})^2 \le (c\eta_0)^{2^n}$. Note that $\eta_0 = c\tilde{\varepsilon}_0/(\alpha_0^2 s_0^{v+v'}) \le c\varepsilon^{m_0}/(\alpha_0^2\rho_0^{v+v'})$. Suppose that ε_* is sufficiently small such that for $0 < \varepsilon < \varepsilon_*$ we have $c\eta_0 \le 1/2$. T_n are affine, so are T^n

with $T^n x = \phi^n(t)x + \psi^n(t)$. By the estimates (3.49) it is easy to prove that $\phi^n(t)$ and $\psi^n(t)$ are convergent and so T^n is actually convergent on the domain $D(r/2, \rho/2)$. Let $T^n \to T_*$ and $T_* x = \phi_*(t)x + \psi_*(t)$. It is easy to see that the estimates for ϕ_* and ψ_* in Theorem 2.1 hold.

Using the estimate for f_n and Cauchy's estimate, we have $|f_n(0,t,\varepsilon)| \le \alpha_n r_n \widetilde{\varepsilon}_n \to 0$ and $|\partial_x f_n(0,t,\varepsilon)| \le \alpha_n \widetilde{\varepsilon}_n \to 0$ as $n \to \infty$. Let $f_n \to f_*$. Then it follows that $f_*(x,t,\varepsilon) = O(x^2)$.

By the estimates (3.50) for β_n we have $\beta_n \to \beta_*$. Thus, by the quasi-periodic symplectic transformation $x = T_* y$, the original system is changed to $\dot{y} = A_* y + f_*(y,t,\varepsilon)$ with $A_* = \beta_* J$.

Estimate of Measure

Let

$$E_n = \left\{ \varepsilon \in (0, \varepsilon_*) \mid \left| \langle \omega, k \rangle - 2\beta_n(\varepsilon) \right| \ge \frac{\alpha_n}{|k|^{\tau'}} \right\}. \tag{3.52}$$

Note that $\beta_n = \beta_1 + \psi$, where $\psi = \sum_{j=1}^{n-1} \beta_{j+1} - \beta_j, \beta_1 = \beta + \widehat{\beta}$, and $\widehat{\beta} J = [Q]_A$. Note that $\widetilde{\varepsilon}_1 = c\widetilde{\varepsilon}_0^2/(\alpha_0^2 s_0^{v+v'})$ and $\widetilde{\varepsilon}_0 = c\varepsilon^{m_0}/\alpha_0$. By the estimates (3.50), we have $\psi(\varepsilon) = O(\varepsilon^{2m_0})$ and $\varepsilon\psi'(\varepsilon) = O(\varepsilon^{2m_0})$. By assumption, $[Q]_A$ is analytic with respect to ε and there exists $m_0 \le N \le 2m_0 - 1$ such that $[Q]_A = \delta\varepsilon^N + O(\varepsilon^{N+1})$ with $\delta \ne 0$. Thus, $\beta_1(\varepsilon) = \beta + \delta\varepsilon^N + O(\varepsilon^{N+1})$. By Lemma 3.4, we have $\text{meas}((0,\varepsilon_*) - E_n) \le c(\alpha_n/\alpha_0^2)\varepsilon_*^{N+1}$. Let $E_* = \bigcap_{n\ge1} E_n$. By $\alpha_n = \alpha_0/2^n$, it follows that $\text{meas}((0,\varepsilon_*) - E_*) \le c\varepsilon_*^{N+1}/\alpha_0$. Thus Lemma 3.5 is proved. □

4. Proof of Theorem 2.1

As we pointed previously, once the non-degeneracy conditions are satisfied in some KAM step, the proof is complete by Lemma 3.5. If the non-degeneracy conditions never happen, the small parameter ε does not involve into the small divisors and so the systems are analytic in ε. To prepare for KAM iteration, we need a preliminary step to change the original system to a suitable form.

Preliminary Step

We first give the preliminary KAM step. Let

$$\dot{x} = Ax + f(x,t,\varepsilon), \tag{4.1}$$

where $A = \beta J$ and $f = J\nabla_x F$. By Lemma 3.3, denote by $\underline{x}$ the solution of $\dot{x} = Ax + f(0,t,\varepsilon)$ on $D_{3\rho/4}$. Under the change of variables $x = T_0 x_+ = \underline{x} + x_+$, the Hamiltonian system (2.1) becomes

$$\dot{x}_+ = Ax_+ + f_1(x_+,t,\varepsilon), \tag{4.2}$$

where $f_1(x_+,t,\varepsilon) = f(\underline{x} + x_+,t,\varepsilon) - f(0,t,\varepsilon)$ satisfying $f_1(0,t,\varepsilon) = O(\varepsilon^2)$ and $\partial_{x_+} f_1(0,t,\varepsilon) = O(\varepsilon)$.

KAM Step

The next step is almost the same as the proof of Lemma 3.5 and even more simple. In the KAM iteration, we only need consider the case that the non-degeneracy condition never happens. In this case, the normal frequency has no shift, which is equivalent to $A_n = A$ for all $n \geq 1$ in the KAM steps in the above nondegenerate case. Moreover, the small divisors conditions are always the initial ones as (2.3) and (2.4) and are independent of the small parameter ε. Thus, we need not delete any parameter. Moreover, the analyticity in ε remains in the KAM steps, which makes the estimate easier. At the first step, we consider $\dot{x} = Ax + f_1(x, t, \varepsilon)$. In the same way as the case of nondegenerate case, let $r_1 = \varepsilon, \rho_1 = 3\rho/4, \varepsilon_1 = \varepsilon_0, D_1 = D(r_1, \rho_1, \varepsilon_1)$, and $\widetilde{\varepsilon}_1 = c\varepsilon/\alpha_0$. Then we have $\|f_1\|_{D_1} \leq \alpha_0 r_1 \widetilde{\varepsilon}_1$.

At nth step, we consider the Hamiltonian system

$$\dot{x} = Ax + f_n(x, t, \varepsilon), \tag{4.3}$$

where f_n is analytic quasi-periodic with respect to t with frequencies ω and real analytic with respect to x and ε on $D_n = D(r_n, \rho_n, \varepsilon_n)$. Moreover, $\|f_n\|_{D_n} \leq \alpha_0 r_n \widetilde{\varepsilon}_n$. Suppose

$$Q_n(t, \varepsilon) = \partial_x f_n(0, t, \varepsilon) = O\left(\varepsilon^{2^{n-1}}\right), \quad f_n(0, t, \varepsilon) = O\left(\varepsilon^{2^n}\right). \tag{4.4}$$

Since Q_n is analytic with respect to ε, it follows that

$$Q_n = \sum_{k=2^{n-1}}^{\infty} Q_n^k \varepsilon^k. \tag{4.5}$$

Truncating the above power series of ε, we let

$$R_n(t, \varepsilon) = \sum_{k=2^{n-1}}^{2^n-1} Q_n^k \varepsilon^k, \qquad \widetilde{Q}_n = Q_n - R_n. \tag{4.6}$$

Because the non-degeneracy conditions do not happen in KAM steps, we must have $[R_n]_A = 0$. In the same way as the proof of Lemma 3.5, we have a quasi-periodic symplectic transformation T_n with $T_n x = \phi_n(t)x + \psi_n(t)$ satisfying (3.49). Let $T^n = T_1 \circ T_2 \cdots \circ T_{n-1}$.

By the transformation $x = T^n y$, the system (4.3) is changed to

$$\dot{y} = Ay + f_{n+1}(y, t, \varepsilon), \tag{4.7}$$

where $f_{n+1} = \widetilde{Q}_n \cdot T_n'' + Q_n' \cdot T_n'' + e^{-P_n} \cdot h_n \circ T_n = \widetilde{Q}_n(\underline{x}_n + y) + Q_n'(\underline{x}_n + y) + e^{-P_n} h_n(e^{P_n}(\underline{x}_n + y))$.

The last two terms can be estimated similarly as those of (3.41). Note that

$$\widetilde{Q}_n = Q_n - R_n = \sum_{k \geq 2^n} Q_n^k \varepsilon^k \tag{4.8}$$

only consists of the higher order terms of ε. So, in the same way as [8, 10], we use the technique of shriek of the domain interval of ε to estimate the first term.

Let $r_1 = \varepsilon, \rho_1 = 3\rho/4, \varepsilon_1 = \varepsilon_0$ and $s_1 = \rho/16$.

Define $s_{n+1} = s_n/2, \rho_{n+1} = \rho_n - 2s_n, \eta_n = (1/8)e^{-(4/3)^n}$, $r_{n+1} = \eta_n r_n$, $\delta_n = 1 - (2/3)^n$ and $\varepsilon_{n+1} = \delta_n \varepsilon_n$. Let $D_{n+1} = D(r_{n+1}, \rho_{n+1}, \varepsilon_{n+1})$.

If $c\tilde{\varepsilon}_n/s_n^{2v} \le \eta_n < (1/8)$, it follows that

$$\|f_{n+1}\|_{D_{n+1}} \le \left(\alpha_0 \tilde{\varepsilon}_n e^{-(4/3)^n} + \left(\frac{c\tilde{\varepsilon}_n}{s_n^v}\right)^2\right)\eta_n r_n + c\alpha_0 r_n \tilde{\varepsilon}_n \eta_n^2 \le \alpha_0 r_{n+1}\tilde{\varepsilon}_{n+1}, \tag{4.9}$$

where $\tilde{\varepsilon}_{n+1} = c\eta_n\tilde{\varepsilon}_n$. Moreover, it is easy to see

$$\partial_x f_{n+1}(0,t,\varepsilon) = O\left(\varepsilon^{2^n}\right), \quad f_{n+1}(0,t,\varepsilon) = O\left(\varepsilon^{2^{n+1}}\right). \tag{4.10}$$

Now we verify $c\tilde{\varepsilon}_n/s_n^{2v} \le \eta_n < 1/8$. Let $G_n = c\tilde{\varepsilon}_n/s_n^{2v}$. By $G_n = ce^{-(4/3)^{n-1}}16^v G_{n-1}$, it follows that

$$G_n = (c16^v)^{n-1} e^{-[(4/3)^{n-1} + (4/3)^{n-2} + \cdots + (4/3)^1]} G_1 = (c16^v)^{n-1} e^4 e^{-4(4/3)^{n-1}} G_1. \tag{4.11}$$

Note that $G_1 = c\tilde{\varepsilon}_1/s_1^{2v}$. If $\tilde{\varepsilon}_1$ is sufficiently small, we have $c\tilde{\varepsilon}_n/s_n^{2v} = G_n \le \eta_n$.

Note that $(cr_n\tilde{\varepsilon}_n/s_n^v) \to 0$ and $(c\tilde{\varepsilon}_n/(\eta_n s_n^v)) \to 0$ as $n \to \infty$, and $\tilde{\varepsilon}_n \le cs_n^{2v}G_n$. Let $\varepsilon_* = \prod_{n\ge1}(1-(2/3)^n)\varepsilon_0$. Thus, in the same way as before we can prove the convergence of the KAM iteration for all $\varepsilon \in (0, \varepsilon_*)$ and obtain the result of Theorem 2.1. We omit the details.

Remark 4.1. As suggested by the referee, we can also introduce an outer parameter to consider the Hamiltonian function $H(x,t,\varepsilon) = \langle \omega, I\rangle + (1/2)(\beta_* + \sigma(\varepsilon))(x_1^2 + x_2^2) + F(x,t,\varepsilon)$, where (θ, I) are the angle variable and the action variable and $x = (x_1, x_2)$ are a pair of normal variables. In the same way as in [11], $\sigma(\varepsilon)$ is the modified term of the normal frequency. Then by some technique as in [11–13], we can also prove Theorem 2.1.

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