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Research Article

Boundedness of a Class of Sublinear Operators and Their Commutators on Generalized Morrey Spaces

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The authors study the boundedness for a large class of sublinear operator T generated by Calderón-Zygmund operator on generalized Morrey spaces $M_{p,\varphi}$. As an application of this result, the boundedness of the commutator of sublinear operators T_a on generalized Morrey spaces is obtained. In the case $a \in \mathrm{BMO}(\mathbb{R}^n)$, $1 < p < \infty$ and T_a is a sublinear operator, we find the sufficient conditions on the pair (φ_1, φ_2) which ensures the boundedness of the operator T_a from one generalized Morrey space M_{p,φ_1} to another M_{p,φ_2}. In all cases, the conditions for the boundedness of T_a are given in terms of Zygmund-type integral inequalities on (φ_1, φ_2), which do not assume any assumption on monotonicity of φ_1, φ_2 in r. Conditions of these theorems are satisfied by many important operators in analysis, in particular pseudodifferential operators, Littlewood-Paley operator, Marcinkiewicz operator, and Bochner-Riesz operator.

1. Introduction

For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x,r)$ the open ball centered at x of radius r, and by ${}^{\complement}B(x,r)$ denote its complement. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$.

Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. The Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y)| dy. \tag{1.1}$$

Let K be a Calderón-Zygmund singular integral operator, briefly a Calderón-Zygmund operator, that is, a linear operator bounded from $L_2(\mathbb{R}^n)$ to $L_2(\mathbb{R}^n)$ taking all infinitely continuously differentiable functions f with compact support to the functions $f \in L_1^{\text{loc}}(\mathbb{R}^n)$ represented by

$$Kf(x) = \int_{\mathbb{R}^n} k(x,y) f(y) dy \quad x \notin \operatorname{supp} f. \tag{1.2}$$

Such operators were introduced in [1]. Here, $k(x,y)$ is a continuous function away from the diagonal which satisfies the standard estimates: there exist $c_1 > 0$ and $0 < \varepsilon \leq 1$ such that

$$|k(x,y)| \leq c_1 |x-y|^{-n}, \tag{1.3}$$

for all $x, y \in \mathbb{R}^n$, $x \neq y$, and

$$|k(x,y) - k(x',y)| + |k(y,x) - k(y,x')| \leq c_1 \left(\frac{|x-x'|}{|x-y|} \right)^{\varepsilon} |x-y|^{-n}, \tag{1.4}$$

whenever $2|x-x'| \leq |x-y|$.

It is well known that maximal operator and Calderón-Zygmund operators play an important role in harmonic analysis (see [2–6]).

Suppose that T represents a linear or a sublinear operator, which satisfies that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin \operatorname{supp} f$

$$|Tf(x)| \leq c_0 \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy, \tag{1.5}$$

where c_0 is independent of f and x.

For a function a, suppose that the commutator operator T_a represents a linear or a sublinear operator, which satisfies that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin \operatorname{supp} f$

$$|T_a f(x)| \leq c_0 \int_{\mathbb{R}^n} |a(x) - a(y)| |x-y|^{-n} |f(y)| dy, \tag{1.6}$$

where c_0 is independent of f and x.

We point out that the condition (1.5) was first introduced by Soria and Weiss in [7]. The condition (1.5) are satisfied by many interesting operators in harmonic analysis, such as the Calderón-Zygmund operators, Carleson's maximal operator, Hardy-Littlewood maximal operator, C. Fefferman's singular multipliers, R. Fefferman's singular integrals, Ricci-Stein's oscillatory singular integrals, and the Bochner-Riesz means (see [7, 8] for details).

In this work, we prove the boundedness of the sublinear operator T satisfies the condition (1.5) generated by Calderón-Zygmund operator from one generalized Morrey space M_{p,φ_1} to another M_{p,φ_2}, $1 < p < \infty$, and from M_{1,φ_1} to the weak space WM_{1,φ_2}. In the case $a \in \mathrm{BMO}(\mathbb{R}^n)$, $1 < p < \infty$ and the commutator operator T_a is a sublinear operator, we find the sufficient conditions on the pair (φ_1, φ_2) which ensures the boundedness of the operators T_a from M_{p,φ_1} to M_{p,φ_2}. Finally, as applications, we apply this result to several particular operators such as the pseudodifferential operators, Littlewood-Paley operator, Marcinkiewicz operator, and Bochner-Riesz operator.

By $A \lesssim B$, we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Morrey Spaces

The classical Morrey spaces $M_{p,\lambda}$ were originally introduced by Morrey Jr. in [9] to study the local behavior of solutions to second-order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [9, 10].

We denote by $M_{p,\lambda} \equiv M_{p,\lambda}(\mathbb{R}^n)$ the Morrey space, the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{p,\lambda}} \equiv \|f\|_{M_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\ r>0} r^{-\lambda/p} \|f\|_{L_p(B(x,r))}, \tag{2.1}$$

where $1 \leq p < \infty$ and $0 \leq \lambda \leq n$.

Note that $M_{p,0} = L_p(\mathbb{R}^n)$ and $M_{p,n} = L_\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $M_{p,\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

We also denote by $WM_{p,\lambda} \equiv WM_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\lambda}} \equiv \|f\|_{WM_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\ r>0} r^{-\lambda/p} \|f\|_{WL_p(B(x,r))} < \infty, \tag{2.2}$$

where $WL_p(B(x,r))$ denotes the weak L_p-space of measurable functions f for which

$$\begin{aligned} \|f\|_{WL_p(B(x,r))} &\equiv \|f\chi_{B(x,r)}\|_{WL_p(\mathbb{R}^n)} \\ &= \sup_{t>0} t\left|\{y \in B(x,r) : \ |f(y)| > t\}\right|^{1/p} \\ &= \sup_{0<t\leq|B(x,r)|} t^{1/p}\left(f\chi_{B(x,r)}\right)^*(t) < \infty. \end{aligned} \tag{2.3}$$

Here, g^* denotes the nonincreasing rearrangement of a function g.

Chiarenza and Frasca [11] studied the boundedness of the maximal operator M in these spaces. Their result can be summarized as follows.

Theorem 2.1. *Let $1 \leq p < \infty$ and $0 \leq \lambda < n$. Then, for $p > 1$ the operator M is bounded on $M_{p,\lambda}$ and for $p = 1 M$ is bounded from $M_{1,\lambda}$ to $WM_{1,\lambda}$.*

Di Fazio and Ragusa [12] studied the boundedness of the Calderón-Zygmund operators in Morrey spaces, and their results imply the following statement for Calderón-Zygmund operators K.

Theorem 2.2. *Let $1 \leq p < \infty$, $0 < \lambda < n$. Then, for $1 < p < \infty$ Calderón-Zygmund operator K is bounded on $M_{p,\lambda}$ and for $p = 1 K$ is bounded from $M_{1,\lambda}$ to $WM_{1,\lambda}$.*

Note that Theorem 2.2 was proved by Peetre [10] in the case of the classical Calderón-Zygmund singular integral operators.

3. Generalized Morrey Spaces

We find it convenient to define the generalized Morrey spaces in the form as follows.

Definition 3.1. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{p,\varphi}} \equiv \|f\|_{M_{p,\varphi}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} \varphi(x,r)^{-1} \, |B(x,r)|^{-1/p} \|f\|_{L_p(B(x,r))}. \tag{3.1}$$

Also, by $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} \equiv \|f\|_{WM_{p,\varphi}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} \varphi(x,r)^{-1} \, |B(x,r)|^{-1/p} \|f\|_{WL_p(B(x,r))} < \infty. \tag{3.2}$$

According to this definition, we recover the spaces $M_{p,\lambda}$ and $WM_{p,\lambda}$ under the choice $\varphi(x,r) = r^{(\lambda-n)/p}$:

$$\begin{aligned} M_{p,\lambda} &= M_{p,\varphi}\big|_{\varphi(x,r)=r^{(\lambda-n)/p}}, \\ WM_{p,\lambda} &= WM_{p,\varphi}\big|_{\varphi(x,r)=r^{(\lambda-n)/p}}. \end{aligned} \tag{3.3}$$

In [13–19], there were obtained sufficient conditions on φ_1 and φ_2 for the boundedness of the maximal operator M and Calderón-Zygmund operator K from M_{p,φ_1} to M_{p,φ_2}, $1 < p < \infty$ (see also [20–23]). In [19], the following condition was imposed on $\varphi(x,r)$:

$$c^{-1}\varphi(x,r) \le \varphi(x,t) \le c\, \varphi(x,r), \tag{3.4}$$

whenever $r \le t \le 2r$, where c ($\ge$1) does not depend on t, r and $x \in \mathbb{R}^n$, jointly with the condition

$$\int_r^\infty \varphi(x,t)^p \frac{dt}{t} \le C\varphi(x,r)^p, \tag{3.5}$$

for the sublinear operator T satisfies the condition (1.5), where C (>0) does not depend on r and $x \in \mathbb{R}^n$.

4. Sublinear Operators Generated by Calderón-Zygmund Operators in the Spaces $M_{p,\varphi}$

In [24] (see, also [25, 26]), the following statements was proved by sublinear operator T satisfies the condition (1.5), containing the result in [18, 19].

Theorem 4.1. *Let $1 < p < \infty$ and $\varphi(x,r)$ satisfy conditions* (3.4) *and* (3.5). *Let T be a sublinear operator satisfies the condition* (1.5) *and bounded on $L_p(\mathbb{R}^n)$. Then, the operator T is bounded on $M_{p,\varphi}$.*

The following statements, containing results obtained in [18, 19] was proved in [13] (see also [14, 15]).

Theorem 4.2. *Let $1 \le p < \infty$, and let (φ_1, φ_2) satisfy the condition*

$$\int_t^{\infty} \varphi_1(x,r)\frac{dr}{r} \le C\varphi_2(x,t), \tag{4.1}$$

where C does not depend on x and t. Then, the operators M and K are bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{1,φ_2}.

In this section, we are going to use the following statement on the boundedness of the Hardy operator:

$$(Hg)(t) := \frac{1}{t}\int_0^t g(r)dr, \quad 0 < t < \infty. \tag{4.2}$$

Theorem 4.3 (see [27]). *The inequality*

$$\operatorname*{ess\,sup}_{t>0} w(t)Hg(t) \le c \operatorname*{ess\,sup}_{t>0} v(t)g(t) \tag{4.3}$$

holds for all nonnegative and nonincreasing g on $(0,\infty)$ if and only if

$$A := \sup_{t>0}\frac{w(t)}{t}\int_0^t \frac{dr}{\operatorname{ess\,sup}_{0<s<r} v(s)} < \infty, \tag{4.4}$$

and $c \approx A$.

Lemma 4.4. *Let $1 \le p < \infty$, T be a sublinear operator which satisfies the condition* (1.5) *bounded on $L_p(\mathbb{R}^n)$ for $p > 1$ and bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$.*

Then, for $1 < p < \infty$,

$$\|Tf\|_{L_p(B)} \lesssim r^{n/p}\int_{2r}^{\infty} t^{-n/p-1}\|f\|_{L_p(B(x_0,t))}dt \tag{4.5}$$

holds for any ball $B = B(x_0, r)$ and for all $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$.

Moreover, for $p = 1$,

$$\|Tf\|_{WL_1(B)} \lesssim r^n \int_{2r}^{\infty} t^{-n-1} \|f\|_{L_1(B(x_0,t))} dt \tag{4.6}$$

holds for any ball $B = B(x_0, r)$ and for all $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Let $p \in (1, \infty)$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r, $2B = B(x_0, 2r)$. We represent f as

$$f = f_1 + f_2, \qquad f_1(y) = f(y)\chi_{2B}(y), \qquad f_2(y) = f(y)\chi_{^{\mathrm{c}}(2B)}(y), \quad r > 0, \tag{4.7}$$

and have

$$\|Tf\|_{L_p(B)} \leq \|Tf_1\|_{L_p(B)} + \|Tf_2\|_{L_p(B)}. \tag{4.8}$$

Since $f_1 \in L_p(\mathbb{R}^n)$, $Tf_1 \in L_p(\mathbb{R}^n)$ and from the boundedness of T in $L_p(\mathbb{R}^n)$, it follows that

$$\|Tf_1\|_{L_p(B)} \leq \|Tf_1\|_{L_p(\mathbb{R}^n)} \leq C\|f_1\|_{L_p(\mathbb{R}^n)} = C\|f\|_{L_p(2B)}, \tag{4.9}$$

where constant $C > 0$ is independent of f.

It is clear that $x \in B$, $y \in {}^{\mathrm{c}}(2\mathrm{B})$ implies $(1/2)|x_0 - y| \leq |x - y| \leq (3/2)|x_0 - y|$. We get

$$|Tf_2(x)| \leq 2^n c_0 \int_{^{\mathrm{c}}(2B)} \frac{|f(y)|}{|x_0 - y|^n} dy. \tag{4.10}$$

By Fubini's theorem, we have

$$\begin{aligned} \int_{^{\mathrm{c}}(2B)} \frac{|f(y)|}{|x_0 - y|^n} dy &\approx \int_{^{\mathrm{c}}(2B)} |f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}} dy \\ &\approx \int_{2r}^{\infty} \int_{2r \leq |x_0-y| < t} |f(y)| dy \frac{dt}{t^{n+1}} \\ &\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)| dy \frac{dt}{t^{n+1}}. \end{aligned} \tag{4.11}$$

Applying Hölder's inequality, we get

$$\int_{^{\mathrm{c}}(2B)} \frac{|f(y)|}{|x_0 - y|^n} dy \lesssim \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{n/p+1}}. \tag{4.12}$$

Moreover, for all $p \in [1,\infty)$,

$$\|Tf_2\|_{L_p(B)} \lesssim r^{n/p} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{n/p+1}} \tag{4.13}$$

is valid. Thus,

$$\|Tf\|_{L_p(B)} \lesssim \|f\|_{L_p(2B)} + r^{n/p} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{n/p+1}}. \tag{4.14}$$

On the other hand,

$$\begin{aligned} \|f\|_{L_p(2B)} &\approx r^{n/p} \|f\|_{L_p(2B)} \int_{2r}^{\infty} \frac{dt}{t^{n/p+1}} \\ &\lesssim r^{n/p} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{n/p+1}}. \end{aligned} \tag{4.15}$$

Thus,

$$\|Tf\|_{L_p(B)} \lesssim r^{n/p} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{n/p+1}}. \tag{4.16}$$

Let $p = 1$. From the weak (1,1) boundedness of T and (4.15), it follows that

$$\begin{aligned} \|Tf_1\|_{WL_1(B)} &\le \|Tf_1\|_{WL_1(\mathbb{R}^n)} \lesssim \|f_1\|_{L_1(\mathbb{R}^n)} \\ &= \|f\|_{L_1(2B)} \lesssim r^n \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)| dy \frac{dt}{t^{n+1}}. \end{aligned} \tag{4.17}$$

Then, by (4.13) and (4.17), we get (4.6). □

Theorem 4.5. *Let $1 \le p < \infty$, and let (φ_1, φ_2) satisfy the condition*

$$\int_r^{\infty} \frac{\text{ess inf}_{t<s<\infty} \varphi_1(x,s) s^{n/p}}{t^{n/p+1}} dt \le C\varphi_2(x,r), \tag{4.18}$$

where C does not depend on x and r. Let T be a sublinear operator which satisfies the condition (1.5) *bounded on $L_p(\mathbb{R}^n)$ for $p > 1$ and bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$. Then, the operator T is bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{1,φ_2}. Moreover, for $p > 1$*

$$\|Tf\|_{M_{p,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}}, \tag{4.19}$$

and for $p = 1$

$$\|Tf\|_{WM_{1,\varphi_2}} \lesssim \|f\|_{M_{1,\varphi_1}}. \tag{4.20}$$

Proof. By Lemma 4.4 and Theorem 4.3, we have for $p > 1$

$$\begin{aligned}
\|Tf\|_{M_{p,\varphi_2}} &\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_r^\infty \|f\|_{L_p(B(x,t))} \frac{dt}{t^{n/p+1}} \\
&\approx \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_0^{r^{-n/p}} \|f\|_{L_p(B(x,t^{-p/n}))} dt \\
&= \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2\left(x,r^{-p/n}\right)^{-1} \int_0^r \|f\|_{L_p(B(x,t^{-p/n}))} dt \\
&\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_1\left(x,r^{-p/n}\right)^{-1} r \|f\|_{L_p(B(x,t))} = \|f\|_{M_{p,\varphi_1}},
\end{aligned} \tag{4.21}$$

and for $p = 1$

$$\begin{aligned}
\|Tf\|_{WM_{1,\varphi_2}} &\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_r^\infty \|f\|_{L_1(B(x,t))} \frac{dt}{t^{n+1}} \\
&\approx \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_0^{r^{-n}} \|f\|_{L_1(B(x,t^{-1/n}))} dt \\
&= \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2\left(x,r^{-1/n}\right)^{-1} \int_0^r \|f\|_{L_1(B(x,t^{-1/n}))} dt \\
&\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_1\left(x,r^{-1/n}\right)^{-1} r \|f\|_{L_1(B(x,r^{-1/n}))} = \|f\|_{M_{1,\varphi_1}}.
\end{aligned} \tag{4.22}$$

□

Corollary 4.6. *Let $1 \le p < \infty$, and (φ_1, φ_2) satisfies the condition* (4.18). *Then, the operators M and K are bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and bounded from M_{1,φ_1} to WM_{1,φ_2}.*

Note that Corollary 4.6 was proved in [28].

5. Commutators of Sublinear Operators Generated by Calderón-Zygmund Operators in the Spaces $M_{p,\varphi}$

Let T be a Calderón-Zygmund singular integral operator and $a \in \mathrm{BMO}(\mathbb{R}^n)$. A well-known result of Coifman et al. [29] states that the commutator operator $[a,T]f = T(af) - a\,Tf$ is bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. The commutator of Calderón-Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, e.g., [12, 30, 31]).

First, we introduce the definition of the space of $\mathrm{BMO}(\mathbb{R}^n)$.

Definition 5.1. Suppose that $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$, and let

$$\|f\|_* = \sup_{x\in\mathbb{R}^n,\, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \left| f(y) - f_{B(x,r)} \right| dy < \infty, \tag{5.1}$$

where

$$f_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) dy. \tag{5.2}$$

Define

$$\mathrm{BMO}(\mathbb{R}^n) = \left\{ f \in L_1^{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_* < \infty \right\}. \tag{5.3}$$

If one regards two functions whose difference is a constant as one, then space $\mathrm{BMO}(\mathbb{R}^n)$ is a Banach space with respect to norm $\|\cdot\|_*$.

Remark 5.2. (1) The John-Nirenberg inequality: there are constants $C_1, C_2 > 0$ such that for all $f \in \mathrm{BMO}(\mathbb{R}^n)$ and $\beta > 0$,

$$\left| \left\{ x \in B : \left| f(x) - f_B \right| > \beta \right\} \right| \le C_1 |B| e^{-C_2 \beta / \|f\|_*}, \quad \forall B \subset \mathbb{R}^n. \tag{5.4}$$

(2) The John-Nirenberg inequality implies that

$$\|f\|_* \approx \sup_{x\in\mathbb{R}^n,\, r>0} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} \left| f(y) - f_{B(x,r)} \right|^p dy \right)^{1/p}, \tag{5.5}$$

for $1 < p < \infty$.

(3) Let $f \in \mathrm{BMO}(\mathbb{R}^n)$. Then, there is a constant $C > 0$ such that

$$\left| f_{B(x,r)} - f_{B(x,t)} \right| \le C \|f\|_* \ln \frac{t}{r} \quad \text{for } 0 < 2r < t, \tag{5.6}$$

where C is independent of f, x, r, and t.

In [24], the following statement was proved for the commutators of sublinear operators, containing the result in [18, 19].

Theorem 5.3. *Let $1 < p < \infty$, $\varphi(x,r)$ which satisfies the conditions* (3.4) *and* (3.5) *and $a \in \mathrm{BMO}(\mathbb{R}^n)$. Suppose that T is a linear operator and satisfies the condition* (1.5)*. If the operator $[a,T]$ is bounded on $L_p(\mathbb{R}^n)$, then the operator $[a,T]$ is bounded on $M_{p,\varphi}$.*

Remark 5.4. Note that Theorem 5.3 in the following form is also valid. Let $1 < p < \infty$, $\varphi(x,r)$ satisfy the conditions (3.4) and (3.5) and $a \in \mathrm{BMO}(\mathbb{R}^n)$. Suppose that T_a is a sublinear operator satisfies the condition (1.6) and bounded on $L_p(\mathbb{R}^n)$, then the operator T_a is bounded on $M_{p,\varphi}$.

Lemma 5.5. *Let $1 < p < \infty$, $a \in BMO(\mathbb{R}^n)$, and a sublinear operator T_a satisfies the condition* (1.6) *and bounded on $L_p(\mathbb{R}^n)$.*

Then,

$$\left\|T_a f\right\|_{L_p(B)} \lesssim \|a\|_* r^{n/p} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) t^{-(n/p)-1} \|f\|_{L_p(B(x_0,t))} dt \tag{5.7}$$

holds for any ball $B = B(x_0, r)$ and for all $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. $1 < p < \infty$, $a \in \mathrm{BMO}(\mathbb{R}^n)$, and a sublinear operator T_a satisfies the condition (1.6). For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{{}^c(2B)}$. Hence,

$$\left\|T_a f\right\|_{L_p(B)} \leq \left\|T_a f_1\right\|_{L_p(B)} + \left\|T_a f_2\right\|_{L_p(B)}. \tag{5.8}$$

From the boundedness of T_a in $L_p(\mathbb{R}^n)$, it follows that

$$\begin{aligned} \left\|T_a f_1\right\|_{L_p(B)} &\leq \left\|T_a f_1\right\|_{L_p(\mathbb{R}^n)} \\ &\lesssim \|a\|_* \left\|f_1\right\|_{L_p(\mathbb{R}^n)} = \|a\|_* \|f\|_{L_p(2B)}. \end{aligned} \tag{5.9}$$

For $x \in B$, we have

$$\begin{aligned} \left|T_a f_2(x)\right| &\lesssim \int_{\mathbb{R}^n} \frac{|a(y) - a(x)|}{|x - y|^n} \left|f_2(y)\right| dy \\ &\approx \int_{{}^c(2B)} \frac{|a(y) - a(x)|}{|x_0 - y|^n} |f(y)| dy. \end{aligned} \tag{5.10}$$

Then,

$$\begin{aligned} \left\|T_a f_2\right\|_{L_p(B)} &\lesssim \left(\int_B \left(\int_{{}^c(2B)} \frac{|a(y) - a(x)|}{|x_0 - y|^n} |f(y)| dy\right)^p dx\right)^{1/p} \\ &\lesssim \left(\int_B \left(\int_{{}^c(2B)} \frac{|a(y) - a_B|}{|x_0 - y|^n} |f(y)| dy\right)^p dx\right)^{1/p} \\ &\quad + \left(\int_B \left(\int_{{}^c(2B)} \frac{|a(x) - a_B|}{|x_0 - y|^n} |f(y)| dy\right)^p dx\right)^{1/p} \\ &= I_1 + I_2. \end{aligned} \tag{5.11}$$

Let us estimate I_1

$$\begin{aligned} I_1 &\approx r^{n/p} \int_{{}^{c}(2B)} \frac{|a(y)-a_B|}{|x_0-y|^n} |f(y)| dy \\ &\approx r^{n/p} \int_{{}^{c}(2B)} |a(y)-a_B| |f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}} dy \\ &\approx r^{n/p} \int_{2r}^{\infty} \int_{2r\le|x_0-y|\le t} |a(y)-a_B| |f(y)| dy \frac{dt}{t^{n+1}} \\ &\lesssim r^{n/p} \int_{2r}^{\infty} \int_{B(x_0,t)} |a(y)-a_B| |f(y)| dy \frac{dt}{t^{n+1}}. \end{aligned} \tag{5.12}$$

Applying Hölder's inequality and by (5.5), (5.6), we get

$$\begin{aligned} I_1 &\lesssim r^{n/p} \int_{2r}^{\infty} \int_{B(x_0,t)} |a(y)-a_{B(x_0,t)}| |f(y)| dy \frac{dt}{t^{n+1}} \\ &\quad + r^{n/p} \int_{2r}^{\infty} |a_{B(x_0,r)} - a_{B(x_0,t)}| \int_{B(x_0,t)} |f(y)| dy \frac{dt}{t^{n+1}} \\ &\lesssim r^{n/p} \int_{2r}^{\infty} \left(\int_{B(x_0,t)} |a(y)-a_{B(x_0,t)}|^{p'} dy \right)^{1/p'} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{n+1}} \\ &\quad + r^{n/p} \int_{2r}^{\infty} |a_{B(x_0,r)} - a_{B(x_0,t)}| \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{n/p+1}} \\ &\lesssim \|a\|_* r^{n/p} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{n/p+1}}. \end{aligned} \tag{5.13}$$

In order to estimate I_2, note that

$$I_2 = \left(\int_B |a(x)-a_B|^p dx \right)^{1/p} \int_{{}^{c}(2B)} \frac{|f(y)|}{|x_0-y|^n} dy. \tag{5.14}$$

By (5.5), we get

$$I_2 \lesssim \|a\|_* r^{n/p} \int_{{}^{c}(2B)} \frac{|f(y)|}{|x_0-y|^n} dy. \tag{5.15}$$

Thus, by (4.12),

$$I_2 \lesssim \|a\|_* r^{n/p} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{n/p+1}}. \tag{5.16}$$

Summing up I_1 and I_2, for all $p \in [1, \infty)$, we get

$$\|T_a f_2\|_{L_p(B)} \lesssim \|a\|_* r^{n/p} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{n/p+1}}. \tag{5.17}$$

Finally,

$$\|T_a f\|_{L_p(B)} \lesssim \|a\|_* \|f\|_{L_p(2B)} + \|a\|_* r^{n/p} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{n/p+1}}, \tag{5.18}$$

and statement of Lemma 5.5 follows by (4.15). □

The following theorem is true.

Theorem 5.6. *Let $1 < p < \infty$, $a \in BMO(\mathbb{R}^n)$ and (φ_1, φ_2) satisfy the condition*

$$\int_{r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \frac{\text{ess inf}_{t<s<\infty} \varphi_1(x,s) s^{n/p}}{t^{n/p+1}} dt \le C \varphi_2(x,r), \tag{5.19}$$

where C does not depend on x and r. Suppose that T_a is a sublinear operator which satisfies the condition (1.6) *and bounded on $L_p(\mathbb{R}^n)$.*

Then, the operator T_a is bounded from M_{p,φ_1} to M_{p,φ_2}. Moreover,

$$\|T_a f\|_{M_{p,\varphi_2}} \lesssim \|a\|_* \|f\|_{M_{p,\varphi_1}}. \tag{5.20}$$

Proof. The statement of Theorem 5.6 is followed by Lemma 5.5 and Theorem 4.3 in the same manner as in the proof of Theorem 4.5. □

For the sublinear commutator of the maximal operator

$$M_a(f)(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |a(x) - a(y)| |f(y)| dy, \tag{5.21}$$

and for the linear commutator of the Calderón-Zygmund operator $[a, K]$ from Theorem 5.6, we get the following new results.

Corollary 5.7. *Let $1 < p < \infty$, (φ_1, φ_2) satisfy the condition* (5.19) *and $a \in BMO(\mathbb{R}^n)$. Then, the sublinear commutator operator M_a is bounded from M_{p,φ_1} to M_{p,φ_2}.*

Corollary 5.8. *Let $1 < p < \infty$, (φ_1, φ_2) satisfy the condition* (5.19) *and $a \in BMO(\mathbb{R}^n)$. Then, Calderón-Zygmund singular integral $Kf(x)$ exists for a.e. $x \in \mathbb{R}^n$ and the operator $[a, K]$ is bounded from M_{p,φ_1} to M_{p,φ_2}.*

Note that when the conditions of Corollary 5.8 are satisfied, the existence of $Kf(x)$ for a.e. $x \in \mathbb{R}^n$ was proved in [28].

6. Some Applications

In this section, we will apply Theorems 4.5 and 5.6 to several particular operators such as the pseudodifferential operators, Littlewood-Paley operator, Marcinkiewicz operator, and Bochner-Riesz operator.

6.1. Pseudodifferential Operators

Pseudodifferential operators are generalizations of differential operators and singular integrals. Let m be real number, $0 \le \delta < 1$ and $0 \le \rho < 1$. Following [32, 33], a symbol in $S^m_{\rho,\delta}$ is a smooth function $\sigma(x,\xi)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ such that for all multi-indices α and β the following estimate holds:

$$\left|D_x^\alpha D_\xi^\beta \sigma(x,\xi)\right| \le C_{\alpha\beta}(1+|\xi|)^{m-\rho|\beta|+\delta|\alpha|}, \tag{6.1}$$

where $C_{\alpha\beta} > 0$ is independent of x and ξ. A symbol in $S^{-\infty}_{\rho,\delta}$ is one which satisfies the above estimates for each real number m.

The operator A given by

$$Af(x) = \int_{\mathbb{R}^n} \sigma(x,\xi) e^{2\pi i x\xi} \widehat{f}(\xi) d\xi \tag{6.2}$$

is called a pseudodifferential operator with symbol $\sigma(x,\xi) \in S^m_{\rho,\delta}$, where f is a Schwartz function and $\widehat{f}$ denotes the Fourier transform of f. As usual, $L^m_{\rho,\delta}$ will denote the class of pseudodifferential operators with symbols in $S^m_{\rho,\delta}$.

Miller [34] showed the boundedness of $L^0_{1,0}$ pseudodifferential operators on weighted $L_p(1 < p < \infty)$ spaces whenever the weight function belongs to Muckenhoupt's class A_p. In [1], it is shown that pseudodifferential operators in $L^0_{1,0}$ are Calderón-Zygmund operators, then from Corollary 5.8, we get the following new results.

Corollary 6.1. *Let $1 \le p < \infty$, and let (φ_1, φ_2) satisfy the condition* (4.18). *If A is a pseudodifferential operator of the Hörmander class $L^0_{1,0}$, then the operator A is bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and bounded from M_{1,φ_1} to WM_{1,φ_2}.*

Corollary 6.2. *Let $1 < p < \infty$, (φ_1, φ_2) satisfy the condition* (5.19) *and $a \in BMO(\mathbb{R}^n)$. Let also A be a pseudodifferential operator of the Hörmander class $L^0_{1,0}$. Then, the commutator operator $[a, A]$ is bounded from M_{p,φ_1} to M_{p,φ_2}.*

6.2. Littlewood-Paley Operator

The Littlewood-Paley functions play an important role in classical harmonic analysis, for example, in the study of nontangential convergence of Fatou type and boundedness of Riesz transforms and multipliers [4–6, 35]. The Littlewood-Paley operator (see [6, 36]) is defined as follows.

Definition 6.3. Suppose that $\varphi \in L_1(\mathbb{R}^n)$ *satisfies*

$$\int_{\mathbb{R}^n} \varphi(x)dx = 0. \tag{6.3}$$

Then, the generalized Littlewood-Paley g function g_φ is defined by

$$g_\varphi(f)(x) = \left(\int_0^\infty \left|F_t(f)(x)\right|^2 \frac{dt}{t}\right)^{1/2}, \tag{6.4}$$

where $\varphi_t(x) = t^{-n}\varphi(x/t)$ for $t > 0$ and $F_t(f) = \varphi_t * f$.

The sublinear commutator of the operator g_φ is defined by

$$[a, g_\varphi](f)(x) = \left(\int_0^\infty \left|F_t^a(f)(x)\right|^2 \frac{dt}{t}\right)^{1/2}, \tag{6.5}$$

where

$$F_t^a(f)(x) = \int_{\mathbb{R}^n} [a(x) - a(y)]\varphi_t(x-y)f(y)dy. \tag{6.6}$$

The following theorem is valid (see [3, Theorem 5.1.2]).

Theorem 6.4. *Suppose that* $\varphi \in L_1(\mathbb{R}^n)$ *satisfies* (6.3) *and the following properties:*

$$\begin{gathered} \left|\varphi(x)\right| \le \frac{C}{(1+|x|)^{n+\alpha}}, \quad x \in \mathbb{R}^n, \\ \int_{\mathbb{R}^n} \left|\varphi(x+h) - \varphi(x)\right| dx \le C|h|^\alpha, \quad h \in \mathbb{R}^n, \end{gathered} \tag{6.7}$$

where C *and* $\alpha > 0$ *are both independent of* x *and* h. *Then,* g_φ *is bounded on* $L_p(\mathbb{R}^n)$ *for all* $1 < p < \infty$, *and bounded from* $L_1(\mathbb{R}^n)$ *to* $WL_1(\mathbb{R}^n)$.

Let H be the space $H = \{h : \|h\| = (\int_0^\infty |h(t)|^2 dt/t)^{1/2} < \infty\}$, then for each fixed $x \in \mathbb{R}^n$, $F_t(f)(x)$ may be viewed as a mapping from $[0, \infty)$ to H, and it is clear that $g_\varphi(f)(x) = \|F_t(f)(x)\|$. In fact, by Minkowski inequality and the conditions on φ, we get

$$\begin{aligned} g_\varphi(f)(x) &\le \int_{\mathbb{R}^n} \left|f(y)\right| \left(\int_0^\infty \left|\varphi_t(x-y)\right|^2 \frac{dt}{t}\right)^{1/2} dy \\ &\le C\int_{\mathbb{R}^n} \left|f(y)\right| \left(\int_0^\infty \frac{t^{-2n}}{(1+\left|x-y\right|/t)^{2(n+1)}} \frac{dt}{t}\right)^{1/2} dy \\ &= C\int_{\mathbb{R}^n} \frac{\left|f(y)\right|}{\left|x-y\right|^n} dy. \end{aligned} \tag{6.8}$$

Thus, we get the following.

Corollary 6.5. *Let $1 \le p < \infty$, (φ_1, φ_2) satisfies the condition* (4.18) *and $\psi \in L_1(\mathbb{R}^n)$ satisfies* (6.3) *and* (6.7)*. Then the operator g_ψ is bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and bounded from M_{1,φ_1} to WM_{1,φ_2}.*

Corollary 6.6. *Let $1 < p < \infty$, (φ_1, φ_2) satisfies the condition* (5.19)*, $a \in BMO(\mathbb{R}^n)$ and $\psi \in L_1(\mathbb{R}^n)$ satisfies* (6.3) *and* (6.7)*. Then the operator $[a, g_\psi]$ is bounded from M_{p,φ_1} to M_{p,φ_2}.*

6.3. Marcinkiewicz Operator

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere in $\mathbb{R}^n$ equipped with the Lebesgue measure $d\sigma$. Suppose that Ω satisfies the following conditions.

(a) Ω is the homogeneous function of degree zero on $\mathbb{R}^n \setminus \{0\}$; that is,

$$\Omega(tx) = \Omega(x), \quad \text{for any } t > 0,\ x \in \mathbb{R}^n \setminus \{0\}. \tag{6.9}$$

(b) Ω has mean zero on S^{n-1}; that is,

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0. \tag{6.10}$$

(c) $\Omega \in \mathrm{Lip}_\gamma(S^{n-1})$, $0 < \gamma \le 1$, that is there exists a constant $M > 0$ such that

$$\left|\Omega(x') - \Omega(y')\right| \le M\left|x' - y'\right|^\gamma \quad \text{for any } x', y' \in S^{n-1}. \tag{6.11}$$

In 1958, Stein [35] defined the Marcinkiewicz integral of higher dimension μ_Ω as

$$\mu_\Omega(f)(x) = \left(\int_0^\infty \left|F_{\Omega,t}(f)(x)\right|^2 \frac{dt}{t^3}\right)^{1/2}, \tag{6.12}$$

where

$$F_{\Omega,t}(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy. \tag{6.13}$$

Since Stein's work in 1958, the continuity of Marcinkiewicz integral has been extensively studied as a research topic and also provides useful tools in harmonic analysis [3–6].

The sublinear commutator of the operator μ_Ω is defined by

$$[a, \mu_\Omega](f)(x) = \left(\int_0^\infty \left|F_{\Omega,t,a}(f)(x)\right|^2 \frac{dt}{t^3}\right)^{1/2}, \tag{6.14}$$

where

$$F_{\Omega,t,a}(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [a(x) - a(y)] f(y) dy. \tag{6.15}$$

Let H be the space $H = \{h : \|h\| = (\int_0^\infty |h(t)|^2 dt/t^3)^{1/2} < \infty\}$. Then, it is clear that $\mu_\Omega(f)(x) = \|F_{\Omega,t}(f)(x)\|$.

By Minkowski inequality and the conditions on Ω, we get

$$\mu_\Omega(f)(x) \le \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f(y)| \left(\int_{|x-y|}^\infty \frac{dt}{t^3} \right)^{1/2} dy \le C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy. \tag{6.16}$$

Thus, μ_Ω satisfies the condition (1.5). It is known that μ_Ω is bounded on $L_p(\mathbb{R}^n)$ for $p > 1$ and bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$ (see [37]), then from Theorems 4.5 and 5.6, we get the following collory.

Corollary 6.7. *Let $1 \le p < \infty$ and (φ_1, φ_2) satisfy the condition (4.18), and let Ω satisfy the conditions (a)–(c). Then, μ_Ω is bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and bounded from M_{1,φ_1} to WM_{1,φ_2}.*

Corollary 6.8. *Let $1 < p < \infty$, (φ_1, φ_2) satisfy the condition (5.19), $a \in BMO(\mathbb{R}^n)$, and Ω satisfy the conditions (a)–(c). Then, $[a, \mu_\Omega]$ is bounded from M_{p,φ_1} to M_{p,φ_2}.*

6.4. Bochner-Riesz Operator

Let $\delta > (n-1)/2$, $B_t^\delta(\widehat{f})(\xi) = (1 - t^2|\xi|^2)_+^\delta \widehat{f}(\xi)$ and $B_t^\delta(x) = t^{-n} B^\delta(x/t)$ for $t > 0$. The maximal Bochner-Riesz operator is defined by (see [38, 39])

$$B_*^\delta(f)(x) = \sup_{t>0} \left| B_t^\delta(f)(x) \right|. \tag{6.17}$$

Let H be the space $H = \{h : \|h\| = \sup_{t>0} |h(t)| < \infty\}$, then it is clear that $B_*^\delta(f)(x) = \|B_t^\delta(f)(x)\|$.

By the condition on B_r^δ (see [2]), we have

$$\begin{aligned} \left| B_r^\delta(x-y) \right| &\le C r^{-n} (1 + |x-y|/r)^{-(\delta+(n+1)/2)} \\ &= C \left(\frac{r}{r + |x-y|} \right)^{\delta-(n-1)/2} \frac{1}{(r + |x-y|)^n} \\ &\le |x-y|^{-n}, \\ B_*^\delta(f)(x) &\le C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy. \end{aligned} \tag{6.18}$$

Thus, B_*^δ satisfies the condition (1.5). It is known that B_*^δ is bounded on $L_p(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$, then from Theorems 4.5 and 5.6, we get the following corollary.

Corollary 6.9. *Let* $1 \le p < \infty$, (φ_1, φ_2) *satisfy the condition* (4.18) *and* $\delta > (n-1)/2$. *Then, the operator* B_*^δ *is bounded from* M_{p,φ_1} *to* M_{p,φ_2} *for* $p > 1$ *and bounded from* M_{1,φ_1} *to* WM_{1,φ_2}.

Corollary 6.10. *Let* $1 < p < \infty$, (φ_1, φ_2) *satisfy the condition* (5.19), $\delta > (n-1)/2$ *and* $a \in BMO(\mathbb{R}^n)$. *Then, the operator* $[a, B_t^\delta]$ *is bounded from* M_{p,φ_1} *to* M_{p,φ_2}.

Remark 6.11. Recall that under the assumption that $\varphi(x, r)$ satisfies the conditions (3.4) and (3.5), the Corollaries 6.9 and 6.10 were proved in [38].

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