

Research Article

Integral Inequalities on Time Scales via the Theory of Isotonic Linear Functionals

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We apply the theory of isotonic linear functionals to derive a series of known inequalities, extensions of known inequalities, and new inequalities in the theory of dynamic equations on time scales.

1. Introduction

A time scale is an arbitrary nonempty closed subset of the real numbers. For an introduction to the theory of dynamic equations on time-scale, we refer to [1]. For functions defined on a time scale, we can consider the derivative and also the integral. For example, when the time scale is the set of all real numbers, the time-scale integral is an ordinary integral; when the time scale is the set of all integers, the time-scale integral is a sum; when the time scale is the set of all integer powers of a fixed number, the time-scale integral is a Jackson integral.

In this paper, we present a series of inequalities for the time-scale integral. Among the inequalities presented, we offer time-scale versions of Jensen's-inequality, Jensen-type inequalities, converses of Jensen's inequality, inequalities for means, Hölder's inequality, Minkowski's inequality, Dresher's inequality, Aczél's inequality, Popoviciu's inequality, and Diaz-Metcalf's inequality.

The monograph [2] contains numerous classical inequalities that are proved for the so-called isotonic linear functionals. Since the time-scale integral is in fact an isotonic linear functional, the results from [2] can be applied to this setting. Our work shows that it is not necessary to prove such kinds of inequalities “from scratch” in the time-scale setting as they can all be obtained easily from well-known inequalities for isotonic linear functionals.

The setup of this paper is as follows. In the next section, we review some known results from the literature concerning Jensen’s inequality on time-scale. Section 3 contains the definition of an isotonic linear functional and the confirmations that the time-scale Cauchy delta, Cauchy nabla, α -diamond, multiple Riemann, and multiple Lebesgue integrals all are indeed isotonic linear functionals. Section 4 then is devoted to the time-scale Jensen’s inequality and some of its generalizations. Some converses of Jensen’s inequality in the form of time-scale Hermite-Hadamard’s inequality and generalizations of it are contained in Section 5. Section 6 presents the multidimensional time-scale versions of Hölder’s and Cauchy-Schwarz’s inequality, followed in Section 7 by Minkowski’s inequality. Section 8 is concerned with Dresher’s inequality, and Section 9 offers time-scale versions of Aczél’s and Popoviciu’s inequalities. Section 10 contains Bellman’s inequality and Section 11 deals with the Diaz-Metcalf inequality and some consequences. Five further converses of Jensen’s inequality are contained in the final Section 12.

2. Known Results Concerning Jensen’s Inequality

Jensen’s inequality is of great interest in the theories of differential and difference equations as well as other areas of mathematics. The original Jensen inequality can be stated as follows.

Theorem 2.1 (Jensen’s inequality [3, Formula (5’)]). *Let $a, b \in \mathbb{R}$ with $a < b$, and suppose $I \subset \mathbb{R}$ is an interval. If $\Phi \in C(I, \mathbb{R})$ is convex and $f \in C([a, b], I)$, then*

$$\Phi\left(\frac{\int_a^b f(t)dt}{b-a}\right) \leq \frac{\int_a^b \Phi(f(t))dt}{b-a}. \quad (2.1)$$

The Jensen inequality on time-scale has been obtained by Agarwal et al. [4].

Theorem 2.2 (Jensen’s inequality [1, Theorem 6.17]). *Let $a, b \in \mathbb{T}$ with $a < b$, and suppose $I \subset \mathbb{R}$ is an interval. If $\Phi \in C(I, \mathbb{R})$ is convex and $f \in C_{\text{rd}}([a, b], I)$, then*

$$\Phi\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right) \leq \frac{\int_a^b \Phi(f(t))\Delta t}{b-a}. \quad (2.2)$$

When $\mathbb{T} = \mathbb{R}$ in Theorem 2.2, we obtain Theorem 2.1. When $\mathbb{T} = \mathbb{Z}$ in Theorem 2.2, we get the usual geometric-arithmetic mean inequality.

The following result is given by Wong et al. in [5]. When $h(t) \equiv 1$ in Theorem 2.3, we obtain Theorem 2.2.

Theorem 2.3 (Jensen’s inequality [5, Theorem 2.2]). *Let $a, b \in \mathbb{T}$ with $a < b$, and suppose $I \subset \mathbb{R}$ is an interval. Assume $h \in C_{\text{rd}}([a, b], \mathbb{R})$ satisfies $\int_a^b |h(t)| \Delta t > 0$. If $\Phi \in C(I, \mathbb{R})$ is convex and $f \in C_{\text{rd}}([a, b], I)$, then*

$$\Phi \left(\frac{\int_a^b |h(t)| f(t) \Delta t}{\int_a^b |h(t)| \Delta t} \right) \leq \frac{\int_a^b |h(t)| \Phi(f(t)) \Delta t}{\int_a^b |h(t)| \Delta t}. \quad (2.3)$$

In [6], Özkan et al. proved that Theorem 2.3 is also true if we use the nabla integral (see [1, Section 8.4]) instead of the delta integral. In [7], Sheng et al. introduced the so-called α -diamond integral, where $0 \leq \alpha \leq 1$. It is a linear combination of the delta integral and the nabla integral. When $\alpha = 1$, we get the usual delta integral, and when $\alpha = 0$, we get the usual nabla integral. The following result concerning the α -diamond integral is given by Ammi et al. in [8] (see also [6]).

Theorem 2.4 (Jensen’s inequality [8, Theorem 3.3]). *Let $\alpha \in [0, 1]$. Let $a, b \in \mathbb{T}$ with $a < b$ and suppose $I \subset \mathbb{R}$ is an interval. Assume $h \in C([a, b], \mathbb{R})$ satisfies $\int_a^b |h(t)| \diamond_{\alpha} t > 0$. If $\Phi \in C(I, \mathbb{R})$ is convex and $f \in C([a, b], I)$, then*

$$\Phi \left(\frac{\int_a^b |h(t)| f(t) \diamond_{\alpha} t}{\int_a^b |h(t)| \diamond_{\alpha} t} \right) \leq \frac{\int_a^b |h(t)| \Phi(f(t)) \diamond_{\alpha} t}{\int_a^b |h(t)| \diamond_{\alpha} t}. \quad (2.4)$$

3. Isotonic Linear Functionals and Time-Scale Integrals

We recall the following definition from [2, page 47].

Definition 3.1 (Isotonic linear functional). Let E be a nonempty set and L be a linear class of real-valued functions $f : E \rightarrow \mathbb{R}$ having the following properties:

- (L₁) If $f, g \in L$ and $a, b \in \mathbb{R}$, then $(af + bg) \in L$.
- (L₂) If $f(t) = 1$ for all $t \in E$, then $f \in L$.

An *isotonic linear functional* is a functional $A : L \rightarrow \mathbb{R}$ having the following properties:

- (A₁) If $f, g \in L$ and $a, b \in \mathbb{R}$, then $A(af + bg) = aA(f) + bA(g)$.
- (A₂) If $f \in L$ and $f(t) \geq 0$ for all $t \in E$, then $A(f) \geq 0$.

When we use the approach of isotonic linear functionals as given in Definition 3.1, it is not necessary to know many details from the calculus of dynamic equations on time-scale. We only need to know that the time-scale integral is such an isotonic linear functional.

Theorem 3.2. *Let \mathbb{T} be a time scale. For $a, b \in \mathbb{T}$ with $a < b$, let*

$$E = [a, b) \cap \mathbb{T}, \quad L = C_{\text{rd}}([a, b), \mathbb{R}). \quad (3.1)$$

Then (L_1) and (L_2) are satisfied. Moreover, let

$$A(f) = \int_a^b f(t) \Delta t, \quad (3.2)$$

where the integral is the Cauchy delta time-scale integral. Then (A_1) and (A_2) are satisfied.

Proof. This follows from [1, Definition 1.58 and Theorem 1.77]. \square

Instead of recalling the formal definition of the time-scale integral and the definition of the set of rd-continuous functions C_{rd} used in Theorem 3.2, which can be found in [1, Section 1.4], we choose to only give a few examples.

Example 3.3. If $\mathbb{T} = \mathbb{R}$ in Theorem 3.2, then $L = C([a, b], \mathbb{R})$ and

$$A(f) = \int_a^b f(t) dt. \quad (3.3)$$

If $\mathbb{T} = \mathbb{Z}$ in Theorem 3.2, then L consists of all real-valued functions defined on $[a, b - 1] \cap \mathbb{Z}$ and

$$A(f) = \sum_{t=a}^{b-1} f(t). \quad (3.4)$$

Let $h > 0$. If $\mathbb{T} = h\mathbb{Z}$ in Theorem 3.2, then L consists of all real-valued functions defined on $[a, b - h] \cap h\mathbb{Z}$ and

$$A(f) = h \sum_{k=a/h}^{b/h-1} f(k). \quad (3.5)$$

Let $q > 1$. If $\mathbb{T} = q^{\mathbb{N}_0}$ in Theorem 3.2, then L consists of all real-valued functions defined on $[a, b/q] \cap q^{\mathbb{N}_0}$ and

$$A(f) = (q - 1) \sum_{k=\log_q(a)}^{\log_q(b)-1} q^k f(q^k). \quad (3.6)$$

Note that Theorem 3.2 also has corresponding versions for the nabla and α -diamond integral, which are given next for completeness.

Theorem 3.4. Let \mathbb{T} be a time scale. For $a, b \in \mathbb{T}$ with $a < b$, let

$$E = (a, b] \cap \mathbb{T}, \quad L = C_{\text{id}}((a, b], \mathbb{R}). \quad (3.7)$$

Then (L_1) and (L_2) are satisfied. Moreover, let

$$A(f) = \int_a^b f(t) \nabla t, \quad (3.8)$$

where the integral is the Cauchy nabla time-scale integral. Then (A_1) and (A_2) are satisfied.

Proof. This follows from [1, Definition 8.43 and Theorem 8.47]. \square

Theorem 3.5. Let \mathbb{T} be a time scale. For $a, b \in \mathbb{T}$ with $a < b$, let

$$E = [a, b] \cap \mathbb{T}, \quad L = C([a, b], \mathbb{R}). \quad (3.9)$$

Then (L_1) and (L_2) are satisfied. Moreover, let

$$A(f) = \int_a^b f(t) \diamond_{\alpha} t, \quad (3.10)$$

where the integral is the Cauchy α -diamond time-scale integral. Then (A_1) and (A_2) are satisfied.

Proof. This follows from [7, Definition 3.2 and Theorem 3.7]. \square

Multiple Riemann integration on time-scale was introduced in [9]. The Riemann integral introduced there is also an isotonic linear functional.

Theorem 3.6. Let $\mathbb{T}_1, \dots, \mathbb{T}_n$ be time-scale. For $a_i, b_i \in \mathbb{T}_i$ with $a_i < b_i$, $1 \leq i \leq n$, let

$$E \subset ([a_1, b_1] \cap \mathbb{T}_1) \times \cdots \times ([a_n, b_n] \cap \mathbb{T}_n) \quad (3.11)$$

be Jordan Δ -measurable and let L be the set of all bounded Δ -integrable functions from E to \mathbb{R} . Then (L_1) and (L_2) are satisfied. Moreover, let

$$A(f) = \int_E f(t) \Delta t, \quad (3.12)$$

where the integral is the multiple Riemann delta time-scale integral. Then (A_1) and (A_2) are satisfied.

Proof. This follows from [9, Definition 4.13 and Theorem 3.4]. \square

From [9, Remark 2.18], it is also clear that a theorem similar to Theorem 3.6 is also true for the nabla case or any mixture of delta and nabla integrals in the multiple variable case.

The multiple Lebesgue integration on time-scale was introduced in [10]. The Lebesgue integral introduced there is also an isotonic linear functional.

Theorem 3.7. Let $\mathbb{T}_1, \dots, \mathbb{T}_n$ be time-scale. For $a_i, b_i \in \mathbb{T}_i$ with $a_i < b_i$, $1 \leq i \leq n$, let

$$E \subset ([a_1, b_1] \cap \mathbb{T}_1) \times \cdots \times ([a_n, b_n] \cap \mathbb{T}_n) \quad (3.13)$$

be Lebesgue Δ -measurable and let L be the set of all Δ -measurable functions from E to \mathbb{R} . Then (L_1) and (L_2) are satisfied. Moreover, let

$$A(f) = \int_E f(t) \Delta t, \quad (3.14)$$

where the integral is the multiple Lebesgue delta time-scale integral. Then (A_1) and (A_2) are satisfied.

Proof. This follows from [10, Section 3]. \square

Theorem 3.8. Under the assumptions of Theorem 3.7, let $A(f)$ be replaced by

$$A(f) = \frac{\int_E |h(t)| f(t) \Delta t}{\int_E |h(t)| \Delta t}, \quad (3.15)$$

where $h : E \rightarrow \mathbb{R}$ is Δ -integrable such that $\int_E |h(t)| \Delta t > 0$. Then A is an isotonic linear functional satisfying $A(1) = 1$.

4. Jensen's Inequality

Jessen in [11] gave the following generalization of Jensen's inequality for isotonic linear functionals.

Theorem 4.1 (Jessen's inequality [2, Theorem 2.4]). Let L satisfy properties (L_1) and (L_2) . Assume $\Phi \in C(I, \mathbb{R})$ is convex, where $I \subset \mathbb{R}$ is an interval. If A satisfies (A_1) and (A_2) such that $A(1) = 1$, then for all $f \in L$ such that $\Phi(f) \in I$, one has $A(f) \in I$ and

$$\Phi(A(f)) \leq A(\Phi(f)). \quad (4.1)$$

Now our first result is the following generalization of Jensen's inequality.

Theorem 4.2 (Jensen's inequality). Assume $\Phi \in C(I, \mathbb{R})$ is convex, where $I \subset \mathbb{R}$ is an interval. Let $E \subset \mathbb{R}^n$ be as in Theorem 3.7 and suppose f is Δ -integrable on E such that $f(E) = I$. Moreover, let $h : E \rightarrow \mathbb{R}$ be Δ -integrable such that $\int_E |h(t)| \Delta t > 0$. Then

$$\Phi\left(\frac{\int_E |h(t)| f(t) \Delta t}{\int_E |h(t)| \Delta t}\right) \leq \frac{\int_E |h(t)| \Phi(f(t)) \Delta t}{\int_E |h(t)| \Delta t}. \quad (4.2)$$

Proof. Just apply Theorems 4.1 and 3.8. \square

Remark 4.3 (Jensen's inequality). Note that the known results from Section 2 follow from Theorem 4.1 in the same way as Theorem 4.2 does: Theorem 2.3 follows as in Theorems 3.2 and 2.4 follows as in Theorem 3.5. Note also that a similar theorem for the multiple Riemann integral can be stated and proved using Theorem 3.6. This will be the case for all inequalities stated in this paper; however, we only explicitly state each time the case for the multiple Lebesgue integral.

5. Hermite-Hadamard's Inequality

Beesack and Pečarić in [12] gave the following generalization of Hermite-Hadamard's inequality for isotonic linear functionals.

Theorem 5.1 (Beesack-Pečarić's inequality [2, Theorem 3.37]). *Let L satisfy properties (L_1) and (L_2) . Assume $\Phi \in C(I, \mathbb{R})$ is convex, where $I = [m, M] \subset \mathbb{R}$ with $m < M$. If A satisfies (A_1) and (A_2) such that $A(1) = 1$, then for all $f \in L$ such that $\Phi(f) \in L$, one has*

$$A(\Phi(f)) \leq \frac{M - A(f)}{M - m} \Phi(m) + \frac{A(f) - m}{M - m} \Phi(M). \tag{5.1}$$

Theorem 5.2 (Hermite-Hadamard's inequality). *Assume $\Phi \in C(I, \mathbb{R})$ is convex, where $I = [m, M] \subset \mathbb{R}$ with $m < M$. Let $E \subset \mathbb{R}^n$ be as in Theorem 3.7, and suppose f is Δ -integrable on E such that $f(E) = I$. Moreover, let $h : E \rightarrow \mathbb{R}$ be Δ -integrable such that $\int_E |h(t)| \Delta t > 0$. Then*

$$\begin{aligned} \frac{\int_E |h(t)| \Phi(f(t)) \Delta t}{\int_E |h(t)| \Delta t} &\leq \frac{M - \int_E |h(t)| f(t) \Delta t / \int_E |h(t)| \Delta t}{M - m} \Phi(m) \\ &+ \frac{\int_E |h(t)| f(t) \Delta t / \int_E |h(t)| \Delta t - m}{M - m} \Phi(M). \end{aligned} \tag{5.2}$$

Proof. Just apply Theorems 5.1 and 3.8. □

Remark 5.3 (Hermite-Hadamard's inequality). Note that the known result [13, Theorem 3.14] (see also [14, 15]) follows from Theorem 5.1 in the same way as Theorem 5.2 does, this time applying Theorem 3.5.

A combination of Theorem 4.1 and Theorem 5.1 in a slightly different form is given by Pečarić and Beesack in [16] as follows.

Theorem 5.4 (Pečarić-Beesack's inequality [2, Theorem 5.13]). *Let L satisfy properties (L_1) and (L_2) . Assume $\Phi \in C(I, \mathbb{R})$ is convex, where $[m, M] \subset I$ with $m < M$ and $I \subset \mathbb{R}$ is an interval. Suppose A satisfies (A_1) and (A_2) such that $A(1) = 1$. Let $f \in L$ such that $f(E) \subset [m, M]$ and $\Phi(f) \in L$, and define $p, q \geq 0$ such that $p + q > 0$ and*

$$A(f) = \frac{pm + qM}{p + q} \tag{5.3}$$

holds. Then

$$\Phi\left(\frac{pm + qM}{p + q}\right) \leq A(\Phi(f)) \leq \frac{p\Phi(m) + q\Phi(M)}{p + q}. \tag{5.4}$$

Theorem 5.5 (Hermite-Hadamard's inequality). *Assume $\Phi \in C(I, \mathbb{R})$ is convex, where $[m, M] \subset I$ with $m < M$ and $I \subset \mathbb{R}$ is an interval. Let $E \subset \mathbb{R}^n$ be as in Theorem 3.7 and suppose f is*

Δ -integrable on E such that $f(E) \subset [m, M]$. Moreover, let $h : E \rightarrow \mathbb{R}$ be Δ -integrable such that $\int_E |h(t)| \Delta t > 0$. Let $p, q \geq 0$ be such that $p + q > 0$ and

$$\frac{\int_E |h(t)| f(t) \Delta t}{\int_E |h(t)| \Delta t} = \frac{pm + qM}{p + q} \quad (5.5)$$

holds. Then

$$\Phi\left(\frac{pm + qM}{p + q}\right) \leq \frac{\int_E |h(t)| \Phi(f(t)) \Delta t}{\int_E |h(t)| \Delta t} \leq \frac{p\Phi(m) + q\Phi(M)}{p + q}. \quad (5.6)$$

Proof. Just apply Theorems 5.4 and 3.8. \square

6. Hölder's Inequality

We first recall Hölder's inequality for isotonic linear functionals as given in [2].

Theorem 6.1 (Hölder's inequality [2, Theorem 4.12]). *Let E, L , and A be such that $(L_1), (L_2), (A_1)$, and (A_2) are satisfied. For $p \neq 1$, define $q = p/(p - 1)$. Assume $|w|f|^p, |w|g|^q, |wfg| \in L$. If $p > 1$, then*

$$A(|wfg|) \leq A^{1/p}(|w|f|^p) A^{1/q}(|w|g|^q). \quad (6.1)$$

This inequality is reversed if $0 < p < 1$ and $A(|w|g|^q) > 0$, and it is also reversed if $p < 0$ and $A(|w|f|^p) > 0$.

Theorem 6.2 (Hölder's inequality). *For $p > 1$, define $q = p/(p - 1)$. Let $E \subset \mathbb{R}^n$ be as in Theorem 3.7. Assume $|w|f|^p, |w|g|^q, |wfg|$ are Δ -integrable on E . If $p > 1$, then*

$$\int_E |w(t)f(t)g(t)| \Delta t \leq \left(\int_E |w(t)||f(t)|^p \Delta t \right)^{1/p} \left(\int_E |w(t)||g(t)|^q \Delta t \right)^{1/q}. \quad (6.2)$$

This inequality is reversed if $0 < p < 1$ and $\int_E |w(t)||g(t)|^q \Delta t > 0$, and it is also reversed if $p < 0$ and $\int_E |w(t)||f(t)|^p \Delta t > 0$.

Proof. Just apply Theorems 6.1 and 3.7. \square

Remark 6.3 (Hölder's inequality). Note that the known results from the time-scale literature follow from Theorem 6.1 in the same way as Theorem 6.2 does: [1, Theorem 6.13] follows as in Theorem 3.2 and [8, Theorem 4.1] (see also [17, 18]) follows as in Theorem 3.5.

Theorem 6.4 (Cauchy-Schwarz's inequality). *Let $E \subset \mathbb{R}^n$ be as in Theorem 3.7. If $|w|f|^2, |w|g|^2, |wfg|$ are Δ -integrable on E , then*

$$\int_E |w(t)f(t)g(t)| \Delta t \leq \sqrt{\left(\int_E |w(t)|f^2(t) \Delta t \right) \left(\int_E |w(t)|g^2(t) \Delta t \right)}. \quad (6.3)$$

Proof. Just let $p = 2$ in Theorem 6.2. \square

7. Minkowski's Inequality

Another classical inequality is Minkowski's inequality. We first recall Minkowski's inequality for isotonic linear functionals as given in [2].

Theorem 7.1 (Minkowski's inequality [2, Theorem 4.13]). *Let E , L , and A be such that (L_1) , (L_2) , (A_1) , and (A_2) are satisfied. For $p \in \mathbb{R}$, assume $|\omega||f|^p, |\omega||g|^p, |\omega||f + g|^p \in L$. If $p > 1$, then*

$$A^{1/p}(|\omega||f + g|^p) \leq A^{1/p}(|\omega||f|^p) + A^{1/p}(|\omega||g|^p). \tag{7.1}$$

This inequality is reversed if $0 < p < 1$ or $p < 0$ provided $A(|\omega||f|^p) > 0$ and $A(|\omega||g|^p) > 0$ hold.

Theorem 7.2 (Minkowski's inequality). *Let $E \subset \mathbb{R}^n$ be as in Theorem 3.7. For $p \in \mathbb{R}$, assume $|\omega||f|^p, |\omega||g|^p, |\omega||f + g|^p$ are Δ -integrable on E . If $p > 1$, then*

$$\begin{aligned} \left(\int_E |\omega(t)||f(t) + g(t)|^p \Delta t \right)^{1/p} &\leq \left(\int_E |\omega(t)||f(t)|^p \Delta t \right)^{1/p} \\ &+ \left(\int_E |\omega(t)||g(t)|^p \Delta t \right)^{1/p}. \end{aligned} \tag{7.2}$$

This inequality is reversed for $0 < p < 1$ or $p < 0$ provided each of the two terms on the right-hand side is positive.

Proof. Just apply Theorems 7.1 and 3.7. □

Remark 7.3 (Minkowski's inequality). Note that the known results from the time-scale literature follow from Theorem 7.1 in the same way as Theorem 7.2 does: [1, Theorem 6.16] follows as in Theorem 3.2 and [8, Theorem 4.4] (see also [17, 18]) follows as in Theorem 3.5.

8. Dresher's Inequality

If $n = 2$ in the result of this section, then one has the so-called Dresher inequality (see [19, Section 7]). We first present the generalization of this inequality for isotonic linear functionals as given in [2].

Theorem 8.1 (Dresher's inequality [2, Theorem 4.21]). *Let E and L be such that (L_1) , (L_2) are satisfied, and suppose that both A and B satisfy (A_1) , (A_2) . If*

$$|\omega||f_i|^p, \quad |\omega|\left(\sum_{i=1}^n |f_i|\right)^p, \quad |\omega||g_i|^r, \quad |\omega|\left(\sum_{i=1}^n |g_i|\right)^r \in L, \tag{8.1}$$

where $p \geq 1 > r > 0$ and $B(|\omega||g_i|^r) > 0$ for $1 \leq i \leq n$, then

$$\left(\frac{A(|\omega|(\sum_{i=1}^n |f_i|)^p)}{B(|\omega|(\sum_{i=1}^n |g_i|)^r)} \right)^{1/(p-r)} \leq \sum_{i=1}^n \left(\frac{A(|\omega||f_i|^p)}{B(|\omega||g_i|^r)} \right)^{1/(p-r)}. \tag{8.2}$$

Theorem 8.2 (Dresher's inequality). Let $E \subset \mathbb{R}^n$ be as in Theorem 3.7. If

$$|\omega||f_i|^p, \quad |\omega|\left(\sum_{i=1}^n |f_i|\right)^p, \quad |\omega||g_i|^r, \quad |\omega|\left(\sum_{i=1}^n |g_i|\right)^r \quad (8.3)$$

are Δ -integrable on E , where $p \geq 1 > r > 0$ and $\int_E |\omega(t)||g_i(t)|^r \Delta t > 0$ for $1 \leq i \leq n$, then

$$\left(\frac{\int_E |\omega(t)|\left(\sum_{i=1}^n |f_i(t)|\right)^p \Delta t}{\int_E |\omega(t)|\left(\sum_{i=1}^n |g_i(t)|\right)^r \Delta t}\right)^{1/(p-r)} \leq \sum_{i=1}^n \left(\frac{\int_E |\omega(t)||f_i(t)|^p \Delta t}{\int_E |\omega(t)||g_i(t)|^r \Delta t}\right)^{1/(p-r)}. \quad (8.4)$$

Proof. Just apply Theorems 8.1 and 3.7. \square

Remark 8.3 (Dresher's inequality). Dresher's inequality on time-scale is new even for the cases of a single-variable Cauchy delta and nabla integral and also for the α -diamond integral.

9. Popoviciu's Inequality

We first recall Popoviciu's inequality for isotonic linear functionals as given in [2].

Theorem 9.1 (Popoviciu's inequality [2, Theorem 4.27]). Let E, L , and A be such that (L_1) , (L_2) , (A_1) , and (A_2) are satisfied. For $p \neq 1$, define $q = p/(p-1)$. Assume $|f|^p, |g|^q, |fg| \in L$. Suppose $f_0, g_0 > 0$ are such that

$$f_0^p - A(|f|^p) > 0, \quad g_0^q - A(|g|^q) > 0. \quad (9.1)$$

If $p > 1$, then

$$\left(f_0^p - A(|f|^p)\right)^{1/p} \left(g_0^q - A(|g|^q)\right)^{1/q} \leq f_0 g_0 - A(|fg|). \quad (9.2)$$

This inequality is reversed if $0 < p < 1$ and $A(|g|^q) > 0$ or if $p < 0$ and $A(|f|^p) > 0$.

Theorem 9.2 (Popoviciu's inequality). Let $E \subset \mathbb{R}^n$ be as in Theorem 3.7. For $p \neq 1$, define $q = p/(p-1)$. Assume $|f|^p, |g|^q, |fg|$ are Δ -integrable on E . Suppose $f_0, g_0 > 0$ are such that

$$f_0^p - \int_E |f(t)|^p \Delta t > 0, \quad g_0^q - \int_E |g(t)|^q \Delta t > 0. \quad (9.3)$$

If $p > 1$, then

$$f_0 g_0 - \int_E |f(t)g(t)| \Delta t \geq \left(f_0^p - \int_E |f(t)|^p \Delta t\right)^{1/p} \left(g_0^q - \int_E |g(t)|^q \Delta t\right)^{1/q}. \quad (9.4)$$

This inequality is reversed if $0 < p < 1$ and $\int_E |g(t)|^q \Delta t > 0$ or if $p < 0$ and $\int_E |f(t)|^p \Delta t > 0$.

Proof. Just apply Theorems 8.1 and 3.7. \square

Theorem 9.3 (Aczél’s inequality). *Let $E \subset \mathbb{R}^n$ be as in Theorem 3.7. Assume $f^2, g^2, |fg|$ are Δ -integrable on E . If $f_0, g_0 > 0$ are such that*

$$f_0^2 - \int_E f^2(t) \Delta t > 0, \quad g_0^2 - \int_E g^2(t) \Delta t > 0, \quad (9.5)$$

then

$$f_0 g_0 - \int_E |f(t)g(t)| \Delta t \geq \sqrt{\left(f_0^2 - \int_E f^2(t) \Delta t\right) \left(g_0^2 - \int_E g^2(t) \Delta t\right)}. \quad (9.6)$$

Proof. Just let $p = 2$ in Theorem 9.2. □

Remark 9.4 (Aczél’s and Popoviciu’s inequalities). Aczél’s and Popoviciu’s inequalities on time-scale are new even for the cases of a single-variable Cauchy delta and nabla integral and also for the α -diamond integral. The original Aczél inequality can be found in [20]. For a version of Aczél’s inequality for isotonic linear functionals, we refer to [2, Theorem 4.26].

10. Bellman’s Inequality

We first recall Bellman’s inequality for isotonic linear functionals as given in [2].

Theorem 10.1 (Bellman’s inequality [2, Theorem 4.29]). *Let E, L , and A be such that $(L_1), (L_2), (A_1)$, and (A_2) are satisfied. For $p \in \mathbb{R}$, assume $|f|^p, |g|^p, (|f| + |g|) \in L$. Suppose $f_0, g_0 > 0$ are such that*

$$f_0^p - A(|f|^p) > 0, \quad g_0^p - A(|g|^p) > 0. \quad (10.1)$$

If $p > 1$, then

$$\left(\left(f_0^p - A(|f|^p) \right)^{1/p} + \left(g_0^p - A(|g|^p) \right)^{1/p} \right)^p \leq (f_0 + g_0)^p - A((|f| + |g|)^p). \quad (10.2)$$

This inequality is reversed if $0 < p < 1$ or $p < 0$ and $A(|f|^p) > 0$.

Theorem 10.2 (Bellman’s inequality). *Let $E \subset \mathbb{R}^n$ be as in Theorem 3.7. For $p \in \mathbb{R}$, assume $|f|^p, |g|^p, (|f| + |g|)$ are Δ -integrable on E . Suppose $f_0, g_0 > 0$ are such that*

$$f_0^p - \int_E |f(t)|^p \Delta t > 0, \quad g_0^p - \int_E |g(t)|^p \Delta t > 0. \quad (10.3)$$

If $p > 1$, then

$$\left(\left(f_0^p - \int_E |f(t)|^p \Delta t \right)^{1/p} + \left(g_0^p - \int_E |g(t)|^p \Delta t \right)^{1/p} \right)^p \leq (f_0 + g_0)^p - \int_E (|f(t)| + |g(t)|)^p \Delta t. \quad (10.4)$$

This inequality is reversed if $0 < p < 1$ or $p < 0$ and $\int_E |f(t)|^p \Delta t > 0$.

Proof. Just apply Theorems 10.1 and 3.7. \square

11. Diaz-Metcalf's Inequality

If $p = q = 2$ and $w = 1$ in the first result of this section, then one has the so-called Diaz-Metcalf inequality. We first present the generalization of this inequality for isotonic linear functionals as given in [2].

Theorem 11.1 (Diaz-Metcalf's inequality [2, Theorem 4.14]). *Let E, L , and A be such that (L_1) , (L_2) , (A_1) , and (A_2) are satisfied. For $p \neq 1$, let $q = p/(p-1)$. Assume $|w||f|^p, |w||g|^q, |wfg| \in L$ and, if $p \neq 0$,*

$$0 < m \leq |f(x)||g(x)|^{-q/p} \leq M, \quad \forall x \in E. \quad (11.1)$$

If $p > 1$, or if $p < 0$ and $A(|w||f|^p) + A(|w||g|^q) > 0$, then

$$(M - m)A(|w||f|^p) + (mM^p - Mm^p)A(|w||g|^q) \leq (M^p - m^p)A(|wfg|). \quad (11.2)$$

This inequality is reversed if $0 < p < 1$ and $A(|w||f|^p) + A(|w||g|^q) > 0$.

Theorem 11.2 (Diaz-Metcalf's inequality). *Let $E \subset \mathbb{R}^n$ be as in Theorem 3.7. For $p \neq 1$, let $q = p/(p-1)$. Assume $|w||f|^p, |w||g|^q, |wfg|$ are Δ -integrable on E and, if $p \neq 0$,*

$$0 < m \leq |f(x)||g(x)|^{-q/p} \leq M, \quad \forall x \in E. \quad (11.3)$$

If $p > 1$, or if $p < 0$ and at least one of the two integrals on the left-hand side of the following inequality is positive, then

$$\begin{aligned} & (M - m) \int_E |w(t)||f(t)|^p \Delta t + (mM^p - Mm^p) \int_E |w(t)||g(t)|^q \Delta t \\ & \leq (M^p - m^p) \int_E |w(t)f(t)g(t)| \Delta t. \end{aligned} \quad (11.4)$$

This inequality is reversed if $0 < p < 1$ and at least one of the two integrals on the left-hand side is positive.

Proof. Just apply Theorems 11.1 and 3.7. \square

The last two results in this section follow from [2, Theorem 4.16 and Theorem 4.18] in the same way as Theorem 11.2 follows from Theorem 11.1.

Theorem 11.3. *Let $E, p, q, \omega, f, g, m, M$ be as in Theorem 11.2. If $p > 1$, then*

$$\int_E |\omega(t)f(t)g(t)| \Delta t \geq |p|^{1/p} |q|^{1/q} \frac{(M-m)^{1/p} (mM^p - Mm^p)^{1/q}}{|M^p - m^p|} \times \left(\int_E |\omega(t)||f(t)|^p \Delta t \right)^{1/p} \left(\int_E |\omega(t)||g(t)|^q \Delta t \right)^{1/q}. \tag{11.5}$$

This inequality is reversed if $p < 0$ or $0 < p < 1$, provided at least one of the two integrals on the right-hand side is positive.

Proof. Just apply [2, Theorem 4.16] and Theorem 3.7. □

Theorem 11.4. *Let $E, p, q, \omega, f, g, m, M$ be as in Theorem 11.2 and assume*

$$0 < m < F(x) \leq M, \quad 0 \leq G(x) \leq M, \quad \forall x \in E, \tag{11.6}$$

where $F = f(f + g)^{-q/p}$ and $G = g(f + g)^{-q/p}$. Let $K(p, q, m, M)$ denote the constant on the right-hand side of the inequality in Theorem 11.3. If $p > 1$, then

$$\left(\int_E |\omega(t)|(|f(t)| + |g(t)|)^p \Delta t \right)^{1/p} \geq K(p, q, m, M) \times \left\{ \left(\int_E |\omega(t)||f(t)|^p \Delta t \right)^{1/p} + \left(\int_E |\omega(t)||g(t)|^p \Delta t \right)^{1/p} \right\}. \tag{11.7}$$

This inequality is reversed if $0 < p < 1$, or if $p < 0$ and the integral on the left-hand side is positive.

Proof. Just apply [2, Theorem 4.18] and Theorem 3.7. □

12. Further Converses of Jensen’s Inequality

Several results from the previous sections are sometimes also called converses of Jensen’s inequality. This section is concerned with some further converses of Jensen’s inequality. The five results presented follow from the specified results in [2] in the same way as Theorem 5.2 follows from Theorem 5.1.

Theorem 12.1. *(a) Assume $\Phi \in C(I, \mathbb{R})$ is convex, where $I = [m, M]$ with $m < M$ such that $\Phi''(x) \geq 0$ with equality for at most isolated points of I . Assume further that either*

- (i) $\Phi(x) > 0$ for all $x \in I$,
- (i') $\Phi(x) > 0$ for all $m < x < M$ with either $\Phi(m) = 0, \Phi'(m) \neq 0$, or $\Phi(M) = 0, \Phi'(M) \neq 0$,

(ii) $\Phi(x) < 0$ for all $x \in I$,

(ii') $\Phi(x) < 0$ for all $m < x < M$ with precisely one of $\Phi(m) = 0, \Phi(M) = 0$.

Let $E \subset \mathbb{R}^n$ be as in Theorem 3.7, and suppose f is Δ -integrable on E such that $f(E) = I$. Moreover, let $h : E \rightarrow \mathbb{R}$ be Δ -integrable such that $\int_E |h(t)| \Delta t > 0$. Then

$$\frac{\int_E |h(t)| \Phi(f(t)) \Delta t}{\int_E |h(t)| \Delta t} \leq \lambda \Phi \left(\frac{\int_E |h(t)| f(t) \Delta t}{\int_E |h(t)| \Delta t} \right) \quad (12.1)$$

holds for some $\lambda > 1$ in cases (i), (i'), or $\lambda \in (0, 1)$ in cases (ii), (ii'). More precisely, a value of λ , depending only on m, M, Φ , may be determined as follows: define $\nu = (\Phi(M) - \Phi(m)) / (M - m)$. If $\nu = 0$, let $\tilde{x} \in (m, M)$ be the unique solution of the equation $\Phi'(x) = 0$; then $\lambda = \Phi(m) / \Phi(\tilde{x})$. If $\nu \neq 0$, let $\tilde{x} \in [m, M]$ be the unique solution of the equation $\nu \Phi(x) - \Phi'(x)(\Phi(m) + \nu(x - m)) = 0$; then $\lambda = \nu / \Phi'(\tilde{x})$. Moreover, one has $\tilde{x} \in (m, M)$ in the cases (i), (ii).

(b) Let all the hypotheses of (a) hold except that Φ is concave on I with $\Phi''(x) \leq 0$ with equality for at most isolated points of I . Then

$$\frac{\int_E |h(t)| \Phi(f(t)) \Delta t}{\int_E |h(t)| \Delta t} \geq \lambda \Phi \left(\frac{\int_E |h(t)| f(t) \Delta t}{\int_E |h(t)| \Delta t} \right), \quad (12.2)$$

where λ is determined as in (a). Furthermore, $\lambda > 1$ holds if $\Phi(x) < 0$ for all $x \in (m, M)$, and $0 < \lambda < 1$ holds if $\Phi(x) > 0$ for all $x \in (m, M)$.

Proof. Just apply [2, Theorem 3.39] and Theorem 3.8. \square

Theorem 12.2. (a) Let E, f, I, m, M, h, ν be as in Theorem 12.1, and let $\Phi \in C(I, \mathbb{R})$ be differentiable such that Φ' is strictly increasing on I . Then

$$\frac{\int_E |h(t)| \Phi(f(t)) \Delta t}{\int_E |h(t)| \Delta t} \leq \lambda + \Phi \left(\frac{\int_E |h(t)| f(t) \Delta t}{\int_E |h(t)| \Delta t} \right) \quad (12.3)$$

for $\lambda = \Phi(m) - \Phi(\tilde{x}) + \nu(\tilde{x} - m) \in (0, (M - m)(\nu - \Phi'(m)))$, where $\tilde{x} \in (m, M)$ is the unique solution of the equation $\Phi'(x) = \nu$.

(b) Let all the hypotheses of (a) hold except that Φ' is strictly decreasing on I . Then

$$\Phi \left(\frac{\int_E |h(t)| f(t) \Delta t}{\int_E |h(t)| \Delta t} \right) \leq \lambda + \frac{\int_E |h(t)| \Phi(f(t)) \Delta t}{\int_E |h(t)| \Delta t} \quad (12.4)$$

for $\lambda = \Phi(\tilde{x}) - \Phi(m) - \nu(\tilde{x} - m) \in (0, (M - m)(\Phi'(m) - \nu))$ with \tilde{x} given in (a).

Proof. Just apply [2, Theorem 3.41] and Theorem 3.8. \square

Theorem 12.3. *In addition to the assumptions of Theorem 5.2, let $J \subset \mathbb{R}$ be an interval such that $J \supset \Phi(I)$ and assume that $F : J \times J \rightarrow \mathbb{R}$ is increasing in the first variable. Then*

$$\begin{aligned} & F\left(\frac{\int_E |h(t)|\Phi(f(t))\Delta t}{\int_E |h(t)|\Delta t}, \Phi\left(\frac{\int_E |h(t)|f(t)\Delta t}{\int_E |h(t)|\Delta t}\right)\right) \\ & \leq \max_{x \in [m, M]} F\left(\frac{M-x}{M-m}\Phi(m) + \frac{x-m}{M-m}\Phi(M), \Phi(x)\right) \\ & = \max_{\theta \in [0,1]} F(\theta\Phi(m) + (1-\theta)\Phi(M), \Phi(\theta m + (1-\theta)M)), \end{aligned} \tag{12.5}$$

and the right-hand side of the inequality is an increasing function of M and a decreasing function of m .

Proof. Just apply [2, Theorem 3.42] and Theorem 3.8. □

Theorem 12.4. *Under the same hypotheses as in Theorem 12.3 except that F is decreasing in its first variable, one has*

$$\begin{aligned} & F\left(\frac{\int_E |h(t)|\Phi(f(t))\Delta t}{\int_E |h(t)|\Delta t}, \Phi\left(\frac{\int_E |h(t)|f(t)\Delta t}{\int_E |h(t)|\Delta t}\right)\right) \\ & \geq \min_{x \in [m, M]} F\left(\frac{M-x}{M-m}\Phi(m) + \frac{x-m}{M-m}\Phi(M), \Phi(x)\right) \\ & = \min_{\theta \in [0,1]} F(\theta\Phi(m) + (1-\theta)\Phi(M), \Phi(\theta m + (1-\theta)M)), \end{aligned} \tag{12.6}$$

and the right-hand side of the inequality is a decreasing function of M and an increasing function of m .

Proof. Just apply [2, Theorem 3.42'] and Theorem 3.8. □

Theorem 12.5. *Assume $\Phi \in C(I, \mathbb{R})$ is convex, where $I \subset \mathbb{R}$ is an interval. Let $E \subset \mathbb{R}^n$ be as in Theorem 3.7. Let $h : E \rightarrow \mathbb{R}$ be Δ -integrable on E such that $0 < \int_E |h(t)|\Delta t < \alpha$ for some $\alpha \in \mathbb{R}$. If $|h|f$ and $|h|(\Phi \circ f)$ are Δ -integrable on E and $a \in I$ is such that*

$$\frac{\alpha a - \int_E |h(t)|f(t)\Delta t}{\alpha - \int_E |h(t)|\Delta t} \in I, \tag{12.7}$$

then

$$\Phi\left(\frac{\alpha a - \int_E |h(t)|f(t)\Delta t}{\alpha - \int_E |h(t)|\Delta t}\right) \geq \frac{\alpha\Phi(a) - \int_E |h(t)|\Phi(f(t))\Delta t}{\alpha - \int_E |h(t)|\Delta t}. \tag{12.8}$$

Proof. Just apply [2, Lemma 4.25] and Theorem 3.7. □

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