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Research Article

Fuzzy Stability of a Functional Equation Deriving from Quadratic and Additive Mappings

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We investigate a fuzzy version of stability for the functional equation $f(2x+y)+f(2x-y)+2f(x)-f(x+y)-f(x-y)-2f(2x)=0$ in the sense of Mirmostafaee and Moslehian.

1. Introduction and Preliminaries

A classical question in the theory of functional equations is "when is it true that a mapping, which approximately satisfies a functional equation, must be somehow close to an exact solution of the equation?". Such a problem, called *a stability problem of the functional equation*, was formulated by Ulam [1] in 1940. In the next year, Hyers [2] gave a partial solution of Ulam's problem for the case of approximate additive mappings. Subsequently, his result was generalized by Aoki [3] for additive mappings, and by Rassias [4] for linear mappings, to considering the stability problem with unbounded Cauchy differences. During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians, see [5–15].

In 1984, Katsaras [16] defined a fuzzy norm on a linear space to construct a fuzzy structure on the space. Since then, some mathematicians have introduced several types of fuzzy norm in different points of view. In particular, Bag and Samanta [17], following Cheng and Mordeson [18], gave an idea of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [19]. In 2008, Mirmostafaee and Moslehian [20] obtained a fuzzy version of stability for *the Cauchy functional equation*

$$f(x+y)-f(x)-f(y)=0. \tag{1.1}$$

In the same year, they [21] proved a fuzzy version of stability for *the quadratic functional equation*

$$f(x+y)+f(x-y)-2f(x)-2f(y)=0. \tag{1.2}$$

We call a solution of (1.1) *an additive mapping* and a solution of (1.2) is called *a quadratic mapping*. Now, we consider the functional equation

$$f(2x+y)+f(2x-y)+2f(x)-f(x+y)-f(x-y)-2f(2x)=0, \tag{1.3}$$

which is called *a functional equation deriving from quadratic and additive mappings*. We call a solution of (1.3) *a general quadratic mapping*. In 2008, Najati and Moghimi [22] obtained a stability of the functional equation (1.3) by taking and composing an additive mapping A and a quadratic mapping Q to prove the existence of a general quadratic mapping F which is close to the given mapping f. In their processing, A is approximate to the odd part $(f(x)-f(-x))/2$ of f, and Q is close to the even part $(f(x)+f(-x))/2-f(0)$ of it, respectively.

In this paper, we get a general stability result of the functional equation deriving from quadratic and additive mappings (1.3) in the fuzzy normed linear space. To do it, we introduce a Cauchy sequence $\{J_n f(x)\}$, starting from a given mapping f, which converges to the desired mapping F in the fuzzy sense. As we mentioned before, in previous studies of stability problem of (1.3), they attempted to get stability theorems by handling the odd and even part of f, respectively. According to our proposal in this paper, we can take the desired approximate solution F at once. Therefore, this idea is a refinement with respect to the simplicity of the proof.

2. Fuzzy Stability of the Functional Equation (1.3)

We use the definition of a fuzzy normed space given in [17] to exhibit a reasonable fuzzy version of stability for the functional equation deriving from quadratic and additive mappings in the fuzzy normed linear space.

Definition 2.1 (see [17]). Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0,1]$ (the so-called fuzzy subset) is said to be *a fuzzy norm on* X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N1) $N(x,c)=0$ for $c \le 0$,

(N2) $x=0$ if and only if $N(x,c)=1$ for all $c>0$,

(N3) $N(cx,t)=N(x,t/|c|)$ if $c \ne 0$,

(N4) $N(x+y,s+t) \ge \min\{N(x,s),N(y,t)\}$,

(N5) $N(x,\cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty}N(x,t)=1$.

The pair (X,N) is called *a fuzzy normed linear space.* Let (X,N) be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in X. Then, $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty}N(x_n-x,t)=1$ for all $t>0$. In this case, x is called *the limit of the sequence* $\{x_n\}$, and we denote it by $N-\lim_{n\to\infty}x_n=x$. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon>0$ and each $t>0$, there exists n_0 such that for all $n \ge n_0$ and all $p>0$ we have $N(x_{n+p}-x_n,t)>1-\varepsilon$. It is known that every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete*, and the fuzzy normed space is called *a fuzzy Banach space*.

Let (X, N) be a fuzzy normed space and (Y, N') a fuzzy Banach space. For a given mapping $f : X \to Y$, we use the abbreviation

$$Df(x,y) := f(2x+y) + f(2x-y) + 2f(x) - f(x+y) - f(x-y) - 2f(2x), \tag{2.1}$$

for all $x, y \in X$. Recall $Df \equiv 0$ means that f is a general quadratic mapping. For given $q > 0$, the mapping f is called *a fuzzy q-almost general quadratic mapping* if

$$N'(Df(x,y), t+s) \geq \min\{N(x, s^q), N(y, t^q)\}, \tag{2.2}$$

for all $x, y \in X \setminus \{0\}$ and all $s, t \in [0, \infty)$. Now, we get the general stability result in the fuzzy normed linear setting.

Theorem 2.2. *Let q be a positive real number with $q \neq 1/2, 1$. And let f be a fuzzy q-almost general quadratic mapping from a fuzzy normed space (X, N) into a fuzzy Banach space (Y, N'). Then, there is a unique general quadratic mapping $F : X \to Y$ such that*

$$N'(F(x) - f(x), t) \geq \sup_{0<t'<t} N\left(x, \frac{t'^q}{((7+2^p+3^p+4^p)/(|4-2^p|3^p) + (5+2\cdot 2^p+3^p)/(2|2-2^p|))^q}\right), \tag{2.3}$$

for each $x \in X$ and $t > 0$, where $p = 1/q$.

Proof. We will prove the theorem in three cases, $q > 1$, $1/2 < q < 1$, and $0 < q < 1/2$.

Case 1. Let $q > 1$. We define a mapping $J_n f : X \to Y$ by

$$J_n f(x) = \frac{1}{2}\left(4^{-n}\left(f(2^n x) + f(-2^n x) - 2f(0)\right) + 2^{-n}\left(f(2^n x) - f(-2^n x)\right)\right) + f(0), \tag{2.4}$$

for all $x \in X$. Then, $J_0 f(x) = f(x)$, $J_j f(0) = f(0)$, and

$$J_j f(x) - J_{j+1} f(x) = \frac{Df(2^j x/3, 2^j x/3)}{4^{j+1}} - \frac{Df(2^j x/3, 2^{j+1} x/3)}{2\cdot 4^{j+1}} - \frac{Df(2^j x/3, 2^j x)}{2\cdot 4^{j+1}}$$
$$- \frac{Df(2^j x/3, 2^{j+2} x/3)}{2\cdot 4^{j+1}} + \frac{Df(-2^j x/3, -2^j x/3)}{4^{j+1}} - \frac{Df(-2^j x/3, -2^{j+1} x/3)}{2\cdot 4^{j+1}}$$

$$-\frac{Df(-2^jx/3,-2^jx)}{2\cdot 4^{j+1}}-\frac{Df(-2^jx/3,-2^{j+2}x/3)}{2\cdot 4^{j+1}}+\frac{Df(2^{j+1}x,2^jx)}{2^{j+2}}$$
$$-\frac{Df(2^jx,3\cdot 2^jx)}{2^{j+2}}+\frac{Df(2^jx,2^jx)}{2^{j+2}}+\frac{Df(2^jx,-2^{j+1}x)}{2^{j+2}}, \tag{2.5}$$

for all $x \in X \setminus \{0\}$ and $j \geq 0$. Together with (N3), (N4), and (2.2), this equation implies that if $n + m > m \geq 0$, then

$$N'\left(J_mf(x)-J_{n+m}f(x),\sum_{j=m}^{n+m-1}\left(\frac{7+2^p+3^p+4^p}{4\cdot 3^p}\left(\frac{2^p}{4}\right)^j+\frac{5+2\cdot 2^p+3^p}{4}\left(\frac{2^p}{2}\right)^j\right)t^p\right)$$
$$\geq \min\bigcup_{j=m}^{n+m-1}\left\{N'\left(J_jf(x)-J_{j+1}f(x),\left(\frac{7+2^p+3^p+4^p}{4^{j+1}\cdot 3^p}+\frac{5+2\cdot 2^p+3^p}{2^{j+2}}\right)2^{jp}t^p\right)\right\}$$
$$\geq \min\bigcup_{j=m}^{n+m-1}\left\{\min\left\{N'\left(\frac{Df(2^jx/3,2^jx/3)}{4^{j+1}},\frac{2^{jp}t^p}{2\cdot 4^j\cdot 3^p}\right),\right.\right.$$
$$N'\left(-\frac{Df(2^jx/3,2^{j+1}x/3)}{2\cdot 4^{j+1}},\frac{2^{jp}(1+2^p)t^p}{2\cdot 4^{j+1}\cdot 3^p}\right),$$
$$N'\left(-\frac{Df(2^jx/3,2^jx)}{2\cdot 4^{j+1}},\frac{2^{jp}(1+3^p)t^p}{2\cdot 4^{j+1}\cdot 3^p}\right),$$
$$N'\left(-\frac{Df(2^jx/3,2^{j+2}x/3)}{2\cdot 4^{j+1}},\frac{2^{jp}(1+4^p)t^p}{2\cdot 4^{j+1}\cdot 3^p}\right),$$
$$N'\left(\frac{Df(-2^jx/3,-2^jx/3)}{4^{j+1}},\frac{2^{jp}t^p}{2\cdot 4^j\cdot 3^p}\right),$$
$$N'\left(-\frac{Df(-2^jx/3,-2^{j+1}x/3)}{2\cdot 4^{j+1}},\frac{2^{jp}(1+2^p)t^p}{2\cdot 4^{j+1}\cdot 3^p}\right),$$
$$N'\left(-\frac{Df(-2^jx/3,-2^jx)}{2\cdot 4^{j+1}},\frac{2^{jp}(1+3^p)t^p}{2\cdot 4^{j+1}\cdot 3^p}\right),$$
$$N'\left(-\frac{Df(-2^jx/3,-2^{j+2}x/3)}{2\cdot 4^{j+1}},\frac{2^{jp}(1+4^p)t^p}{2\cdot 4^{j+1}\cdot 3^p}\right),$$
$$N'\left(\frac{Df(2^{j+1}x,2^jx)}{2^{j+2}},\frac{2^{jp}(1+2^p)t^p}{2^{j+2}}\right),$$
$$N'\left(-\frac{Df(2^jx,3\cdot 2^jx)}{2^{j+2}},\frac{2^{jp}(1+3^p)t^p}{2^{j+2}}\right),$$
$$N'\left(\frac{Df(2^jx,2^jx)}{2^{j+2}},\frac{2^{jp}t^p}{2^{j+1}}\right),$$
$$\left.\left.N'\left(\frac{Df(2^jx,-2^{j+1}x)}{2^{j+2}},\frac{2^{jp}(1+2^p)t^p}{2^{j+2}}\right)\right\}\right\}$$

$$\begin{aligned}
&\geq \min \bigcup_{j=m}^{n+m-1} \left\{ \min\left\{ N\left(2^j x, 2^j t\right), N\left(2^{j+1} x, 2^{j+1} t\right), N\left(3 \cdot 2^j x, 3 \cdot 2^j t\right), \right.\right. \\
&\qquad\qquad \left.\left. N\left(\frac{2^j x}{3}, \frac{2^j t}{3}\right), N\left(\frac{2^{j+1} x}{3}, \frac{2^{j+1} t}{3}\right), N\left(\frac{2^{j+2} x}{3}, \frac{2^{j+2} t}{3}\right) \right\}\right\} \\
&= N(x,t),
\end{aligned} \tag{2.6}$$

for all $x \in X \setminus \{0\}$ and $t > 0$. Let $\varepsilon > 0$ be given. Since $\lim_{t\to\infty} N(x,t) = 1$, there is $t_0 > 0$ such that

$$N(x,t_0) \geq 1 - \varepsilon. \tag{2.7}$$

We observe that for some $\widetilde{t} > t_0$, the series $\sum_{j=0}^{\infty}((7 + 2^p + 3^p + 4^p)/(4^{j+1} \cdot 3^p) + (5 + 2 \cdot 2^p + 3^p)/(2^{j+2}))2^{jp}\widetilde{t}^p$ converges for $p = 1/q < 1$. It guarantees that for an arbitrary given $c > 0$, there exists $n_0 \geq 0$ such that

$$\sum_{j=m}^{n+m-1} \left(\frac{7 + 2^p + 3^p + 4^p}{4^{j+1} \cdot 3^p} + \frac{5 + 2 \cdot 2^p + 3^p}{2^{j+2}} \right) 2^{jp}\widetilde{t}^p < c, \tag{2.8}$$

for each $m \geq n_0$ and $n > 0$. By (N5) and (2.6) we have

$$\begin{aligned}
&N'(J_m f(x) - J_{n+m} f(x), c) \\
&\quad \geq N'\left(J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1} \left(\frac{7 + 2^p + 3^p + 4^p}{4^{j+1} \cdot 3^p} + \frac{5 + 2 \cdot 2^p + 3^p}{2^{j+2}} \right) 2^{jp}\widetilde{t}^p \right) \\
&\quad \geq N\left(x, \widetilde{t}\right) \geq N(x,t_0) \geq 1 - \varepsilon,
\end{aligned} \tag{2.9}$$

for all $x \in X \setminus \{0\}$. Recall $J_n f(0) = f(0)$ for all $n > 0$. Thus, $\{J_n f(x)\}$ becomes a Cauchy sequence for all $x \in X$. Since (Y, N') is complete, we can define a mapping $F : X \to Y$ by

$$F(x) := N' - \lim_{n\to\infty} J_n f(x), \tag{2.10}$$

for all $x \in X$. Moreover, if we put $m = 0$ in (2.6), we have

$$\begin{aligned}
&N'(f(x) - J_n f(x), t) \\
&\quad \geq N\left(x, \frac{t^q}{\left(\sum_{j=0}^{n-1}((7 + 2^p + 3^p + 4^p)/(4^{j+1} \cdot 3^p) + (5 + 2 \cdot 2^p + 3^p)/2^{j+2})2^{jp}\right)^q} \right),
\end{aligned} \tag{2.11}$$

for all $x \in X$. Next, we will show that F is a general quadratic mapping. Using (N4), we have

$$\begin{aligned}
&N'(DF(x,y),t)\\
&\quad \geq \min\Big\{ N'\Big(F(2x+y) - J_n f(2x+y), \frac{t}{16}\Big),\\
&\qquad N'\Big(-F(x+y) + J_n f(x+y), \frac{t}{16}\Big), N'\Big(-F(x-y) + J_n f(x-y), \frac{t}{16}\Big),\\
&\qquad N'\Big(2F(x) - 2J_n f(x), \frac{t}{8}\Big), N'\Big(-2F(2x) + 2J_n f(2x), \frac{t}{8}\Big),\\
&\qquad N'\Big(F(2x-y) + J_n f(2x-y), \frac{t}{16}\Big), N'\Big(DJ_n f(x,y), \frac{t}{2}\Big)\Big\},
\end{aligned} \tag{2.12}$$

for all $x, y \in X \setminus \{0\}$ and $n \in \mathbb{N}$. The first six terms on the right hand side of (2.12) tend to 1 as $n \to \infty$ by the definition of F and (N2), and the last term holds

$$\begin{aligned}
N'\Big(DJ_n f(x,y), \frac{t}{2}\Big) \geq \min\Big\{ & N'\Big(\frac{Df(2^n x, 2^n y)}{2 \cdot 4^n}, \frac{t}{8}\Big), N'\Big(\frac{Df(-2^n x, -2^n y)}{2 \cdot 4^n}, \frac{t}{8}\Big),\\
& N'\Big(\frac{Df(2^n x, 2^n y)}{2 \cdot 2^n}, \frac{t}{8}\Big), N'\Big(\frac{Df(-2^n x, -2^n y)}{2 \cdot 2^n}, \frac{t}{8}\Big)\Big\},
\end{aligned} \tag{2.13}$$

for all $x, y \in X \setminus \{0\}$. By (N3) and (2.2), we obtain

$$\begin{aligned}
N'\Big(\frac{Df(\pm 2^n x, \pm 2^n y)}{2 \cdot 4^n}, \frac{t}{8}\Big) &= N'\Big(Df(\pm 2^n x, \pm 2^n y), \frac{2 \cdot 4^n t}{8}\Big)\\
&\geq \min\Big\{ N\Big(\pm 2^n x, \Big(\frac{4^n t}{8}\Big)^q\Big), N\Big(\pm 2^n y, \Big(\frac{4^n t}{8}\Big)^q\Big)\Big\}\\
&\geq \min\Big\{ N\Big(x, 2^{(2q-1)n - 3q} t^q\Big), N\Big(y, 2^{(2q-1)n - 3q} t^q\Big)\Big\},\\
N'\Big(\frac{Df(\pm 2^n x, \pm 2^n y)}{2 \cdot 2^n}, \frac{t}{8}\Big) &\geq \min\Big\{ N\Big(x, 2^{(q-1)n - 3q} t^q\Big), N\Big(y, 2^{(q-1)n - 3q} t^q\Big)\Big\},
\end{aligned} \tag{2.14}$$

for all $x, y \in X \setminus \{0\}$ and $n \in \mathbb{N}$. Since $q > 1$, together with (N5), we can deduce that the last term of (2.12) also tends to 1 as $n \to \infty$. It follows from (2.12) that

$$N'(DF(x,y),t) = 1, \tag{2.15}$$

for all $x, y \in X \setminus \{0\}$ and $t > 0$. Since $DF(0,0) = 0$, $DF(x,0) = 0$ and $DF(0,y) = 0$ for all $x, y \in X \setminus \{0\}$, this means that $DF(x,y) = 0$ for all $x, y \in X$ by (N2).

Now, we approximate the difference between f and F in a fuzzy sense. For an arbitrary fixed $x \in X$ and $t > 0$, choose $0 < \varepsilon < 1$ and $0 < t' < t$. Since F is the limit of $\{J_n f(x)\}$, there is $n \in \mathbb{N}$ such that $N'(F(x) - J_n f(x), t - t') \geq 1 - \varepsilon$. By (2.11), we have

$$\begin{aligned}
&N'(F(x) - f(x), t) \\
&\quad \geq \min\{N'(F(x) - J_n f(x), t - t'), N'(J_n f(x) - f(x), t')\} \\
&\quad \geq \min\left\{1 - \varepsilon, N\left(x, \frac{t'^q}{\left(\sum_{j=0}^{n-1}((7+2^p+3^p+4^p)/(4^{j+1}\cdot 3^p)+(5+2\cdot 2^p+3^p)/2^{j+2})2^{jp}\right)^q}\right)\right\} \\
&\quad \geq \min\left\{1 - \varepsilon, N\left(x, \frac{t'^q}{((7+2^p+3^p+4^p)/(4-2^p)3^p+(5+2\cdot 2^p+3^p)/2(2-2^p))^q}\right)\right\}.
\end{aligned} \tag{2.16}$$

Because $0 < \varepsilon < 1$ is arbitrary and $F(0) = f(0)$, we get (2.3) in this case.

Finally, to prove the uniqueness of F, let $F' : X \to Y$ be another general quadratic mapping satisfying (2.3). Then, by (2.5), we get

$$\begin{aligned}
F(x) - J_n F(x) &= \sum_{j=0}^{n-1}(J_j F(x) - J_{j+1}F(x)) = 0, \\
F'(x) - J_n F'(x) &= \sum_{j=0}^{n-1}(J_j F'(x) - J_{j+1}F'(x)) = 0,
\end{aligned} \tag{2.17}$$

for all $x \in X$ and $n \in \mathbb{N}$. Together with (N4) and (2.3), this implies that

$$\begin{aligned}
N'(F(x) - F'(x), t) &= N'(J_n F(x) - J_n F'(x), t) \\
&\geq \min\left\{N'\left(J_n F(x) - J_n f(x), \frac{t}{2}\right), N'\left(J_n f(x) - J_n F'(x), \frac{t}{2}\right)\right\} \\
&\geq \min\left\{N'\left(\frac{(F-f)(2^n x)}{2\cdot 4^n}, \frac{t}{8}\right), N'\left(\frac{(f-F')(2^n x)}{2\cdot 4^n}, \frac{t}{8}\right),\right. \\
&\qquad N'\left(\frac{(F-f)(-2^n x)}{2\cdot 4^n}, \frac{t}{8}\right), N'\left(\frac{(f-F')(-2^n x)}{2\cdot 4^n}, \frac{t}{8}\right), \\
&\qquad N'\left(\frac{(F-f)(2^n x)}{2\cdot 2^n}, \frac{t}{8}\right), N'\left(\frac{(f-F')(2^n x)}{2\cdot 2^n}, \frac{t}{8}\right),
\end{aligned}$$

$$N'\left(\frac{(F-f)(-2^n x)}{2\cdot 2^n}, \frac{t}{8}\right), N'\left(\frac{(f-F')(-2^n x)}{2\cdot 2^n}, \frac{t}{8}\right)\Bigg\}$$

$$\geq \sup_{t'<t} N\left(x, \frac{2^{(q-1)n-2q}t'^q}{((7+2^p+3^p+4^p)/(4-2^p)3^p + (5+2\cdot 2^p+3^p)/2(2-2^p))^q}\right), \tag{2.18}$$

for all $x \in X$ and $n \in \mathbb{N}$. Observe that for $q = 1/p$, the last term of the above inequality tends to 1 as $n \to \infty$ by (N5). This implies that $N'(F(x) - F'(x), t) = 1$, and so we get

$$F(x) = F'(x), \tag{2.19}$$

for all $x \in X$ by (N2).

Case 2. Let $1/2 < q < 1$, and let $J_n f : X \to Y$ be a mapping defined by

$$J_n f(x) = \frac{1}{2}\left(4^{-n}\left(f(2^n x) + f(-2^n x) - 2f(0)\right) + 2^n\left(f\left(\frac{x}{2^n}\right) - f\left(-\frac{x}{2^n}\right)\right)\right) + f(0), \tag{2.20}$$

for all $x \in X$. Then, we have $J_0 f(x) = f(x)$, $J_j f(0) = f(0)$, and

$$\begin{aligned} J_j f(x) - J_{j+1} f(x) = {} & \frac{Df(2^j x/3, 2^j x/3)}{4^{j+1}} - \frac{Df(2^j x/3, 2^{j+1} x/3)}{2\cdot 4^{j+1}} - \frac{Df(2^j x/3, 2^j x)}{2\cdot 4^{j+1}} \\ & - \frac{Df(2^j x/3, 2^{j+2} x/3)}{2\cdot 4^{j+1}} + \frac{Df(-2^j x/3, -2^j x/3)}{4^{j+1}} - \frac{Df(-2^j x/3, -2^{j+1} x/3)}{2\cdot 4^{j+1}} \\ & - \frac{Df(-2^j x/3, -2^j x)}{2\cdot 4^{j+1}} - \frac{Df(-2^j x/3, -2^{j+2} x/3)}{2\cdot 4^{j+1}} \\ & - 2^{j-1}\left(Df\left(\frac{x}{2^j}, \frac{x}{2^{j+1}}\right) - Df\left(\frac{x}{2^{j+1}}, \frac{3x}{2^{j+1}}\right) + Df\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right)\right. \\ & \left. + Df\left(\frac{x}{2^{j+1}}, \frac{-x}{2^j}\right)\right), \end{aligned} \tag{2.21}$$

for all $x \in X$ and $j \geq 0$. If $n + m > m \geq 0$, then we have

$$N'\left(J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1}\left(\frac{7+2^p+3^p+4^p}{4\cdot 3^p}\left(\frac{2^p}{4}\right)^j + \frac{5+2\cdot 2^p+3^p}{2\cdot 2^p}\left(\frac{2}{2^p}\right)^j\right)t^p\right)$$

$$\geq \min \bigcup_{j=m}^{n+m-1}\left\{\min\left\{N'\left(\frac{Df(2^j x/3, 2^j x/3)}{4^{j+1}}, \frac{2\cdot 2^{jp} t^p}{4^{j+1}\cdot 3^p}\right),\right.\right.$$

$$
\begin{aligned}
&N'\left(\frac{Df(-2^jx/3,-2^jx/3)}{4^{j+1}},\frac{2\cdot 2^{jp}t^p}{4^{j+1}\cdot 3^p}\right),\\
&N'\left(-\frac{Df(2^jx/3,2^{j+1}x/3)}{2\cdot 4^{j+1}},\frac{2^{jp}(1+2^p)t^p}{2\cdot 4^{j+1}\cdot 3^p}\right),\\
&N'\left(-\frac{Df(2^jx/3,2^jx)}{2\cdot 4^{j+1}},\frac{2^{jp}(1+3^p)t^p}{2\cdot 4^{j+1}\cdot 3^p}\right),\\
&N'\left(-\frac{Df(2^jx/3,2^{j+2}x/3)}{2\cdot 4^{j+1}},\frac{2^{jp}(1+4^p)t^p}{2\cdot 4^{j+1}\cdot 3^p}\right),\\
&N'\left(-\frac{Df(-2^jx/3,-2^{j+1}x/3)}{2\cdot 4^{j+1}},\frac{2^{jp}(1+2^p)t^p}{2\cdot 4^{j+1}\cdot 3^p}\right),\\
&N'\left(-\frac{Df(-2^jx/3,-2^jx)}{2\cdot 4^{j+1}},\frac{2^{jp}(1+3^p)t^p}{2\cdot 4^{j+1}\cdot 3^p}\right),\\
&N'\left(-\frac{Df(-2^jx/3,-2^{j+2}x/3)}{2\cdot 4^{j+1}},\frac{2^{jp}(1+4^p)t^p}{2\cdot 4^{j+1}\cdot 3^p}\right),\\
&N'\left(-2^{j-1}Df\left(\frac{x}{2^j},\frac{x}{2^{j+1}}\right),\frac{2^{j-1}(1+2^p)t^p}{2^{(j+1)p}}\right),\\
&N'\left(2^{j-1}Df\left(\frac{x}{2^{j+1}},\frac{3x}{2^{j+1}}\right),\frac{2^{j-1}(1+3^p)t^p}{2^{(j+1)p}}\right),\\
&N'\left(-2^{j-1}Df\left(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}}\right),\frac{2^jt^p}{2^{(j+1)p}}\right),\\
&N'\left(-2^{j-1}Df\left(\frac{x}{2^{j+1}},-\frac{x}{2^j}\right),\frac{2^{j-1}(1+2^p)t^p}{2^{(j+1)p}}\right)\Bigg\}\Bigg\}\\
\geq \min\bigcup_{j=m}^{n+m-1}\Bigg\{\min\Bigg\{&N\left(\frac{x}{2^{j+1}},\frac{t}{2^{j+1}}\right),N\left(\frac{x}{2^j},\frac{t}{2^j}\right),N\left(\frac{3x}{2^{j+1}},\frac{3t}{2^{j+1}}\right),\\
&N\left(\frac{2^jx}{3},\frac{2^jt}{3}\right),N\left(\frac{2^{j+1}x}{3},\frac{2^{j+1}t}{3}\right),N\left(\frac{2^{j+2}x}{3},\frac{2^{j+2}t}{3}\right)\Bigg\}\Bigg\}\\
= N(x,t),&
\end{aligned}
\tag{2.22}
$$

for all $x \in X$ and $t > 0$. In the similar argument following (2.6) of the previous case, we can define the limit $F(x) := N' - \lim_{n\to\infty} J_n f(x)$ of the Cauchy sequence $\{J_n f(x)\}$ in the Banach fuzzy space Y. Moreover, putting $m = 0$ in the above inequality, we have

$$
\begin{aligned}
&N'(f(x)-J_nf(x),t)\\
&\quad\geq N\left(x,\frac{t^q}{\left(\sum_{j=0}^{n-1}\left((7+2^p+3^p+4^p)/(4\cdot 3^p)(2^p/4)^j+(5+2\cdot 2^p+3^p)/(2\cdot 2^p)(2/2^p)^j\right)\right)^q}\right),
\end{aligned}
\tag{2.23}
$$

for all $x \in X$ and $t > 0$. To prove that F is a general quadratic mapping, we have enough to show that the last term of (2.12) in Case 1 tends to 1 as $n \to \infty$. By (N3) and (2.2), we get

$$\begin{aligned}
N'\left(DJ_nf(x,y),\frac{t}{2}\right) &\geq \min\left\{N'\left(\frac{Df(2^nx,2^ny)}{2\cdot 4^n},\frac{t}{8}\right),N'\left(\frac{Df(-2^nx,-2^ny)}{2\cdot 4^n},\frac{t}{8}\right),\right.\\
&\qquad\left.N'\left(2^{n-1}Df\left(\frac{x}{2^n},\frac{y}{2^n}\right),\frac{t}{8}\right),N'\left(2^{n-1}Df\left(\frac{-x}{2^n},\frac{-y}{2^n}\right),\frac{t}{8}\right)\right\}\\
&\geq \min\left\{N\left(x,2^{(2q-1)n-4q}t^q\right),N\left(y,2^{(2q-1)n-4q}t^q\right),\right.\\
&\qquad\left.N\left(x,2^{(1-q)n-4q}t^q\right),N\left(y,2^{(1-q)n-4q}t^q\right)\right\},
\end{aligned} \tag{2.24}$$

for all $x, y \in X \setminus \{0\}$ and $t > 0$. Observe that all the terms on the right hand side of the above inequality tend to 1 as $n \to \infty$, since $1/2 < q < 1$. Hence, together with the similar argument after (2.12), we can say that $DF(x,y) = 0$ for all $x, y \in X$. Recall that in Case 1, (2.3) follows from (2.11). By the same reasoning, we get (2.3) from (2.23) in this case. Now, to prove the uniqueness of F, let F' be another general quadratic mapping satisfying (2.3). Then, together with (N4), (2.3), and (2.17), we have

$$\begin{aligned}
&N'(F(x)-F'(x),t)\\
&\quad= N'(J_nF(x)-J_nF'(x),t)\\
&\quad\geq \min\left\{N'\left(J_nF(x)-J_nf(x),\frac{t}{2}\right),N'\left(J_nf(x)-J_nF'(x),\frac{t}{2}\right)\right\}\\
&\quad\geq \min\left\{N'\left(\frac{(F-f)(2^nx)}{2\cdot 4^n},\frac{t}{8}\right),\left(\frac{(f-F')(2^nx)}{2\cdot 4^n},\frac{t}{8}\right),\right.\\
&\qquad\quad N'\left(\frac{(F-f)(-2^nx)}{2\cdot 4^n},\frac{t}{8}\right),N'\left(\frac{(f-F')(-2^nx)}{2\cdot 4^n},\frac{t}{8}\right),\\
&\qquad\quad N'\left(2^{n-1}\left((F-f)\left(\frac{x}{2^n}\right)\right),\frac{t}{8}\right),N'\left(2^{n-1}\left((f-F')\left(\frac{x}{2^n}\right)\right),\frac{t}{8}\right),\\
&\qquad\quad\left.N'\left(2^{n-1}\left((F-f)\left(\frac{-x}{2^n}\right)\right),\frac{t}{8}\right),N'\left(2^{n-1}\left((f-F')\left(\frac{-x}{2^n}\right)\right),\frac{t}{8}\right)\right\}\\
&\quad\geq \min\left\{N\left(x,\frac{2^{(2q-1)n-2q}t'^q}{((7+2^p+3^p+4^p)/(4-2^p)3^p+(5+2\cdot 2^p+3^p)/2(2^p-2))^q}\right),\right.\\
&\qquad\quad\left.N\left(x,\frac{2^{(1-q)n-2q}t'^q}{((7+2^p+3^p+4^p)/(4-2^p)3^p+((5+2\cdot 2^p+3^p)/2(2^p-2)))^q}\right)\right\},
\end{aligned} \tag{2.25}$$

for all $x \in X$ and $n \in \mathbb{N}$. Since $\lim_{n\to\infty}2^{(2q-1)n-2q} = \lim_{n\to\infty}2^{(1-q)n-2q} = \infty$ in this case, both terms on the right-hand side of the above inequality tend to 1 as $n \to \infty$ by (N5). This implies that $N'(F(x)-F'(x),t) = 1$, and so $F(x) = F'(x)$ for all $x \in X$ by (N2).

Case 3. Finally, we take $0 < q < 1/2$ and define $J_n f : X \to Y$ by

$$J_n f(x) = \frac{1}{2}\left(4^n\left(f(2^{-n}x) + f(-2^{-n}x) - 2f(0)\right) + 2^n\left(f\left(\frac{x}{2^n}\right) - f\left(-\frac{x}{2^n}\right)\right)\right) + f(0), \tag{2.26}$$

for all $x \in X$. Then, we have $J_0 f(x) = f(x)$, $J_j f(0) = f(0)$, and

$$\begin{aligned}
J_j f(x) - J_{j+1} f(x) = &-4^j Df\left(\frac{x}{3\cdot 2^{j+1}}, \frac{x}{3\cdot 2^{j+1}}\right) - 4^j Df\left(\frac{-x}{3\cdot 2^{j+1}}, \frac{-x}{3\cdot 2^{j+1}}\right) \\
&+ \frac{4^j}{2} Df\left(\frac{x}{3\cdot 2^{j+1}}, \frac{x}{3\cdot 2^{j}}\right) + \frac{4^j}{2} Df\left(\frac{x}{3\cdot 2^{j+1}}, \frac{x}{2^{j+1}}\right) \\
&+ \frac{4^j}{2} Df\left(\frac{x}{3\cdot 2^{j+1}}, \frac{x}{3\cdot 2^{j-1}}\right) + \frac{4^j}{2} Df\left(\frac{-x}{3\cdot 2^{j+1}}, \frac{-x}{3\cdot 2^{j}}\right) \\
&+ \frac{4^j}{2} Df\left(\frac{-x}{3\cdot 2^{j+1}}, \frac{-x}{2^{j+1}}\right) + \frac{4^j}{2} Df\left(\frac{-x}{3\cdot 2^{j+1}}, \frac{-x}{3\cdot 2^{j-1}}\right) \\
&- 2^{j-1}\left(Df\left(\frac{x}{2^j}, \frac{x}{2^{j+1}}\right) - Df\left(\frac{x}{2^{j+1}}, \frac{3x}{2^{j+1}}\right) + Df\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right)\right. \\
&\qquad \left. + Df\left(\frac{x}{2^{j+1}}, \frac{-x}{2^{j}}\right)\right),
\end{aligned} \tag{2.27}$$

which implies that if $n + m > m \geq 0$, then

$$\begin{aligned}
N'\Bigg(J_m f(x) - J_{n+m} f(x), &\sum_{j=m}^{n+m-1}\left(\frac{7 + 2^p + 3^p + 4^p}{2^p\cdot 3^p}\left(\frac{4}{2^p}\right)^j + \frac{5 + 2\cdot 2^p + 3^p}{2\cdot 2^p}\left(\frac{2}{2^p}\right)^j\right)t^p\Bigg) \\
\geq \min \bigcup_{j=m}^{n+m-1}\Bigg\{\min\Bigg\{ &N'\left(-4^j Df\left(\frac{x}{3\cdot 2^{j+1}}, \frac{x}{3\cdot 2^{j+1}}\right), \frac{2\cdot 4^j t^p}{2^{(j+1)p}\cdot 3^p}\right), \\
&N'\left(-4^j Df\left(\frac{-x}{3\cdot 2^{j+1}}, \frac{-x}{3\cdot 2^{j+1}}\right), \frac{2\cdot 4^j t^p}{2^{(j+1)p}\cdot 3^p}\right), \\
&N'\left(\frac{4^j Df\left(x/(3\cdot 2^{j+1}), x/(3\cdot 2^j)\right)}{2}, \frac{4^j(1+2^p)t^p}{2\cdot 2^{(j+1)p}\cdot 3^p}\right), \\
&N'\left(\frac{4^j Df\left(x/(3\cdot 2^{j+1}), x/2^{j+1}\right)}{2}, \frac{4^j(1+3^p)t^p}{2\cdot 2^{(j+1)p}\cdot 3^p}\right), \\
&N'\left(\frac{4^j Df\left(x/(3\cdot 2^{j+1}), x/(3\cdot 2^{j-1})\right)}{2}, \frac{4^j(1+4^p)t^p}{2\cdot 2^{(j+1)p}\cdot 3^p}\right), \\
&N'\left(\frac{4^j Df\left(-x/(3\cdot 2^{j+1}), -x/(3\cdot 2^j)\right)}{2}, \frac{4^j(1+2^p)t^p}{2\cdot 2^{(j+1)p}\cdot 3^p}\right),
\end{aligned}$$

$$
\begin{aligned}
&N'\left(\frac{4^jDf(-x/(3\cdot 2^{j+1}),-x/2^{j+1})}{2},\frac{4^j(1+3^p)t^p}{2\cdot 2^{(j+1)p}\cdot 3^p}\right),\\
&N'\left(\frac{4^jDf(-x/(3\cdot 2^{j+1}),-x/(3\cdot 2^{j-1}))}{2},\frac{4^j(1+4^p)t^p}{2\cdot 2^{(j+1)p}\cdot 3^p}\right),\\
&N'\left(-2^{j-1}Df\left(\frac{x}{2^j},\frac{x}{2^{j+1}}\right),\frac{2^{j-1}(1+2^p)t^p}{2^{(j+1)p}}\right),\\
&N'\left(2^{j-1}Df\left(\frac{x}{2^{j+1}},\frac{3x}{2^{j+1}}\right),\frac{2^{j-1}(1+3^p)t^p}{2^{(j+1)p}}\right),\\
&N'\left(-2^{j-1}Df\left(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}}\right),\frac{2^jt^p}{2^{(j+1)p}}\right),\\
&N'\left(-2^{j-1}Df\left(\frac{x}{2^{j+1}},-\frac{x}{2^j}\right),\frac{2^{j-1}(1+2^p)t^p}{2^{(j+1)p}}\right)\Bigg\}\Bigg\}\\
&\geq \min\bigcup_{j=m}^{n+m-1}\left\{\min\left\{N\left(\frac{x}{2^{j+1}},\frac{t}{2^{j+1}}\right),N\left(\frac{x}{2^j},\frac{t}{2^j}\right),N\left(\frac{3x}{2^{j+1}},\frac{3t}{2^{j+1}}\right),N\left(\frac{x}{2^{j+1}},\frac{t}{2^{j+1}}\right),\right.\right.\\
&\qquad\left.\left.N\left(\frac{x}{3\cdot 2^{j+1}},\frac{t}{3\cdot 2^{j+1}}\right),N\left(\frac{x}{3\cdot 2^j},\frac{t}{3\cdot 2^j}\right),N\left(\frac{x}{3\cdot 2^{j-1}},\frac{t}{3\cdot 2^{j-1}}\right)\right\}\right\}\\
&= N(x,t),
\end{aligned}
\tag{2.28}
$$

for all $x \in X \setminus \{0\}$ and $t > 0$. Similar to the previous cases, it leads us to define the mapping $F : X \to Y$ by $F(x) := N' - \lim_{n\to\infty} J_n f(x)$. Putting $m = 0$ in the above inequality, we have

$$
\begin{aligned}
&N'(f(x)-J_nf(x),t)\\
&\geq N\left(x,\frac{t^q}{\left(\sum_{j=0}^{n-1}\left((7+2^p+3^p+4^p)/(2^p\cdot 3^p)(4/2^p)^j+(5+2\cdot 2^p+3^p)/(2\cdot 2^p)(2/2^p)^j\right)\right)^q}\right),
\end{aligned}
\tag{2.29}
$$

for all $x \in X$. Notice that

$$
\begin{aligned}
N'\left(DJ_nf(x,y),\frac{t}{2}\right) &\geq \min\left\{N'\left(\frac{4^n}{2}Df\left(\frac{x}{2^n},\frac{y}{2^n}\right),\frac{t}{8}\right),N'\left(\frac{4^n}{2}Df\left(\frac{-x}{2^n},\frac{-y}{2^n}\right),\frac{t}{8}\right),\right.\\
&\qquad\left.N'\left(2^{n-1}Df\left(\frac{x}{2^n},\frac{y}{2^n}\right),\frac{t}{8}\right),N'\left(2^{n-1}Df\left(\frac{-x}{2^n},\frac{-y}{2^n}\right),\frac{t}{8}\right)\right\}\\
&\geq \min\left\{N\left(x,2^{(1-2q)n-3q}t^q\right),N\left(y,2^{(1-2q)n-3q}t^q\right)\right\},
\end{aligned}
\tag{2.30}
$$

for all $x, y \in X \setminus \{0\}$ and $t > 0$. Since $0 < q < 1/2$, both terms on the right-hand side tend to 1 as $n \to \infty$, which implies that the last term of (2.12) tends to 1 as $n \to \infty$. Therefore, we can say that $DF \equiv 0$. Moreover, using the similar argument after (2.12) in Case 1, we get

(2.3) from (2.29) in this case. To prove the uniqueness of F, let $F' : X \to Y$ be another general quadratic mapping satisfying (2.3). Then, by (2.17), we get

$$\begin{aligned}
&N'(F(x)-F'(x),t)\\
&\quad\ge \min\left\{N'\left(J_nF(x)-J_nf(x),\frac{t}{2}\right),N'\left(J_nf(x)-J_nF'(x),\frac{t}{2}\right)\right\}\\
&\quad\ge \min\left\{N'\left(\frac{4^n}{2}\left((F-f)\left(\frac{x}{2^n}\right)\right),\frac{t}{8}\right),N'\left(\frac{4^n}{2}\left((f-F')\left(\frac{x}{2^n}\right)\right),\frac{t}{8}\right),\right.\\
&\qquad N'\left(\frac{4^n}{2}\left((F-f)\left(-\frac{x}{2^n}\right)\right),\frac{t}{8}\right),N'\left(\frac{4^n}{2}\left((f-F')\left(-\frac{x}{2^n}\right)\right),\frac{t}{8}\right),\\
&\qquad N'\left(2^{n-1}\left((F-f)\left(\frac{x}{2^n}\right)\right),\frac{t}{8}\right),N'\left(2^{n-1}\left((f-F')\left(\frac{x}{2^n}\right)\right),\frac{t}{8}\right),\\
&\qquad \left.N'\left(2^{n-1}\left((F-f)\left(\frac{-x}{2^n}\right)\right),\frac{t}{8}\right),N'\left(2^{n-1}\left((f-F')\left(\frac{-x}{2^n}\right)\right),\frac{t}{8}\right)\right\}\\
&\quad\ge \sup_{t'<t}N\left(x,\frac{2^{(1-2q)n-2q}t'^q}{((7+2^p+3^p+4^p)/(2^p-4)3^p+(5+2\cdot 2^p+3^p)/2(2^p-2))^q}\right),
\end{aligned} \tag{2.31}$$

for all $x \in X$ and $n \in \mathbb{N}$. Observe that for $0 < q < 1/2$, the last term tends to 1 as $n \to \infty$ by (N5). This implies that $N'(F(x)-F'(x),t) = 1$ and $F(x) = F'(x)$ for all $x \in X$ by (N2). This completes the proof. □

Remark 2.3. Consider a mapping $f : X \to Y$ satisfying (2.2) for all $x, y \in X \setminus \{0\}$ and a real number $q < 0$. Take any $t > 0$. If we choose a real number s with $0 < 2s < t$, then we have

$$N'(Df(x,y),t) \ge N'(Df(x,y),2s) \ge \min\{N(x,s^q),N(y,s^q)\}, \tag{2.32}$$

for all $x, y \in X \setminus \{0\}$. Since $q < 0$, we have $\lim_{s\to 0^+} s^q = \infty$. This implies that

$$\lim_{s\to 0^+} N(x,s^q) = \lim_{s\to 0^+} N(y,s^q) = 1, \tag{2.33}$$

and so

$$N'(Df(x,y),t) = 1, \tag{2.34}$$

for all $t > 0$ and $x, y \in X \setminus \{0\}$. Since $DF(0,0) = 0$, $DF(x,0) = 0$, and $DF(0,y) = 0$ for all $x, y \in X \setminus \{0\}$, this means that $DF(x,y) = 0$ for all $x, y \in X$ by (N2). In other words, f is itself a general quadratic mapping if f is a fuzzy q-almost general quadratic mapping for the case $q < 0$.

We can use Theorem 2.2 to get a classical result in the framework of normed spaces. Let $(X, \|\cdot\|)$ be a normed linear space. Then, we can define a fuzzy norm N_X on X by

$$N_X(x,t) = \begin{cases} 0, & t \leq \|x\|, \\ 1, & t > \|x\|, \end{cases} \tag{2.35}$$

where $x \in X$ and $t \in \mathbb{R}$ [21]. Suppose that $f : X \to Y$ is a mapping into a Banach space $(Y, |||\cdot|||)$ such that

$$\left|\left\|Df(x,y)\right\|\right| \leq \|x\|^p + \|y\|^p, \tag{2.36}$$

for all $x, y \in X$, where $p > 0$ and $p \neq 1, 2$. Let N_Y be a fuzzy norm on Y. Then, we get

$$N_Y(Df(x,y), t+s) = \begin{cases} 0, & t+s \leq \left|\left\|Df(x,y)\right\|\right|, \\ 1, & t+s > \left|\left\|Df(x,y)\right\|\right|, \end{cases} \tag{2.37}$$

for all $x, y \in X$ and $s, t \in \mathbb{R}$. Consider the case $N_Y(Df(x,y), t+s) = 0$. This implies that

$$\|x\|^p + \|y\|^p \geq \left|\left\|Df(x,y)\right\|\right| \geq t+s, \tag{2.38}$$

and so, either $\|x\|^p \geq t$ or $\|y\|^p \geq s$ in this case. Hence, for $q = 1/p$, we have

$$\min\{N_X(x, s^q), N_X(y, t^q)\} = 0, \tag{2.39}$$

for all $x, y \in X$ and $s, t > 0$. Therefore, in every case,

$$N_Y(Df(x,y), t+s) \geq \min\{N_X(x, s^q), N_X(y, t^q)\} \tag{2.40}$$

holds. It means that f is a fuzzy q-almost general quadratic mapping, and by Theorem 2.2, we get the following stability result.

Corollary 2.4. *Let $(X, \|\cdot\|)$ be a normed linear space, and let $(Y, |||\cdot|||)$ be a Banach space. If $f : X \to Y$ satisfies*

$$\left|\left\|Df(x,y)\right\|\right| \leq \|x\|^p + \|y\|^p, \tag{2.41}$$

for all $x, y \in X$, where $p > 0$ and $p \neq 1, 2$, then there is a unique general quadratic mapping $F : X \to Y$ such that

$$\left|\left\|F(x) - f(x)\right\|\right| \leq \left(\frac{2(7 + 2^p + 3^p + 4^p)}{3^p|4 - 2^p|} + \frac{5 + 2\cdot 2 + 3^p}{|2 - 2^p|}\right)\|x\|^p, \tag{2.42}$$

for all $x \in X$.

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