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Research Article

Translation Invariant Spaces and Asymptotic Properties of Variational Equations

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We present a new perspective concerning the study of the asymptotic behavior of variational equations by employing function spaces techniques. We give a complete description of the dichotomous behaviors of the most general case of skew-product flows, without any assumption concerning the flow, the cocycle or the splitting of the state space, our study being based only on the solvability of some associated control systems between certain function spaces. The main results do not only point out new necessary and sufficient conditions for the existence of uniform and exponential dichotomy of skew-product flows, but also provide a clear chart of the connections between the classes of translation invariant function spaces that play the role of the input or output classes with respect to certain control systems. Finally, we emphasize the significance of each underlying hypothesis by illustrative examples and present several interesting applications.

1. Introduction

Starting from a collection of open questions related to the modeling of the equations of mathematical physics in the unified setting of dynamical systems, the study of their qualitative properties became a domain of large interest and with a wide applicability area. In this context, the interaction between the modern methods of pure mathematics and questions arising naturally from mathematical physics created a very active field of research (see [1–18] and the references therein). In recent years, some interesting unsolved problems concerning the long-time behavior of dynamical systems were identified, whose potential results would be of major importance in the process of understanding, clarifying, and solving some of the essential problems belonging to a wide range of scientific domains, among, we mention: fluid mechanics, aeronautics, magnetism, ecology, population dynamics, and so forth. Generally, the asymptotic behavior of the solutions of nonlinear evolution equations arising in mathematical physics can be described in terms of attractors, which are often studied by constructing the skew-product flows of the dynamical processes.

It was natural then to independently consider and analyze the asymptotic behavior of variational systems modeled by skew-product flows (see [3–5, 14–19]). In this framework, two of the most important asymptotic properties are described by uniform dichotomy and exponential dichotomy. Both properties focus on the decomposition of the state space into a direct sum of two closed invariant subspaces such that the solution on these subspaces (uniformly or exponentially) decays backward and forward in time, and the splitting holds at every point of the flow's domain. Precisely, these phenomena naturally lead to the study of the existence of stable and unstable invariant manifolds. It is worth mentioning that starting with the remarkable works of Coppel [20], Daleckii and Krein [21], and Massera and Schäffer [22] the study of the dichotomy had a notable impact on the development of the qualitative theory of dynamical systems (see [1–9, 13, 14, 17, 18, 23]).

A very important step in the infinite-dimensional asymptotic theory of dynamical systems was made by Van Minh et al. in [7] where the authors proposed a unified treatment of the stability, instability, and dichotomy of evolution families on the half-line via input-output techniques. Their paper carried out a beautiful connection between the classical techniques originating in the pioneering works of Perron [11] and Maĭzel [24] and the natural requests imposed by the development of the infinite-dimensional systems theory. They extended the applicability area of the so-called admissibility techniques developed by Massera and Schäffer in [22], from differential equations in infinite-dimensional spaces to general evolutionary processes described by propagators. The authors pointed out that instead of characterizing the behavior of a homogeneous equation in terms of the solvability of the associated inhomogeneous equation (see [20–22]) one may detect the asymptotic properties by analyzing the existence of the solutions of the associated integral system given by the variation of constants formula. These new methods technically moved the central investigation of the qualitative properties into a different sphere, where the study strongly relied on control-type arguments. It is important to mention that the control-type techniques have been also successfully used by Palmer (see [9]) and by Rodrigues and Ruas-Filho (see [13]) in order to formulate characterizations for exponential dichotomy in terms of the Fredholm Alternative. Starting with these papers, the interaction between control theory and the asymptotic theory of dynamical systems became more profound, and the obtained results covered a large variety of open problems (see, e.g., [1, 2, 12, 14–17, 23] and the references therein).

Despite the density of papers devoted to the study of the dichotomy in the past few years and in contrast with the apparent impression that the phenomenon is well understood, a large number of unsolved problems still raise in this topic, most of them concerning the variational case. In the present paper, we will provide a complete answer to such an open question. We start from a natural problem of finding suitable conditions for the existence of uniform dichotomy as well as of exponential dichotomy using control-type methods, emphasizing on the identification of the essential structures involved in such a construction, as the input-output system, the eligible spaces, the interplay between their main properties, the specific lines that make the differences between a necessary and a sufficient condition, and the proper motivation of each underlying condition.

In this paper, we propose an inedit link between the theory of function spaces and the dichotomous behavior of the solutions of infinite dimensional variational systems, which offers a deeper understanding of the subtle mechanisms that govern the control-type approaches in the study of the existence of the invariant stable and unstable manifolds.

We consider the general setting of variational equations described by skew-product flows, and we associate a control system on the real line. Beside obtaining new conditions for the existence of uniform or exponential dichotomy of skew-product flows, the main aim is to clarify the chart of the connections between the classes of translation invariant function spaces that play the role of the input class or of the output class with respect to the associated control system, proposing a merger between the functional methods proceeding from interpolation theory and the qualitative techniques from the asymptotic theory of dynamical systems in infinite dimensional spaces.

We consider the most general case of skew-product flows, without any assumption concerning the flow or the cocycle, without any invertibility property, and we work without assuming any initial splitting of the state space and without imposing any invariance property. Our central aim is to establish the existence of the dichotomous behaviors with all their properties (see Definitions 3.5 and 4.1) based only on the minimal solvability of an associated control system described at every point of the base space by an integral equation on the real line. First, we deduce conditions for the existence of uniform dichotomy of skew-product flows and we discuss the technical consequences implied by the solvability of the associated control system between two appropriate translation invariant spaces. We point out, for the first time, that an adequate solvability on the real line of the associated integral control system (see Definition 3.6) implies both the existence of the uniform dichotomy projections as well as their uniform boundedness. Next, the attention focuses on the exponential behavior on the stable and unstable manifold, preserving the solvability concept from the previous section and modifying the properties of the input and the output spaces. Thus, we deduce a clear overview on the representative classes of function spaces which should be considered in the detection of the exponential dichotomy of skew-product flows in terms of the solvability of associated control systems on the real line. The obtained results provide not only new necessary and sufficient conditions for exponential dichotomy, but also a complete diagram of the specific delimitations between the classes of function spaces which may be considered in the study of the exponential dichotomy compared with those from the uniform dichotomy case. Moreover, we point out which are the specific properties of the underlying spaces which make a difference between the sufficient hypotheses and the necessary conditions for the existence of exponential dichotomy of skew-product flows. Finally, we motivate our techniques by illustrative examples and present several interesting applications of the main theorems which generalize the input-output type results of previous research in this topic, among, we mention the well-known theorems due to Perron [11], Daleckii and Krein [21], Massera and Schäffer [22], Van Minh et al. [7], and so forth.

2. Banach Function Spaces: Basic Notations and Preliminaries

In this section, for the sake of clarity, we recall several definitions and properties of Banach function spaces, and, also, we establish the notations that will be used throughout the paper.

Let $\mathbb{R}$ denote the set of real numbers, let $\mathbb{R}_+ = \{t \in \mathbb{R} : t \geq 0\}$, and let $\mathbb{R}_- = \{t \in \mathbb{R} : t \leq 0\}$. For every $A \subset \mathbb{R}$, χ_A denotes the characteristic function of the set A. Let $\mathcal{M}(\mathbb{R}, \mathbb{R})$ be the linear space of all Lebesgue measurable functions $u : \mathbb{R} \to \mathbb{R}$ identifying the functions which are equal almost everywhere.

Definition 2.1. A linear subspace $B \subset \mathcal{M}(\mathbb{R}, \mathbb{R})$ is called *normed function space* if there is a mapping $|\cdot|_B : B \to \mathbb{R}_+$ such that the following properties hold:

(i) $|u|_B = 0$ if and only if $u = 0$ a.e.;

(ii) $|\alpha u|_B = |\alpha||u|_B$, for all $(\alpha, u) \in \mathbb{R} \times B$;

(iii) $|u + v|_B \le |u|_B + |v|_B$, for all $u, v \in B$;

(iv) if $|u(t)| \le |v(t)|$ a.e. $t \in \mathbb{R}$ and $v \in B$, then $u \in B$ and $|u|_B \le |v|_B$.

If $(B, |\cdot|_B)$ is complete, then B is called a *Banach function space*.

Remark 2.2. If $(B, |\cdot|_B)$ is a Banach function space and $u \in B$, then also $|u(\cdot)| \in B$.

Definition 2.3. A Banach function space $(B, |\cdot|_B)$ is said to be *invariant under translations* if for every $(u, t) \in B \times \mathbb{R}$ the function $u_t : \mathbb{R} \to \mathbb{R}$, $u_t(s) = u(s - t)$ belongs to B and $|u_t|_B = |u|_B$.

Let $\mathcal{C}_c(\mathbb{R}, \mathbb{R})$ be the linear space of all continuous functions $v : \mathbb{R} \to \mathbb{R}$ with compact support. We denote by $\mathcal{T}(\mathbb{R})$ the class of all Banach function spaces B which are invariant under translations, $\mathcal{C}_c(\mathbb{R}, \mathbb{R}) \subset B$ and

(i) for every $t > 0$ there is $c(t) > 0$ such that $\int_0^t |u(\tau)| d\tau \le c(t)|u|_B$, for all $u \in B$;

(ii) if $B \setminus L^1(\mathbb{R}, \mathbb{R}) \neq \emptyset$, then there is a continuous function $\gamma \in B \setminus L^1(\mathbb{R}, \mathbb{R})$.

Remark 2.4. Let $B \in \mathcal{T}(\mathbb{R})$. Then, the following properties hold:

(i) if $J \subset \mathbb{R}$ is a bounded interval, then $\chi_J \in B$.

(ii) if $u_n \to u$ in B, then there is a subsequence $(u_{k_n}) \subset (u_n)$ which converges to u a.e. (see, e.g., [25]).

Remark 2.5. Let $B \in \mathcal{T}(\mathbb{R})$. If $\nu > 0$ and $e_\nu : \mathbb{R} \to \mathbb{R}$ is defined by

$$e_\nu(t) = \begin{cases} e^{-\nu t}, & t \ge 0, \\ 0, & t < 0, \end{cases} \tag{2.1}$$

then it is easy to see that

$$e_\nu(t) = \sum_{n=0}^{\infty} e^{-\nu t} \chi_{[n,n+1)}(t) \le \sum_{n=0}^{\infty} e^{-\nu n} \chi_{[n,n+1)}(t), \quad \forall t \in \mathbb{R}. \tag{2.2}$$

It follows that $e_\nu \in B$ and $|e_\nu|_B \le |\chi_{[0,1)}|_B / (1 - e^{-\nu})$.

Example 2.6. (i) If $p \in [1, \infty)$, then $L^p(\mathbb{R}, \mathbb{R}) = \{u \in \mathcal{M}(\mathbb{R}, \mathbb{R}) : \int_{\mathbb{R}} |u(t)|^p dt < \infty\}$, with respect to the norm $\|u\|_p = (\int_{\mathbb{R}} |u(t)|^p dt)^{1/p}$, is a Banach function space which belongs to $\mathcal{T}(\mathbb{R})$.

(ii) The linear space $L^\infty(\mathbb{R}, \mathbb{R})$ of all measurable essentially bounded functions $u : \mathbb{R} \to \mathbb{R}$ with respect to the norm $\|u\|_\infty = \text{ess sup}_{t \in \mathbb{R}} |u(t)|$ is a Banach function space which belongs to $\mathcal{T}(\mathbb{R})$.

Example 2.7 (Orlicz spaces). Let $\varphi : \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ be a nondecreasing left continuous function which is not identically 0 or ∞ on $(0, \infty)$, and let $Y_\varphi(t) := \int_0^t \varphi(s) ds$. If $u \in \mathcal{M}(\mathbb{R}, \mathbb{R})$ let

$$M_\varphi(u) := \int_{\mathbb{R}} Y_\varphi(|u(s)|) ds. \tag{2.3}$$

The linear space $O_\varphi(\mathbb{R},\mathbb{R}) := \{u \in \mathcal{M}(\mathbb{R},\mathbb{R}) : \exists k > 0$ such that $M_\varphi(ku) < \infty\}$, with respect to the norm

$$|u|_\varphi := \inf\left\{k > 0 : M_\varphi\left(\frac{u}{k}\right) \leq 1\right\}, \tag{2.4}$$

is a Banach function space called the *Orlicz space* associated to φ. It is easy to see that $O_\varphi(\mathbb{R},\mathbb{R})$ is invariant under translations.

Remark 2.8. A remarkable example of Orlicz space is represented by $L^p(\mathbb{R},\mathbb{R})$, for every $p \in [1,\infty]$. This can be obtained for $\varphi(t) = pt^{p-1}$, if $p \in [1,\infty)$ and for

$$\varphi(t) = \begin{cases} 0, & t \in [0,1], \\ \infty, & t > 1, \end{cases} \quad \text{if } p = \infty. \tag{2.5}$$

Lemma 2.9. *If $\varphi(1) < \infty$, then $O_\varphi(\mathbb{R},\mathbb{R}) \in \mathfrak{T}(\mathbb{R})$.*

Proof. Let $v \in \mathcal{C}_c(\mathbb{R},\mathbb{R})$. Then, there are $a,\ b \in \mathbb{R}, a < b$ such that $v(t) = 0$, for all $t \in \mathbb{R} \setminus (a,b)$. Since v is continuous on $[a,b]$, there is $M > 0$ such that $|v(t)| \leq M$, for all $t \in [a,b]$. Then, we have that

$$|v(t)| \leq M\chi_{[a,b]}(t), \quad \forall t \in \mathbb{R}. \tag{2.6}$$

We observe that

$$M_\varphi(\chi_{[a,b]}) = \int_{\mathbb{R}} Y_\varphi(\chi_{[a,b]}(\tau))d\tau = (b-a)Y_\varphi(1) \leq (b-a)\varphi(1) < \infty. \tag{2.7}$$

This implies that $\chi_{[a,b]} \in O_\varphi(\mathbb{R},\mathbb{R})$. Using (2.6), we deduce that $v \in O_\varphi(\mathbb{R},\mathbb{R})$. So, $\mathcal{C}_c(\mathbb{R},\mathbb{R}) \subset O_\varphi(\mathbb{R},\mathbb{R})$.

Since Y_φ is nondecreasing with $\lim_{t\to\infty} Y_\varphi(t) = \infty$, there is $q > 0$ such that $Y_\varphi(t) > 1$, for all $t \geq q$.

Let $t \geq 1$ and let $u \in O_\varphi(\mathbb{R},\mathbb{R}) \setminus \{0\}$. Taking into account that Y_φ is a convex function and using Jensen's inequality (see, e.g., [26]), we deduce that

$$Y_\varphi\left(\frac{1}{t}\int_0^t \frac{|u(\tau)|}{|u|_\varphi}d\tau\right) \leq \frac{1}{t}\int_0^t Y_\varphi\left(\frac{|u(\tau)|}{|u|_\varphi}\right)d\tau \leq M_\varphi\left(\frac{u}{|u|_\varphi}\right) \leq 1. \tag{2.8}$$

This implies that

$$\frac{1}{t}\int_0^t \frac{|u(\tau)|}{|u|_\varphi}d\tau \leq q, \quad \forall t \geq 1. \tag{2.9}$$

In addition, using (2.9), we have that

$$\int_0^t |u(\tau)| d\tau \leq \int_0^1 |u(\tau)| d\tau \leq q|u|_\varphi, \quad \forall t \in [0,1). \tag{2.10}$$

Taking $c : (0,\infty) \to (0,\infty)$, $c(t) = \max\{qt, q\}$, from relations (2.9) and (2.10), it follows that

$$\int_0^t |u(\tau)| d\tau \leq c(t)|u|_\varphi, \quad \forall t \geq 0. \tag{2.11}$$

Since the function c does not depend on u, we obtain that $O_\varphi(\mathbb{R},\mathbb{R}) \in \mathcal{T}(\mathbb{R})$. □

Example 2.10. If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $\varphi(0) = 0$, $\varphi(t) = 1$, for $t \in (0,1]$ and $\varphi(t) = e^{t-1}$, for $t > 1$, then according to Lemma 2.9 we have that the Orlicz space $O_\varphi(\mathbb{R},\mathbb{R}) \in \mathcal{T}(\mathbb{R})$. Moreover, it is easy to see that $O_\varphi(\mathbb{R},\mathbb{R})$ is a proper subspace of $L^1(\mathbb{R},\mathbb{R})$.

Example 2.11. Let $p \in [1,\infty)$ and let $M^p(\mathbb{R},\mathbb{R})$ be the linear space of all $u \in \mathcal{M}(\mathbb{R},\mathbb{R})$ with $\sup_{t\in\mathbb{R}} \int_t^{t+1} |u(s)|^p ds < \infty$. With respect to the norm

$$\|u\|_{M^p} := \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} |u(s)|^p ds \right)^{1/p}, \tag{2.12}$$

this is a Banach function space which belongs to $\mathcal{T}(\mathbb{R})$.

Remark 2.12. If $B \in \mathcal{T}(\mathbb{R})$, then $B \subset M^1(\mathbb{R},\mathbb{R})$.

Indeed, let $c(1) > 0$ be such that $\int_0^1 |u(\tau)| d\tau \leq c(1)|u|_B$, for all $u \in B$. If $u \in B$ we observe that

$$\int_t^{t+1} |u(\tau)| d\tau = \int_0^1 |u_t(\xi)| d\xi \leq c(1)|u_t|_B = c(1)|u|_B, \quad \forall t \in \mathbb{R}, \tag{2.13}$$

so $u \in M^1(\mathbb{R},\mathbb{R})$.

In what follows, we will introduce three remarkable subclasses of $\mathcal{T}(\mathbb{R})$, which will have an essential role in the study of the existence of dichotomy from the next sections. To do this, we first need the following.

Definition 2.13. Let $B \in \mathcal{T}(\mathbb{R})$. The mapping $F_B : (0,\infty) \to \mathbb{R}_+$, $F_B(t) = |\chi_{[0,t)}|_B$ is called *the fundamental function* of the space B.

Remark 2.14. If $B \in \mathcal{T}(\mathbb{R})$, then the fundamental function F_B is nondecreasing.

Notation 1. We denote by $\mathcal{Q}(\mathbb{R})$ the class of all Banach function spaces $B \in \mathcal{T}(\mathbb{R})$ with the property that $\sup_{t>0} F_B(t) = \infty$.

Lemma 2.15. *If $\varphi(t) \in (0, \infty)$, for all $t > 0$, then $O_\varphi(\mathbb{R}, \mathbb{R}) \in \mathcal{Q}(\mathbb{R})$.*

Proof. It is easy to see that Y_φ is strictly increasing, continuous with $Y_\varphi(0) = 0$ and $Y_\varphi(t) \geq (t-1)\varphi(1)$, for all $t > 1$, so $\lim_{t\to\infty} Y_\varphi(t) = \infty$. Hence, Y_φ is bijective.

Let $t > 0$. Since

$$M_\varphi\left(\frac{1}{k}\chi_{[0,t)}\right) = tY_\varphi\left(\frac{1}{k}\right), \quad \forall k > 0, \tag{2.14}$$

it follows that $M_\varphi((1/k)\chi_{[0,t)}) \leq 1$ if and only if $1/Y_\varphi^{-1}(1/t) \leq k$. This implies that

$$F_{O_\varphi(\mathbb{R},\mathbb{R})}(t) = \frac{1}{Y_\varphi^{-1}(1/t)}, \quad \forall t > 0. \tag{2.15}$$

Since $Y_\varphi^{-1}(0) = 0$, from (2.15), we obtain that $O_\varphi(\mathbb{R}, \mathbb{R}) \in \mathcal{Q}(\mathbb{R})$. □

Another distinctive subclass of $\mathcal{T}(\mathbb{R})$ is introduced in the following.

Notation 2. Let $\mathcal{L}(\mathbb{R})$ denote the class of all Banach function spaces $B \in \mathcal{T}(\mathbb{R})$ with the property that $B \setminus L^1(\mathbb{R}, \mathbb{R}) \neq \emptyset$.

Remark 2.16. According to Remark 2.2, we have that if $B \in \mathcal{L}(\mathbb{R})$, then there is a continuous function $\gamma : \mathbb{R} \to \mathbb{R}_+$ such that $\gamma \in B \setminus L^1(\mathbb{R}, \mathbb{R})$.

We will also see, in this paper, that the necessary conditions for the existence of exponential dichotomy should be expressed using another remarkable subclass of $\mathcal{T}(\mathbb{R})$—the rearrangement invariant spaces, see the definitions below.

Definition 2.17. Let $u, v \in \mathcal{M}(\mathbb{R}, \mathbb{R})$. We say that u and v are *equimeasurable* if for every $t > 0$ the sets $\{s \in \mathbb{R} : |u(s)| > t\}$ and $\{s \in \mathbb{R} : |v(s)| > t\}$ have the same measure.

Definition 2.18. A Banach function space $(B, |\cdot|_B)$ is *rearrangement invariant* if for every equimeasurable functions u, v with $u \in B$, we have that $v \in B$ and $|u|_B = |v|_B$.

Notation 3. We denote by $\mathcal{R}(\mathbb{R})$ the class of all Banach function spaces $B \in \mathcal{T}(\mathbb{R})$ which are rearrangement invariant.

Remark 2.19. If $B \in \mathcal{R}(\mathbb{R})$, then B is an interpolation space between $L^1(\mathbb{R},\mathbb{R})$ and $L^\infty(\mathbb{R}, \mathbb{R})$ (see [27, Theorem 2.2, page 106]).

Remark 2.20. The Orlicz spaces are rearrangement invariant (see [27]). Using Lemma 2.9, we deduce that if $\varphi(1) < \infty$, then $O_\varphi(\mathbb{R}, \mathbb{R}) \in \mathcal{R}(\mathbb{R})$.

Lemma 2.21. *Let $B \in \mathcal{R}(\mathbb{R})$ and let $\nu > 0$. Then for every $u \in B$, the functions $\varphi_u, \psi_u : \mathbb{R} \to \mathbb{R}$ defined by*

$$\varphi_u(t) = \int_{-\infty}^{t} e^{-\nu(t-\tau)} u(\tau) d\tau, \qquad \psi_u(t) = \int_{t}^{\infty} e^{-\nu(\tau-t)} u(\tau) d\tau \tag{2.16}$$

belong to B. Moreover, there is $\gamma_{B,\nu} > 0$ which depends only on B and ν such that

$$\left|\varphi_u\right|_B \le \gamma_{B,\nu}|u|_B, \qquad \left|\psi_u\right|_B \le \gamma_{B,\nu}|u|_B, \quad \forall u \in B. \tag{2.17}$$

Proof. We consider the operators

$$\begin{aligned} Z &: L^\infty(\mathbb{R},\mathbb{R}) \longrightarrow L^\infty(\mathbb{R},\mathbb{R}), \quad (Z(u))(t) = \int_{-\infty}^{t} e^{-\nu(t-\tau)}u(\tau)d\tau, \\ W &: L^\infty(\mathbb{R},\mathbb{R}) \longrightarrow L^\infty(\mathbb{R},\mathbb{R}), \quad (W(u))(t) = \int_{t}^{\infty} e^{-\nu(\tau-t)}u(\tau)d\tau. \end{aligned} \tag{2.18}$$

We have that Z and W are correctly defined bounded linear operators. Moreover, the restrictions $Z_| : L^1(\mathbb{R},\mathbb{R}) \to L^1(\mathbb{R},\mathbb{R})$ and $W_| : L^1(\mathbb{R},\mathbb{R}) \to L^1(\mathbb{R},\mathbb{R})$ are correctly defined and bounded linear operators. Since $B \in \mathcal{R}(\mathbb{R})$, then, from Remark 2.19, we have that B is an interpolation space between $L^1(\mathbb{R},\mathbb{R})$ and $L^\infty(\mathbb{R},\mathbb{R})$. This implies that the restrictions $Z_{|B} : B \to B$ and $W_{|B} : B \to B$ are correctly defined and bounded linear operators. Setting $\gamma_{B,\nu} = \max\{\|Z_{|B}\|, \|W_{|B}\|\}$, the proof is complete. □

Notations

If X is a Banach space, for every Banach function space $B \in \mathcal{T}(\mathbb{R})$, we denote by $B(\mathbb{R}, X)$ the space of all Bochner measurable functions $v : \mathbb{R} \to X$ with the property that the mapping $N_v : \mathbb{R} \to \mathbb{R}_+$, $N_v(t) = \|v(t)\|$ belongs to B. With respect to the norm

$$\|v\|_{B(\mathbb{R},X)} := |N_v|_B, \tag{2.19}$$

$B(\mathbb{R}, X)$ is a Banach space. We also denote by $C_{0,c}(\mathbb{R}, X)$ the linear space of all continuous functions $v : \mathbb{R} \to X$ with compact support contained in $(0,\infty)$. It is easy to see that $C_{0,c}(\mathbb{R}, X) \subset B(\mathbb{R}, X)$, for all $B \in \mathcal{T}(\mathbb{R})$.

3. Uniform Dichotomy for Skew-Product Flows

In this section, we start our investigation by studying the existence of by the upper and lower uniform boundedness of the solution in a uniform way on certain complemented subspaces. We will employ a control-type technique and we will show that the use of the function spaces, from the class $\mathcal{T}(\mathbb{R})$ introduced in the previous section, provides several interesting conclusions concerning the qualitative behavior of the solutions of variational equations.

Let X be a real or complex Banach space and let I_d denote the identity operator on X. The norm on X and on $\mathcal{B}(X)$—the Banach algebra of all bounded linear operators on X, will be denoted by $\|\cdot\|$. Let (Θ, d) be a metric space.

Definition 3.1. A continuous mapping $\sigma : \Theta \times \mathbb{R} \to \Theta$ is called a *flow* on Θ if $\sigma(\theta, 0) = \theta$ and $\sigma(\theta, s + t) = \sigma(\sigma(\theta, s), t)$, for all $(\theta, s, t) \in \Theta \times \mathbb{R}^2$.

Definition 3.2. A pair $\pi = (\Phi, \sigma)$ is called a *skew-product flow* on $X \times \Theta$ if σ is a flow on Θ and the mapping $\Phi : \Theta \times \mathbb{R}_+ \to \mathcal{B}(X)$ called *cocycle*, satisfies the following conditions:

(i) $\Phi(\theta,0) = I_d$ and $\Phi(\theta,t+s) = \Phi(\sigma(\theta,s),t)\Phi(\theta,s)$, for all $(\theta,t,s) \in \Theta \times \mathbb{R}_+^2$;

(ii) there are $M \geq 1$ and $\omega > 0$ such that $\|\Phi(\theta,t)\| \leq Me^{\omega t}$, for all $(\theta,t) \in \Theta \times \mathbb{R}_+$;

(iii) for every $(x,\theta) \in X \times \Theta$, the mapping $t \mapsto \Phi(\theta,t)x$ is continuous on $\mathbb{R}_+$.

Example 3.3 (Particular cases). The class described by skew-product flows generalizes the autonomous systems as well as the nonautonomous systems, as the following examples show:

(i) If $\Theta = \mathbb{R}$, then let $\tilde{\sigma}(\theta,t) = \theta + t$ and let $\{U(t,s)\}_{t\geq s}$ be an evolution family on the Banach space X. Setting $\Phi_U(\theta,t) := U(\theta+t,\theta)$, we observe that $\pi_U = (\Phi_U,\tilde{\sigma})$ is a skew-product flow.

(ii) Let $\{T(t)\}_{t\geq 0}$ be a $\mathcal{C}_0$-semigroup on the Banach space X and let Θ be a metric space.

$(\text{ii})_1$ If σ is an arbitrary flow on Θ and $\Phi_T(\theta,t) := T(t)$, then $\pi_T = (\Phi_T,\sigma)$ is a skew-product flow.

$(\text{ii})_2$ Let $\hat{\sigma} : \Theta \times \mathbb{R} \to \Theta$, $\hat{\sigma}(\theta,t) = \theta$ be the projection flow on Θ and let $\{P(\theta)\}_{\theta\in\Theta} \subset \mathcal{B}(X)$ be a uniformly bounded family of projections such that $P(\theta)T(t) = T(t)P(\theta)$, for all $(\theta,t) \in \Theta \times \mathbb{R}_+$. If $\Phi_P(\theta,t) := P(\theta)T(t)$, then $\pi_P = (\Phi_P,\hat{\sigma})$ is a skew-product flow.

Starting with the remarkable work of Foias et al. (see [19]), the qualitative theory of dynamical systems acquired a new perspective concerning the connections between bifurcation theory and the mathematical modeling of nonlinear equations. In [19], the authors proved that classical equations like Navier-Stokes, Taylor-Couette, and Bubnov-Galerkin can be modeled and studied in the unified setting of skew-product flows. In this context, it was pointed out that the skew-product flows often proceed from the linearization of nonlinear equations. Thus, classical examples of skew-product flows arise as operator solutions for variational equations.

Example 3.4 (The variational equation). Let Θ be a locally compact metric space and let σ be a flow on Θ. Let X be a Banach space and let $\{A(\theta) : D(A(\theta)) \subseteq X \to X : \theta \in \Theta\}$ be a family of densely defined closed operators. We consider the variational equation

$$\dot{x}(t) = A(\sigma(\theta,t))x(t), \quad (\theta,t) \in \Theta \times \mathbb{R}_+. \tag{A}$$

A cocycle $\Phi : \Theta \times \mathbb{R}_+ \to \mathcal{B}(X)$ is said to be *a solution of* (A) if for every $\theta \in \Theta$, there is a dense subset $D_\theta \subset D(A(\theta))$ such that for every initial condition $x_\theta \in D_\theta$ the mapping $t \mapsto x(t) := \Phi(\theta,t)x_\theta$ is differentiable on $\mathbb{R}_+$, for every $t \in \mathbb{R}_+ x(t) \in D(A(\sigma(\theta,t)))$ and the mapping $t \mapsto x(t)$ satisfies (A).

An important asymptotic behavior of skew-product flows is described by the uniform dichotomy, which relies on the splitting of the Banach space X at every point $\theta \in \Theta$ into a direct sum of two invariant subspaces such that on the first subspace the trajectory solution is uniformly stable, on the second subspace the restriction of the cocycle is reversible and also the trajectory solution is uniformly unstable on the second subspace. This is given by the following.

Definition 3.5. A skew-product flow $\pi = (\Phi, \sigma)$ is said to be *uniformly dichotomic* if there exist a family of projections $\{P(\theta)\}_{\theta\in\Theta} \subset \mathcal{B}(X)$ and a constant $K \geq 1$ such that the following properties hold:

(i) $\Phi(\theta, t)P(\theta) = P(\sigma(\theta, t))\Phi(\theta, t)$, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$;

(ii) $\|\Phi(\theta, t)x\| \leq K\|x\|$, for all $t \geq 0$, all $x \in \operatorname{Range} P(\theta)$ and all $\theta \in \Theta$;

(iii) the restriction $\Phi(\theta, t)_| : \operatorname{Ker} P(\theta) \to \operatorname{Ker} P(\sigma(\theta, t))$ is an isomorphism, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$;

(iv) $\|\Phi(\theta, t)y\| \geq (1/K)\|y\|$, for all $t \geq 0$, all $y \in \operatorname{Ker} P(\theta)$ and all $\theta \in \Theta$;

(v) $\sup_{\theta\in\Theta}\|P(\theta)\| < \infty$.

In what follows, our main attention will focus on finding suitable conditions for the existence of uniform dichotomy for skew-product flows. To do this, we will introduce an integral control system associated with a skew-product flow such that the input and the output spaces of the system belong to the general class $\mathcal{T}(\mathbb{R})$. We will emphasize that the class $\mathcal{T}(\mathbb{R})$ has an essential role in the study of the dichotomous behavior of variational equations.

Let I, O be two Banach function spaces with $I, O \in \mathcal{T}(\mathbb{R})$. Let $\pi = (\Phi, \sigma)$ be a skew-product flow on $X \times \Theta$. We associate with π the input-output control system $E_\pi = (E_\theta)_{\theta\in\Theta}$, where for every $\theta \in \Theta$

$$f(t) = \Phi(\sigma(\theta, s), t - s)f(s) + \int_s^t \Phi(\sigma(\theta, \tau), t - \tau)v(\tau)d\tau, \quad \forall t \geq s, \tag{E_θ}$$

such that the input function $v \in C_{0,c}(\mathbb{R}, X)$ and the output function $f \in O(\mathbb{R}, X)$.

Definition 3.6. The pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ is said to be *uniformly admissible* for the system (E_π) if there is $L > 0$ such that for every $\theta \in \Theta$, the following properties hold:

(i) for every $v \in C_{0,c}(\mathbb{R}, X)$ there exists $f \in O(\mathbb{R}, X)$ such that the pair (f, v) satisfies (E_θ);

(ii) if $v \in C_{0,c}(\mathbb{R}, X)$ and $f \in O(\mathbb{R}, X)$ are such that the pair (f, v) satisfies (E_θ), then $\|f\|_{O(\mathbb{R},X)} \leq L\|v\|_{I(\mathbb{R},X)}$.

Remark 3.7. (i) According to this admissibility concept, it is sufficient to choose all the input functions from the space $C_{0,c}(\mathbb{R}, X)$, and, thus, we point out that $C_{0,c}(\mathbb{R}, X)$ is in fact *the smaller possible input space* that can be used in the input-output study of the dichotomy.

(ii) It is also interesting to see that the norm estimation from (ii) reflects the presence (and implicitly the structure) of the space $I(\mathbb{R}, X)$. Actually, condition (ii) shows that the norm of each output function in the space $O(\mathbb{R}, X)$ is bounded by the norm of the input function in the space $I(\mathbb{R}, X)$ uniformly with respect to $\theta \in \Theta$.

(iii) In the admissibility concept, there is no need to require the uniqueness of the output function in the property (i), because this follows from condition (ii). Indeed, if the pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ is uniformly admissible for the system (E_π), then from (ii) we deduce that for every $\theta \in \Theta$ and every $v \in C_{0,c}(\mathbb{R}, X)$ there exists *a unique* $f \in O(\mathbb{R}, X)$ such that the pair (f, v) satisfies (E_θ).

In what follows we will analyze the implications of the uniform admissibility of the pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ with $I, O \in \mathcal{T}(\mathbb{R})$ concerning the asymptotic behavior of skew-product flows. With this purpose we introduce two category of subspaces (stable and unstable) and we will point out their role in the detection of the uniform dichotomy.

For every $(x,\theta) \in X \times \Theta$, we consider the function

$$\lambda_{x,\theta} : \mathbb{R} \longrightarrow X, \quad \lambda_{x,\theta}(t) = \begin{cases} \Phi(\theta,t)x, & t \geq 0, \\ 0, & t < 0, \end{cases} \tag{3.1}$$

called *the trajectory* determined by the vector x and the point $\theta \in \Theta$.

For every $\theta \in \Theta$, we denote by $\mathcal{F}(\theta)$ the linear space of all functions $\varphi : \mathbb{R} \to X$ with the property that

$$\varphi(t) = \Phi(\sigma(\theta,s),t-s)\varphi(s), \quad \forall s \leq t \leq 0. \tag{3.2}$$

For every $\theta \in \Theta$, we consider the *stable subset*

$$\mathcal{S}(\theta) = \{x \in X : \lambda_{x,\theta} \in O(\mathbb{R},X)\} \tag{3.3}$$

and, respectively, the *unstable subset*

$$\mathcal{U}(\theta) = \{x \in X : \exists\varphi \in O(\mathbb{R},X) \cap \mathcal{F}(\theta) \text{ with } \varphi(0) = x\}. \tag{3.4}$$

Remark 3.8. It is easy to see that for every $\theta \in \Theta$, $\mathcal{S}(\theta)$, and $\mathcal{U}(\theta)$ are linear subspaces. Therefore, in all what follows, we will refer $\mathcal{S}(\theta)$ as the stable subspace and, respectively, $\mathcal{U}(\theta)$ as the unstable subspace, for each $\theta \in \Theta$.

Proposition 3.9. *For every $(\theta,t) \in \Theta \times \mathbb{R}_+$, the following assertions hold:*

(i) $\Phi(\theta,t)\mathcal{S}(\theta) \subseteq \mathcal{S}(\sigma(\theta,t))$;

(ii) $\Phi(\theta,t)\mathcal{U}(\theta) = \mathcal{U}(\sigma(\theta,t))$.

Proof. The property (i) is immediate. To prove the assertion (ii) let $M,\omega > 0$ be given by Definition 3.2(ii). Let $(\theta,t) \in \Theta \times (0,\infty)$. Let $x \in \mathcal{U}(\theta)$. Then, there is $\varphi \in O(\mathbb{R},X) \cap \mathcal{F}(\theta)$ with $\varphi(0) = x$. We set $y = \Phi(\theta,t)x$, and we consider

$$\psi : \mathbb{R} \longrightarrow X, \quad \psi(s) = \begin{cases} 0, & s > t, \\ \Phi(\theta,s)x, & s \in [0,t], \\ \varphi(s), & s < 0. \end{cases} \tag{3.5}$$

We observe that $\|\psi(s)\| \leq \|\varphi(s)\| + Me^{\omega t}\chi_{[0,t]}(s)\|x\|$, for all $s \in \mathbb{R}$, and since $\varphi \in O(\mathbb{R},X)$, we deduce that $\psi \in O(\mathbb{R},X)$. Using the fact that $\varphi \in \mathcal{F}(\theta)$, we obtain that

$$\psi(s) = \Phi(\sigma(\theta,\tau),s-\tau)\psi(\tau), \quad \forall\tau \leq s \leq t. \tag{3.6}$$

Then, we define the function $\delta : \mathbb{R} \to X$, $\delta(s) = \varphi(s+t)$ and since $O(\mathbb{R}, X)$ is invariant under translations, we deduce that $\delta \in O(\mathbb{R}, X)$. Moreover, from (3.6), it follows that

$$\delta(r) = \Phi(\sigma(\theta, \xi + t), r - \xi)\delta(\xi) = \Phi(\sigma(\sigma(\theta, t), \xi), r - \xi)\delta(\xi), \quad \forall \xi \leq r \leq 0. \tag{3.7}$$

The relation (3.7) implies that $\delta \in \mathcal{F}(\sigma(\theta, t))$, so $y = \delta(0) \in \mathcal{U}(\sigma(\theta, t))$.

Conversely, let $z \in \mathcal{U}(\sigma(\theta, t))$. Then, there is $h \in \mathcal{F}(\sigma(\theta, t)) \cap O(\mathbb{R}, X)$ with $h(0) = z$. Taking $q : \mathbb{R} \to X$, $q(s) = h(s - t)$, we have that $q \in O(\mathbb{R}, X)$ and

$$q(s) = \Phi(\sigma(\theta, \tau), s - \tau)q(\tau), \quad \forall \tau \leq s \leq t. \tag{3.8}$$

In particular, for $\tau \leq s \leq 0$, from (3.8), we deduce that $q \in \mathcal{F}(\theta)$. This implies that $q(0) \in \mathcal{U}(\theta)$. Then, $z = h(0) = q(t) = \Phi(\theta, t)q(0) \in \Phi(\theta, t)\mathcal{U}(\theta)$ and the proof is complete. $\square$

Remark 3.10. From Proposition 3.9(ii), we have that for every $(\theta, t) \in \Theta \times \mathbb{R}_+$ the restriction $\Phi(\theta, t)_| : \mathcal{U}(\theta) \to \mathcal{U}(\sigma(\theta, t))$ is surjective. We also note that according to Proposition 3.9 one may deduce that, the stable subspace and the unstable subspace are candidates for the possible splitting of the main space X required by any dichotomous behavior.

In what follows, we will study the behavior of the cocycle on the stable subspace and also on the unstable subspace and we will deduce several interesting properties of these subspaces in the hypothesis that a pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ of spaces from the class $\mathcal{T}(\mathbb{R})$ is admissible for the control system associated with the skew-product flow.

Theorem 3.11 (The behavior on the stable subspace). *If the pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ is uniformly admissible for the system (E_π), then the following assertions hold:*

(i) *there is $K > 0$ such that $\|\Phi(\theta, t)x\| \leq K\|x\|$, for all $t \geq 0$, all $x \in \mathcal{S}(\theta)$ and all $\theta \in \Theta$;*

(ii) *$\mathcal{S}(\theta)$ is a closed linear subspace, for all $\theta \in \Theta$.*

Proof. Let $L > 0$ be given by Definition 3.6 and let $M, \omega > 0$ be given by Definition 3.2. Let $\alpha : \mathbb{R} \to [0, 2]$ be a continuous function with $\operatorname{supp} \alpha \subset (0, 1)$ and $\int_0^1 \alpha(\tau)d\tau = 1$.

(i) Let $\theta \in \Theta$ and let $x \in \mathcal{S}(\theta)$. We consider the functions

$$v : \mathbb{R} \longrightarrow X, \quad v(t) = \alpha(t)\Phi(\theta, t)x,$$
$$f : \mathbb{R} \longrightarrow X, \quad f(t) = \begin{cases} \Phi(\theta, t)x, & t \geq 1, \\ \displaystyle\int_0^t \alpha(\tau)d\tau\, \Phi(\theta, t)x, & t \in [0, 1), \\ 0, & t < 0. \end{cases} \tag{3.9}$$

Then, $v \in C_{0c}(\mathbb{R}, X)$ and

$$\|f(t)\| \leq \|\lambda_{x,\theta}(t)\|, \quad \forall t \in \mathbb{R}. \tag{3.10}$$

Since $x \in \mathcal{S}(\theta)$, we have that $\lambda_{x,\theta} \in O(\mathbb{R}, X)$. Then, from (3.10), we obtain that $f \in O(\mathbb{R}, X)$. An easy computation shows that the pair (f, v) satisfies (E_θ). Then,

$$\|f\|_{O(\mathbb{R},X)} \leq L\|v\|_{I(\mathbb{R},X)}. \tag{3.11}$$

From $\|v(t)\| \leq \alpha(t)Me^{\omega}\|x\|$, for all $t \in \mathbb{R}$, we obtain that $\|v\|_{I(\mathbb{R},X)} \leq Me^{\omega}|\alpha|_I\|x\|$.

Let $t \geq 2$. From

$$\|\Phi(\theta, t)x\| \leq Me^{\omega}\|\Phi(\theta, s)x\|, \quad \forall s \in [t-1, t), \tag{3.12}$$

it follows that

$$\|\Phi(\theta, t)x\|\chi_{[t-1,t)}(s) \leq Me^{\omega}\|f(s)\|, \quad \forall s \in \mathbb{R}. \tag{3.13}$$

Since O is invariant under translations, we deduce that

$$\|\Phi(\theta, t)x\|F_O(1) \leq Me^{\omega}\|f\|_{O(\mathbb{R},X)}. \tag{3.14}$$

Using relations (3.11) and (3.14), we have that

$$\|\Phi(\theta, t)x\| \leq M^2e^{2\omega}\frac{L|\alpha|_I}{F_O(1)}\|x\|, \quad \forall t \geq 2. \tag{3.15}$$

Since $\|\Phi(\theta, t)x\| \leq Me^{2\omega}\|x\|$, for all $t \in [0, 2)$, setting $K := \max\{(M^2e^{2\omega}L|\alpha|_I)/F_O(1), Me^{2\omega}\}$ we deduce that $\|\Phi(\theta, t)x\| \leq K\|x\|$, for all $t \geq 0$. Taking into account that K does not depend on θ or x, it follows that

$$\|\Phi(\theta, t)x\| \leq K\|x\|, \quad \forall t \geq 0,\ \forall x \in \mathcal{S}(\theta),\ \forall \theta \in \Theta. \tag{3.16}$$

(ii) Let $\theta \in \Theta$ and let $(x_n) \subset \mathcal{S}(\theta)$ with $x_n \underset{n\to\infty}{\longrightarrow} x$. For every $n \in \mathbb{N}$, we consider the sequence

$$\begin{gathered} v_n : \mathbb{R} \longrightarrow X, \quad v_n(t) = \alpha(t)\Phi(\theta, t)x_n, \\ f_n : \mathbb{R} \longrightarrow X, \quad f_n(t) = \begin{cases} \Phi(\theta, t)x_n, & t \geq 1, \\ \displaystyle\int_0^t \alpha(\tau)d\tau\, \Phi(\theta, t)x_n, & t \in [0, 1), \\ 0, & t < 0. \end{cases} \end{gathered} \tag{3.17}$$

We have that $v_n \in C_{0c}(\mathbb{R}, X)$, for all $n \in \mathbb{N}$ and using similar arguments with those used in relation (3.10), we obtain that $f_n \in O(\mathbb{R}, X)$, for all $n \in \mathbb{N}$. An easy computation shows that the pair (f_n, v_n) satisfies (E_θ). Let $v : \mathbb{R} \to X$, $v(t) = \alpha(t)\Phi(\theta, t)x$. Then, $v \in C_{0c}(\mathbb{R}, X)$. According to our hypothesis there is, $f \in O(\mathbb{R}, X)$ such that the pair (f, v) satisfies (E_θ).

Taking $u_n = v_n - v$ and $g_n = f_n - f$ we observe that $u_n \in \mathcal{C}_{0c}(\mathbb{R}, X)$, $g_n \in O(\mathbb{R}, X)$, and the pair (g_n, u_n) satisfies (E_θ). This implies that

$$\|f_n - f\|_{O(\mathbb{R},X)} \le L\|v_n - v\|_{I(\mathbb{R},X)}, \quad \forall n \in \mathbb{N}. \tag{3.18}$$

From $\|v_n(t) - v(t)\| \le \alpha(t) M e^{\omega} \|x_n - x\|$, for all $t \in \mathbb{R}$ and all $n \in \mathbb{N}$, we deduce that

$$\|v_n - v\|_{I(\mathbb{R},X)} \le M e^{\omega} |\alpha|_I \|x_n - x\|, \quad \forall n \in \mathbb{N}. \tag{3.19}$$

From (3.18) and (3.19), it follows that $f_n \underset{n\to\infty}{\longrightarrow} f$ in $O(\mathbb{R}, X)$. From Remark 2.4(ii), we have that there is a subsequence (f_{k_n}) and a negligible set $A \subset \mathbb{R}$ such that $f_{k_n}(t) \underset{n\to\infty}{\longrightarrow} f(t)$, for all $t \in \mathbb{R} \setminus A$. In particular, it follows that there is $r > 1$ such that

$$f(r) = \lim_{n\to\infty} f_{k_n}(r) = \lim_{n\to\infty} \Phi(\theta, r) x_{k_n} = \Phi(\theta, r) x. \tag{3.20}$$

Because the pair (f, v) satisfies (E_θ), we obtain that

$$f(t) = \Phi(\sigma(\theta, r), t - r) f(r) = \Phi(\theta, t) x, \quad \forall t \ge r. \tag{3.21}$$

This shows that $f(t) = \lambda_{x,\theta}(t)$, for all $t \ge r$. Then, from

$$\|\lambda_{x,\theta}(t)\| \le \|f(t)\| + M e^{\omega r} \|x\| \chi_{[0,r)}(t), \quad \forall t \in \mathbb{R}, \tag{3.22}$$

using the fact that $f \in O(\mathbb{R}, X)$ and Remark 2.4(i), we obtain that $\lambda_{x,\theta} \in O(\mathbb{R}, X)$, so $x \in \mathcal{S}(\theta)$.

In conclusion, $\mathcal{S}(\theta)$ is a closed linear subspace, for all $\theta \in \Theta$. □

Theorem 3.12 (The behavior on the unstable subspace). *If the pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ is uniformly admissible for the system (E_π), then the following assertions hold:*

(i) *there is $K > 0$ such that $\|\Phi(\theta, t) y\| \ge (1/K)\|y\|$, for all $t \ge 0$, all $y \in \mathcal{U}(\theta)$ and all $\theta \in \Theta$;*

(ii) *$\mathcal{U}(\theta)$ is a closed linear subspace, for all $\theta \in \Theta$.*

Proof. Let $L > 0$ be given by Definition 3.6 and let $M, \omega > 0$ be given by Definition 3.2. Let $\alpha : \mathbb{R} \to [0, 2]$ be a continuous function with $\operatorname{supp} \alpha \subset (0, 1)$ and $\int_0^1 \alpha(\tau) d\tau = 1$.

(i) Let $\theta \in \Theta$ and let $y \in \mathcal{U}(\theta)$. Then, there is $\varphi \in \mathcal{F}(\theta) \cap O(\mathbb{R}, X)$ with $\varphi(0) = y$. Let $t > 0$. We consider the functions

$$v : \mathbb{R} \longrightarrow X, \quad v(s) = -\alpha(s - t)\Phi(\theta, s) y,$$
$$f : \mathbb{R} \longrightarrow X, \quad f(s) = \begin{cases} \displaystyle\int_s^\infty \alpha(\tau - t) d\tau \, \Phi(\theta, s) y, & s \ge 0, \\ \varphi(s), & s < 0. \end{cases} \tag{3.23}$$

We have that $v \in \mathcal{C}_{0c}(\mathbb{R}, X)$ and f is continuous. Let $m = \sup_{s\in[0,t+1]}\|f(s)\|$. Then, we have that

$$\|f(s)\| \leq \|\varphi(s)\| + m\chi_{[0,t+1]}(s), \quad \forall s \in \mathbb{R}. \tag{3.24}$$

From (3.24) and Remark 2.4(i), we deduce that $f \in O(\mathbb{R}, X)$. An easy computation shows that the pair (f, v) satisfies (E_θ). Then, according to our hypothesis, we have that

$$\|f\|_{O(\mathbb{R},X)} \leq L\|v\|_{I(\mathbb{R},X)}. \tag{3.25}$$

From $\|v(s)\| \leq \alpha(s-t)Me^{\omega}\|\Phi(\theta, t)y\|$, for all $s \in \mathbb{R}$, we obtain that

$$\|v\|_{I(\mathbb{R},X)} \leq |\alpha|_I Me^{\omega}\|\Phi(\theta, t)y\|. \tag{3.26}$$

Since $y = \varphi(0) = \Phi(\sigma(\theta, s), -s)\varphi(s)$, for all $s \in [-1, 0)$, we have that

$$\|y\|\chi_{[-1,0)}(s) \leq Me^{\omega}\|\varphi(s)\|\chi_{[-1,0)}(s) \leq Me^{\omega}\|f(s)\|, \quad \forall s \in \mathbb{R}. \tag{3.27}$$

Using the invariance under translations of the space O from relation (3.27), we obtain that

$$\|y\|F_O(1) \leq Me^{\omega}\|f\|_{O(\mathbb{R},X)}. \tag{3.28}$$

Taking $K = (M^2e^{2\omega}L|\alpha|_I)/F_O(1)$ from relations (3.25), (3.26), and (3.28), it follows that $\|\Phi(\theta, t)y\| \geq (1/K)\|y\|$. Taking into account that K does not depend on t, y or θ, we conclude that

$$\|\Phi(\theta, t)y\| \geq \frac{1}{K}\|y\|, \quad \forall t \geq 0,\ \forall y \in \mathcal{U}(\theta),\ \forall \theta \in \Theta. \tag{3.29}$$

(ii) Let $\theta \in \Theta$ and let $(y_n) \subset \mathcal{U}(\theta)$ with $y_n \to y$. Then, for every $n \in \mathbb{N}$, there is $\varphi_n \in O(\mathbb{R}, X) \cap \mathcal{F}(\theta)$ with $\varphi_n(0) = y_n$. For every $n \in \mathbb{N}$, we consider the functions

$$\begin{gathered} v_n : \mathbb{R} \longrightarrow X, \quad v_n(t) = -\alpha(t)\Phi(\theta, t)y_n, \\ f_n : \mathbb{R} \longrightarrow X, \quad f_n(t) = \begin{cases} \displaystyle\int_t^\infty \alpha(\tau)d\tau\, \Phi(\theta, t)y_n, & t \geq 0, \\ \varphi_n(t), & t < 0. \end{cases} \end{gathered} \tag{3.30}$$

We have that $v_n \in \mathcal{C}_{0c}(\mathbb{R}, X)$, and, using similar arguments with those used in relation (3.24), we deduce that $f_n \in O(\mathbb{R}, X)$, for all $n \in \mathbb{N}$. An easy computation shows that the pair (f_n, v_n) satisfies (E_θ). Let

$$v : \mathbb{R} \longrightarrow X, \quad v(t) = -\alpha(t)\Phi(\theta, t)y. \tag{3.31}$$

According to our hypothesis, there is $f \in O(\mathbb{R}, X)$ such that the pair (f, v) satisfies (E_θ). In particular, this implies that $f \in \mathcal{F}(\theta)$. Moreover, for every $n \in \mathbb{N}$, the pair $(f_n - f, v_n - v)$ satisfies (E_θ). According to our hypothesis, it follows that

$$\|f_n - f\|_{O(\mathbb{R},X)} \leq L\|v_n - v\|_{I(\mathbb{R},X)}, \quad \forall n \in \mathbb{N}. \tag{3.32}$$

We have that $\|v_n(t) - v(t)\| \leq \alpha(t)Me^{\omega}\|y_n - y\|$, for all $t \in \mathbb{R}$ and all $n \in \mathbb{N}$, so

$$\|v_n - v\|_{I(\mathbb{R},X)} \leq Me^{\omega}|\alpha|_I\|y_n - y\|, \quad \forall n \in \mathbb{N}. \tag{3.33}$$

From (3.32) and (3.33) it follows that $f_n \underset{n\to\infty}{\longrightarrow} f$ in $O(\mathbb{R}, X)$. Then, from Remark 2.4(ii), there is a subsequence $(f_{k_n}) \subset (f_n)$ and a negligible set $A \subset \mathbb{R}$ such that $f_{k_n}(t) \underset{n\to\infty}{\longrightarrow} f(t)$, for all $t \in \mathbb{R} \setminus A$. In particular, there is $h < 0$ such that $f_{k_n}(h) \underset{n\to\infty}{\longrightarrow} f(h)$. Since $f,\ f_{k_n} \in \mathcal{F}(\theta)$, we successively deduce that

$$y = \lim_{n\to\infty} y_{k_n} = \lim_{n\to\infty} f_{k_n}(0) = \lim_{n\to\infty} \Phi(\sigma(\theta,h),-h)f_{k_n}(h) = \Phi(\sigma(\theta,h),-h)f(h) = f(0). \tag{3.34}$$

This implies that $y \in \mathcal{U}(\theta)$, so $\mathcal{U}(\theta)$ is a closed linear subspace. □

Taking into account the above results it makes sense to study whether the uniform admissibility of a pair of function spaces from the class $\mathcal{T}(\mathbb{R})$ is a sufficient condition for the existence of the uniform dichotomy. Thus, the main result of this section is as follows.

Theorem 3.13 (Sufficient condition for uniform dichotomy). *Let $O, I \in \mathcal{T}(\mathbb{R})$ and let $\pi = (\Phi, \sigma)$ be a skew-product flow on $X \times \Theta$. If the pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ is uniformly admissible for the system (E_π), then π is uniformly dichotomic.*

Proof. Let $L > 0$ be given by Definition 3.6. Let $M, \omega > 0$ be given by Definition 3.2. Let $\alpha : \mathbb{R} \to [0, 2]$ be a continuous function with $\operatorname{supp}\alpha \subset (0, 1)$ and $\int_0^1 \alpha(\tau)d\tau = 1$. □

Step 1. We prove that $\mathcal{S}(\theta) \cap \mathcal{U}(\theta) = \{0\}$, for all $\theta \in \Theta$.

Let $\theta \in \Theta$ and let $x \in \mathcal{S}(\theta) \cap \mathcal{U}(\theta)$. Then, there is $\varphi \in O(\mathbb{R}, X) \cap \mathcal{F}(\theta)$ with $\varphi(0) = x$. We consider the function

$$f : \mathbb{R} \to X, \qquad f(t) = \begin{cases} \Phi(\theta, t)x, & t \geq 0, \\ \varphi(t), & t < 0. \end{cases} \tag{3.35}$$

Then, $\|f(t)\| \leq \|\varphi(t)\| + \|\lambda_{x,\theta}(t)\|$, for all $t \in \mathbb{R}$. This implies that $f \in O(\mathbb{R}, X)$. An easy computation shows that the pair $(f, 0)$ satisfies (E_θ). Then, according to our hypothesis, it follows that $\|f\|_{O(\mathbb{R},X)} = 0$, so $f(t) = 0$ a.e. $t \in \mathbb{R}$. Observing that f is continuous, we obtain that $f(t) = 0$, for all $t \in \mathbb{R}$. In particular, we have that $x = f(0) = 0$.

Step 2. We prove that $\mathcal{S}(\theta) + \mathcal{U}(\theta) = X$, for all $\theta \in \Theta$.

Let $\theta \in \Theta$ and let $x \in X$. Let $v : \mathbb{R} \to X$, $v(t) = \alpha(t)\Phi(\theta, t)x$. Then, $v \in \mathcal{C}_{0c}(\mathbb{R}, X)$, so there is $f \in O(\mathbb{R}, X)$ such that the pair (f, v) satisfies (E_θ). In particular, this implies that $f \in \mathcal{F}(\theta)$, so $f(0) \in \mathcal{U}(\theta)$. In addition, we observe that

$$f(t) = \Phi(\theta, t)f(0) + \left(\int_0^t \alpha(\tau)d\tau\right)\Phi(\theta, t)x = \Phi(\theta, t)\big(f(0) + x\big), \quad \forall t \geq 1. \tag{3.36}$$

Setting $z_x = f(0) + x$ from (3.36), we have that $\lambda_{z_x,\theta}(t) = f(t)$, for all $t \geq 1$. It follows that

$$\|\lambda_{z_x,\theta}(t)\| \leq \|f(t)\| + Me^{\omega}\|z_x\|\chi_{[0,1)}(t), \quad \forall t \in \mathbb{R}. \tag{3.37}$$

From relation (3.37) and Remark 2.4(i) we obtain that $\lambda_{z_x,\theta} \in O(\mathbb{R}, X)$, so $z_x \in \mathcal{S}(\theta)$. This shows that $x = z_x - f(0) \in \mathcal{S}(\theta) + \mathcal{U}(\theta)$, so $\mathcal{S}(\theta) + \mathcal{U}(\theta) = X$.

According to Steps 1 and 2, Theorem 3.11(ii), and Theorem 3.12(ii), we deduce that

$$\mathcal{S}(\theta) \oplus \mathcal{U}(\theta) = X, \quad \forall \theta \in \Theta. \tag{3.38}$$

For every $\theta \in \Theta$ we denote by $P(\theta)$ the projection with the property that

$$\text{Range}\, P(\theta) = \mathcal{S}(\theta), \qquad \text{Ker}\, P(\theta) = \mathcal{U}(\theta). \tag{3.39}$$

Using Proposition 3.9 we obtain that

$$\Phi(\theta, t)P(\theta) = P(\sigma(\theta, t))\Phi(\theta, t), \quad \forall (\theta, t) \in \Theta \times \mathbb{R}_+. \tag{3.40}$$

Let $(\theta, t) \in \Theta \times \mathbb{R}_+$. From Proposition 3.9(ii), it follows that the restriction $\Phi(\theta, t)_| : \text{Ker}\, P(\theta) \to \text{Ker}\, P(\sigma(\theta, t))$ is correctly defined and surjective. According to Theorem 3.12(ii) we have that $\Phi(\theta, t)_|$ is also injective, so this is an isomorphism, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$.

Step 3. We prove that $\sup_{\theta\in\Theta}\|P(\theta)\| < \infty$.

Let $\theta \in \Theta$ and let $x \in X$. Let $x_s^\theta = P(\theta)x$ and let $x_u^\theta = (I - P(\theta))x$. Since $x_u^\theta \in \text{Ker}\, P(\theta) = \mathcal{U}(\theta)$, there is $\varphi \in \mathcal{F}(\theta) \cap O(\mathbb{R}, X)$ with $\varphi(0) = x_u^\theta$. We consider the functions

$$\begin{gathered} v : \mathbb{R} \longrightarrow X, \quad v(t) = \alpha(t)\Phi(\theta, t)x, \\ f : \mathbb{R} \to X, \quad f(t) = \begin{cases} \Phi(\theta, t)x_s^\theta, & t \geq 1, \\ -\Phi(\theta, t)x_u^\theta + \left(\displaystyle\int_0^t \alpha(\tau)d\tau\right)\Phi(\theta, t)x, & t \in [0, 1), \\ -\varphi(t), & t < 0. \end{cases} \end{gathered} \tag{3.41}$$

We have that $v \in \mathcal{C}_{0c}(\mathbb{R}, X)$ and f is continuous. From $x_s^\theta \in \operatorname{Range} P(\theta) = \mathcal{S}(\theta)$, we have that the function $\lambda_{x_s^\theta,\theta}$ belongs to $O(\mathbb{R}, X)$. Setting $m = \sup_{t\in[0,1]} \|f(t)\|$ and observing that

$$\|f(t)\| \leq \|\varphi(t)\| + m\chi_{[0,1]}(t) + \|\lambda_{x_s^\theta,\theta}(t)\|, \quad \forall t \in \mathbb{R}, \tag{3.42}$$

from (3.42), we deduce that $f \in O(\mathbb{R}, X)$. An easy computation shows that the pair (f, v) satisfies (E_θ). This implies that

$$\|f\|_{O(\mathbb{R},X)} \leq L\|v\|_{I(\mathbb{R},X)}. \tag{3.43}$$

Since $\varphi \in \mathcal{F}(\theta)$, we have that $x_u^\theta = \varphi(0) = \Phi(\sigma(\theta, s), -s)\varphi(s)$, for all $s \in [-1, 0)$. This implies that

$$\|x_u^\theta\| \leq Me^\omega\|\varphi(s)\| = Me^\omega\|f(s)\|, \quad \forall s \in [-1, 0), \tag{3.44}$$

and we obtain that

$$\|x_u^\theta\|\chi_{[-1,0)}(s) \leq Me^\omega\|f(s)\|, \quad \forall s \in \mathbb{R}. \tag{3.45}$$

Using the invariance under translations of the space O, from relation (3.45) we deduce that

$$\|x_u^\theta\|F_O(1) \leq Me^\omega\|f\|_{O(\mathbb{R},X)}. \tag{3.46}$$

In addition, from

$$\|v(t)\| \leq \alpha(t)Me^\omega\|x\|, \quad \forall t \in \mathbb{R}, \tag{3.47}$$

we obtain that

$$\|v\|_{I(\mathbb{R},X)} \leq |\alpha|_I Me^\omega\|x\|. \tag{3.48}$$

Setting $\gamma := [L|\alpha|_I M^2 e^{2\omega}/F_O(1)]$ from relations (3.43), (3.46), and (3.48), we have that

$$\|(I - P(\theta))x\| = \|x_u^\theta\| \leq \gamma\|x\|. \tag{3.49}$$

This implies that

$$\|P(\theta)x\| \leq (1 + \gamma)\|x\|. \tag{3.50}$$

Taking into account that γ does not depend on θ or x, it follows that relation (3.50) holds, for all $\theta \in \Theta$ and all $x \in X$, so $\|P(\theta)\| \leq 1 + \gamma$, for all $\theta \in \Theta$.

Finally, from Theorem 3.11(i) and Theorem 3.12(i), we conclude that π is uniformly dichotomic.

Remark 3.14. Relation (3.39) shows that the stable subspace and the instable subspace play a central role in the detection of the dichotomous behavior of a skew-product flow and gives a comprehensible motivation for their usual appellation.

4. Exponential Dichotomy of Skew-Product Flows

In the previous section, we have obtained sufficient conditions for the uniform dichotomy of a skew-product flow $\pi = (\Phi, \sigma)$ on $X \times \Theta$ in terms of the uniform admissibility of the pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ for the associated control system (E_π), where $O, I \in \mathcal{T}(\mathbb{R})$. The natural question arises: which are the additional (preferably minimal) hypotheses under which this admissibility may provide the existence of the *exponential* dichotomy? In this context, the main purpose of this section is to establish which are the most general classes of Banach function spaces where O or I may belong to, such that the uniform admissibility of the pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ for the control system (E_π) is a sufficient (and also a necessary) condition for the existence of exponential dichotomy.

Let X be a real or complex Banach space and let (Θ, d) be a metric space. Let $\pi = (\Phi, \sigma)$ be a skew-product flow on $X \times \Theta$.

Definition 4.1. A skew-product flow $\pi = (\Phi, \sigma)$ is said to be *exponentially dichotomic* if there exist a family of projections $\{P(\theta)\}_{\theta \in \Theta} \subset \mathcal{B}(X)$ and two constants $K \geq 1$ and $\nu > 0$ such that the following properties hold:

(i) $\Phi(\theta, t)P(\theta) = P(\sigma(\theta, t))\Phi(\theta, t)$, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$;

(ii) $\|\Phi(\theta, t)x\| \leq Ke^{-\nu t}\|x\|$, for all $t \geq 0$, all $x \in \operatorname{Range} P(\theta)$ and all $\theta \in \Theta$;

(iii) the restriction $\Phi(\theta, t)_| : \operatorname{Ker} P(\theta) \to \operatorname{Ker} P(\sigma(\theta, t))$ is an isomorphism, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$;

(iv) $\|\Phi(\theta, t)y\| \geq (1/K)e^{\nu t}\|y\|$, for all $t \geq 0$, all $y \in \operatorname{Ker} P(\theta)$ and all $\theta \in \Theta$.

Before proceeding to the next steps, we need a technical lemma.

Lemma 4.2. *If a skew-product flow π is exponentially dichotomic with respect to a family of projections $\{P(\theta)\}_{\theta \in \Theta}$, then $\sup_{\theta \in \Theta}\|P(\theta)\| < \infty$.*

Proof. Let $K, \nu > 0$ be given by Definition 4.1 and let $M, \omega > 0$ be given by Definition 3.2. For every $(x, \theta) \in X \times \Theta$ and every $t \geq 0$, we have that

$$\begin{aligned} \frac{1}{K}e^{\nu t}\|(I - P(\theta))x\| &\leq \|\Phi(\theta, t)(I - P(\theta))x\| \leq Me^{\omega t}\|x\| + Ke^{-\nu t}\|P(\theta)x\| \\ &\leq \left(Me^{\omega t} + K\right)\|x\| + Ke^{-\nu t}\|(I - P(\theta))x\|, \end{aligned} \tag{4.1}$$

which implies that

$$\left(e^{2\nu t} - K^2\right)\frac{e^{-\nu t}}{K}\|(I - P(\theta))x\| \leq \left(Me^{\omega t} + K\right)\|x\|, \quad \forall t \geq 0,\ \forall (x, \theta) \in X \times \Theta. \tag{4.2}$$

Let $h > 0$ be such that $e^{2\nu h} - K^2 > 0$. Setting $\alpha := (e^{2\nu h} - K^2)e^{-\nu h}/K$ and $\delta := (Me^{\omega h} + K)$, it follows that $\|(I-P(\theta))x\| \le (\delta/\alpha)\|x\|$, for all $(x,\theta) \in X\times\Theta$. This implies that $\|I-P(\theta)\| \le \delta/\alpha$, for all $\theta \in \Theta$, so $\|P(\theta)\| \le 1 + (\delta/\alpha)$, for all $\theta \in \Theta$, and the proof is complete. □

Remark 4.3. (i) Using Lemma 4.2, we deduce that if a skew-product flow π is exponentially dichotomic with respect to a family of projections $\{P(\theta)\}_{\theta\in\Theta}$, then π is uniformly dichotomic with respect to the same family of projections.

(ii) If a skew-product flow π is exponentially dichotomic with respect to a family of projections $\{P(\theta)\}_{\theta\in\Theta}$, then this family is uniquely determined (see, e.g., [18], Remark 2.5).

Remark 4.4. In the description of any dichotomous behavior, the properties (i) and (iii) are inherent, because beside the splitting of the space ensured by the presence of the dichotomy projections, these properties reflect both the invariance with respect to the decomposition induced by each projection as well as the reversibility of the cocycle restricted to the kernel of each projection.

In this context, it is extremely important to note that if in the detection of the dichotomy one assumes from the very beginning that there exist a projection family such that the invariance property (i) and the reversibility condition (iii) hold, then the dichotomy concept is resumed to a stability property (ii) and to an instability condition (iv), which via (iii) will consist only of a double stability. Thus, if in the study of the dichotomy one considers (i) and (iii) as working hypotheses, then the entire investigation is reduced to a quasitrivial case of (double) stability.

In conclusion, in the study of the existence of (uniform or) exponential dichotomy, it is essential to determine conditions *which imply the existence of the projection family* and also the fulfillment of *all* the conditions from Definition 4.1.

Now let O, I be two Banach function spaces such that $O, I \in \mathcal{T}(\mathbb{R})$. According to the main result in the previous section (see Theorem 3.13), if the pair $(O(\mathbb{R},X), I(\mathbb{R},X))$ is uniformly admissible for the system (E_π), then π is uniformly dichotomic with respect to a family of projections $\{P(\theta)\}_{\theta\in\Theta}$ with the property that

$$\operatorname{Range} P(\theta) = \mathcal{S}(\theta), \qquad \operatorname{Ker} P(\theta) = \mathcal{U}(\theta), \quad \forall\theta \in \Theta. \tag{4.3}$$

In what follows, we will see that by imposing some conditions *either* on the output space O *or* on the input space I, the admissibility becomes a sufficient condition for the exponential dichotomy.

Theorem 4.5 (The behavior on the stable subspace). *Let O, I be two Banach function spaces such that either $O \in \mathcal{Q}(\mathbb{R})$ or $I \in \mathcal{L}(\mathbb{R})$. If the pair $(O(\mathbb{R},X),I(\mathbb{R},X))$ is uniformly admissible for the system (E_π), then there are $K, \nu > 0$ such that*

$$\|\Phi(\theta,t)x\| \le Ke^{-\nu t}\|x\|, \quad \forall t \ge 0,\ \forall x \in \operatorname{Range} P(\theta),\ \forall\theta \in \Theta. \tag{4.4}$$

Proof. Let $\delta > 0$ be such that

$$\|\Phi(\theta,t)x\| \le \delta\|x\|, \quad \forall t \ge 0,\ \forall x \in \operatorname{Range} P(\theta),\ \forall\theta \in \Theta. \tag{4.5}$$

We prove that there is $h > 0$ such that

$$\|\Phi(\theta, h)x\| \leq \frac{1}{e}\|x\|, \quad \forall x \in \text{Range}\, P(\theta),\ \forall \theta \in \Theta. \tag{4.6}$$

Let $L > 0$ be given by Definition 3.6 and let $M, \omega > 0$ be given by Definition 3.2.

Case 1. Suppose that $O \in \mathcal{Q}(\mathbb{R})$. Let $\alpha : \mathbb{R} \longrightarrow [0, 2]$ be a continuous function with $\operatorname{supp} \alpha \subset (0, 1)$ such that $\int_0^1 \alpha(\tau)d\tau = 1$. Since $\sup_{t>0} F_O(t) = \infty$, there is $r > 0$ such that

$$F_O(r) \geq e\delta^2 L|\alpha|_I. \tag{4.7}$$

Let $\theta \in \Theta$ and let $x \in \text{Range}\, P(\theta)$. If $\Phi(\theta, 1)x \neq 0$, then we consider the functions

$$\begin{gathered} v : \mathbb{R} \longrightarrow X, \quad v(t) = \alpha(t)\frac{\Phi(\theta, t)x}{\|\Phi(\theta, t)x\|}, \\ f : \mathbb{R} \longrightarrow X, \quad f(t) = \begin{cases} a\Phi(\theta, t)x, & t \geq 1, \\ \displaystyle\int_0^t \frac{\alpha(\tau)}{\|\Phi(\theta, \tau)x\|}d\tau\, \Phi(\theta, t)x, & t \in [0, 1], \\ 0, & t < 0, \end{cases} \end{gathered} \tag{4.8}$$

where

$$a := \int_0^1 \frac{\alpha(\tau)}{\|\Phi(\theta, \tau)x\|}d\tau. \tag{4.9}$$

We observe that f is continuous and

$$\|f(t)\| \leq a\|\lambda_{x,\theta}(t)\|, \quad \forall t \in \mathbb{R}. \tag{4.10}$$

Since $x \in \text{Range}\, P(\theta) = \mathcal{S}(\theta)$, we have that $\lambda_{x,\theta} \in O(\mathbb{R}, X)$. Then using Remark 2.4(i), we deduce that $f \in O(\mathbb{R}, X)$. In addition, we have that $v \in C_{0c}(\mathbb{R}, X)$ and an easy computation shows that the pair (f, v) satisfies (E_θ). Then, according to our hypothesis, it follows that

$$\|f\|_{O(\mathbb{R},X)} \leq L\|v\|_{I(\mathbb{R},X)}. \tag{4.11}$$

Because $\|v(t)\| = \alpha(t)$, for all $t \in \mathbb{R}$, the relation (4.11) becomes

$$\|f\|_{O(\mathbb{R},X)} \leq L|\alpha|_I. \tag{4.12}$$

Using relation (4.5), we deduce that

$$\|\Phi(\theta, r + 1)x\| \leq \delta\|\Phi(\theta, t)x\| = \frac{\delta}{a}\|f(t)\|, \quad \forall t \in [1, r + 1), \tag{4.13}$$

so

$$\|\Phi(\theta, r+1)x\| \chi_{[1,r+1)}(t) \leq \frac{\delta}{a}\|f(t)\|, \quad \forall t \in \mathbb{R}. \tag{4.14}$$

Using the invariance under translations of the space O from relation (4.14), we obtain that

$$\|\Phi(\theta, r+1)x\| F_O(r) \leq \frac{\delta}{a}\|f\|_{O(\mathbb{R},X)}. \tag{4.15}$$

Setting $h := r+1$ from relations (4.12) and (4.15), it follows that

$$\|\Phi(\theta, h)x\| F_O(r) \leq \frac{\delta L|\alpha|_I}{a}. \tag{4.16}$$

Moreover, from relation (4.5), we have that $\|\Phi(\theta, \tau)x\| \leq \delta\|x\|$, for all $\tau \in [0,1)$, so

$$a = \int_0^1 \frac{\alpha(\tau)}{\|\Phi(\theta, \tau)x\|} d\tau \geq \frac{1}{\delta\|x\|}. \tag{4.17}$$

From relations (4.7), (4.16), and (4.17), it follows that

$$\|\Phi(\theta, h)x\| \leq \frac{1}{e}\|x\|. \tag{4.18}$$

If $\Phi(\theta, 1)x = 0$, then $\Phi(\theta, h)x = 0$, so the above relation holds. Taking into account that h does not depend on θ or x, we obtain that in this case, there is $h > 0$ such that relation (4.6) holds.

Case 2. Suppose that $I \in \mathcal{L}(\mathbb{R})$. In this situation, from Remark 2.16, we have that there is a continuous function $\gamma : \mathbb{R} \to \mathbb{R}_+$ such that $\gamma \in I \setminus L^1(\mathbb{R}, \mathbb{R})$. Since the space I is invariant under translations, we may assume that there is $r > 1$ such that

$$\int_1^r \gamma(\tau) d\tau \geq \frac{eL\delta^2|\gamma|_I}{F_O(1)}. \tag{4.19}$$

Let $\beta : \mathbb{R} \to [0,1]$ be a continuous function with $\operatorname{supp}\beta \subset (0, r+1)$ and $\beta(t) = 1$, for all $t \in [1, r]$.

Let $\theta \in \Theta$ and let $x \in \operatorname{Range} P(\theta)$. We consider the functions

$$\begin{aligned}
&v : \mathbb{R} \longrightarrow X, \quad v(t) = \beta(t)\gamma(t)\Phi(\theta, t)x, \\
&f : \mathbb{R} \longrightarrow X, \quad f(t) = \begin{cases} q\Phi(\theta, t)x, & t \geq r+1, \\ \displaystyle\int_0^t \beta(\tau)\gamma(\tau) d\tau\, \Phi(\theta, t)x, & t \in [0, r+1), \\ 0, & t < 0, \end{cases}
\end{aligned} \tag{4.20}$$

where

$$q = \int_0^{r+1} \beta(\tau)\gamma(\tau)d\tau. \tag{4.21}$$

We have that $v \in C_{0c}(\mathbb{R}, X)$, f is continuous, and $\|f(t)\| \leq q\|\lambda_{x,\theta}(t)\|$, for all $t \in \mathbb{R}$. Using similar arguments with those used in relation (4.10), we deduce that $f \in O(\mathbb{R}, X)$. An easy computation shows that the pair (f, v) satisfies (E_θ). Then, we have that

$$\|f\|_{O(\mathbb{R},X)} \leq L\|v\|_{I(\mathbb{R},X)}. \tag{4.22}$$

Using relation (4.5), we obtain that

$$\|v(t)\| \leq \delta\gamma(t)\|x\|, \quad \forall t \in \mathbb{R}, \tag{4.23}$$

which implies that

$$\|v\|_{I(\mathbb{R},X)} \leq \delta|\gamma|_I\|x\|. \tag{4.24}$$

In addition, from $\|\Phi(\theta, r+2)x\| \leq \delta\|\Phi(\theta, t)x\|$, for all $t \in [r+1, r+2)$, we deduce that

$$\|\Phi(\theta, r+2)x\|\chi_{[r+1,r+2)}(t) \leq \frac{\delta}{q}\|f(t)\|, \quad \forall t \in \mathbb{R}. \tag{4.25}$$

Using the invariance under translations of the space O from relations (4.25), (4.22), and (4.24) we have that

$$q\|\Phi(\theta, r+2)x\|F_O(1) \leq \delta\|f\|_{O(\mathbb{R},X)} \leq L\delta^2|\gamma|_I\|x\|. \tag{4.26}$$

Since $q \geq \int_1^r \gamma(\tau)d\tau$, from relations (4.19), (4.21), and (4.26), it follows that

$$\|\Phi(\theta, r+2)x\| \leq \frac{1}{e}\|x\|. \tag{4.27}$$

Setting $h = r+2$ and taking into account that h does not depend on θ or x, we obtain that relation (4.6) holds.

In conclusion, in both situations, there is $h > 0$ such that

$$\|\Phi(\theta, h)x\| \leq \frac{1}{e}\|x\|, \quad \forall x \in \operatorname{Range} P(\theta),\ \forall \theta \in \Theta. \tag{4.28}$$

Let $\nu := 1/h$ and let $K = \delta e$. Let $\theta \in \Theta$ and let $x \in \operatorname{Range} P(\theta)$. Let $t > 0$. Then, there are $k \in \mathbb{N}$ and $\tau \in [0, h)$ such that $t = kh + \tau$. Using relations (4.5) and (4.6), we successively deduce that

$$\|\Phi(\theta, t)x\| \leq \delta \|\Phi(\theta, kh)x\| \leq \delta e^{-k}\|x\| \leq K e^{-\nu t}\|x\|. \tag{4.29}$$

□

Theorem 4.6 (The behavior on the unstable subspace). *Let O, I be two Banach function spaces such that either $O \in \mathcal{Q}(\mathbb{R})$ or $I \in \mathcal{L}(\mathbb{R})$. If the pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ is uniformly admissible for the system (E_π), then, there are $K, \nu > 0$ such that*

$$\|\Phi(\theta, t)y\| \geq \frac{1}{K} e^{\nu t}\|y\|, \quad \forall t \geq 0, \ \forall y \in \operatorname{Ker} P(\theta), \ \forall \theta \in \Theta. \tag{4.30}$$

Proof. Let $\delta > 0$ be such that

$$\|\Phi(\theta, t)y\| \geq \frac{1}{\delta}\|y\|, \quad \forall t \geq 0, \ \forall y \in \operatorname{Ker} P(\theta), \ \forall \theta \in \Theta. \tag{4.31}$$

Let $L > 0$ be given by Definition 3.6 and let $M, \omega > 0$ be given by Definition 3.2. We prove that there is $h > 0$ such that

$$\|\Phi(\theta, h)y\| \geq e\|y\|, \quad \forall y \in \operatorname{Ker} P(\theta), \ \forall \theta \in \Theta. \tag{4.32}$$

Case 1. Suppose that $O \in \mathcal{Q}(\mathbb{R})$. Let $\alpha : \mathbb{R} \longrightarrow [0, 2]$ be a continuous function with $\operatorname{supp} \alpha \subset (0, 1)$ and $\int_0^1 \alpha(\tau)d\tau = 1$. In this case, there is $r > 0$ such that

$$F_O(r) \geq e\delta^2 L|\alpha|_I. \tag{4.33}$$

Let $\theta \in \Theta$ and let $y \in \operatorname{Ker} P(\theta) \setminus \{0\}$. Then, $\Phi(\theta, t)y \neq 0$, for all $t \geq 0$. Since $y \in \operatorname{Ker} P(\theta) = \mathcal{U}(\theta)$, there is $\varphi \in \mathcal{F}(\theta) \cap O(\mathbb{R}, X)$ with $\varphi(0) = y$. We consider the functions

$$\begin{gathered} v : \mathbb{R} \longrightarrow X, \quad v(t) = -\alpha(t - r)\frac{\Phi(\theta, t)y}{\|\Phi(\theta, t)y\|} \\ f : \mathbb{R} \longrightarrow X, \quad f(t) = \begin{cases} \displaystyle\int_t^\infty \frac{\alpha(\tau - r)}{\|\Phi(\theta, \tau)y\|} d\tau \, \Phi(\theta, t)y, & t \geq r, \\ a\Phi(\theta, t)y, & t \in [0, r), \\ a\varphi(t), & t < 0, \end{cases} \end{gathered} \tag{4.34}$$

where

$$a := \int_r^{r+1} \frac{\alpha(\tau - r)}{\|\Phi(\theta, \tau)y\|} d\tau. \tag{4.35}$$

We have that $v \in \mathcal{C}_{0c}(\mathbb{R}, X)$ and f is continuous. Moreover, from

$$\|f(t)\| \leq a\|\varphi(t)\| + aMe^{\omega(r+1)}\|y\|\chi_{[0,r+1)}(t), \quad \forall t \in \mathbb{R}, \tag{4.36}$$

we obtain that $f \in O(\mathbb{R}, X)$. An easy computation shows that the pair (f, v) satisfies (E_θ), so

$$\|f\|_{O(\mathbb{R},X)} \leq L\|v\|_{I(\mathbb{R},X)}. \tag{4.37}$$

Observing that $\|v(t)\| = \alpha(t-r)$, for all $t \in \mathbb{R}$, the relation (4.37) becomes

$$\|f\|_{O(\mathbb{R},X)} \leq L|\alpha|_I. \tag{4.38}$$

From relation (4.31), we have that

$$\|\Phi(\theta, r+1)y\| \geq \frac{1}{\delta}\|\Phi(\theta,\tau)y\|, \quad \forall \tau \in [r, r+1]. \tag{4.39}$$

This implies that

$$a \geq \frac{1}{\delta\|\Phi(\theta, r+1)y\|}. \tag{4.40}$$

In addition, from relation (4.31), we have that

$$\|\Phi(\theta,t)y\| \geq \frac{1}{\delta}\|y\|, \quad \forall t \in [0,r) \tag{4.41}$$

which implies that

$$\|y\|\chi_{[0,r)}(t) \leq \delta\|\Phi(\theta,t)y\|\chi_{[0,r)}(t) \leq \frac{\delta}{a}\|f(t)\|, \quad \forall t \in \mathbb{R}. \tag{4.42}$$

From relation (4.42), it follows that

$$\|y\|F_O(r) \leq \frac{\delta}{a}\|f\|_{O(\mathbb{R},X)}. \tag{4.43}$$

From relations (4.38), (4.40), and (4.43), we deduce that

$$\|y\|F_O(r) \leq \frac{\delta L|\alpha|_I}{a} \leq \delta^2 L|\alpha|_I\|\Phi(\theta, r+1)y\|. \tag{4.44}$$

From relations (4.44) and (4.33), we have that

$$\|\Phi(\theta, r+1)y\| \geq e\|y\|. \tag{4.45}$$

Setting $h := r + 1$ and taking into account that h does not depend on y or θ we obtain that relation (4.32) holds.

Case 2. Suppose that $I \in \mathcal{L}(\mathbb{R})$. In this situation, using Remark 2.16 and the translation invariance of the space I, we have that there is a continuous function $\gamma : \mathbb{R} \to \mathbb{R}_+$ with $\gamma \in I \setminus L^1(\mathbb{R}, \mathbb{R})$ and $r > 1$ such that

$$\int_1^r \gamma(\tau)d\tau \geq e^{\omega+1}\frac{LM\delta|\gamma|_I}{F_O(1)}. \tag{4.46}$$

Let $\beta : \mathbb{R} \to [0,1]$ be a continuous function with $\operatorname{supp}\beta \subset (0, r+1)$ and $\beta(t) = 1$, for all $t \in [1, r]$.

Let $\theta \in \Theta$ and let $y \in \operatorname{Ker} P(\theta)$. Since $\operatorname{Ker} P(\theta) = \mathcal{U}(\theta)$ there is $\varphi \in \mathcal{F}(\theta) \cap O(\mathbb{R}, X)$ with $\varphi(0) = y$. We consider the functions

$$\begin{gathered} v : \mathbb{R} \longrightarrow X, \quad v(t) = -\beta(t)\gamma(t)\Phi(\theta, t)y, \\ f : \mathbb{R} \longrightarrow X, \quad f(t) = \begin{cases} \displaystyle\int_t^\infty \beta(\tau)\gamma(\tau)d\tau\Phi(\theta, t)y, & t \geq 0, \\ q\varphi(t), & t < 0, \end{cases} \end{gathered} \tag{4.47}$$

where $q := \int_0^{r+1} \beta(\tau)\gamma(\tau)d\tau$. We have that $v \in C_{0c}(\mathbb{R}, X)$, and, using similar arguments with those from Case 1, we obtain that $f \in O(\mathbb{R}, X)$. An easy computation shows that the pair (f, v) satisfies (E_θ), so

$$\|f\|_{O(\mathbb{R},X)} \leq L\|v\|_{I(\mathbb{R},X)}. \tag{4.48}$$

From (4.31), we have that $\|\Phi(\theta, r+1)y\| \geq (1/\delta)\|\Phi(\theta, t)y\|$, for all $t \in [0, r+1]$. This implies that

$$\|v(t)\| \leq \gamma(t)\delta\|\Phi(\theta, r+1)y\|, \quad \forall t \in \mathbb{R}, \tag{4.49}$$

so

$$\|v\|_{I(\mathbb{R},X)} \leq |\gamma|_I\delta\|\Phi(\theta, r+1)y\|. \tag{4.50}$$

Since $\varphi \in \mathcal{F}(\theta)$, we have that

$$\|y\| = \|\varphi(0)\| = \|\Phi(\sigma(\theta, t), -t)\varphi(t)\| \leq Me^\omega\|\varphi(t)\|, \quad \forall t \in [-1, 0). \tag{4.51}$$

From relation (4.51), it follows that

$$\|y\|\chi_{[-1,0)}(t) \leq Me^\omega\|\varphi(t)\|\chi_{[-1,0)}(t) \leq \frac{Me^\omega}{q}\|f(t)\|, \quad \forall t \in \mathbb{R}. \tag{4.52}$$

Using the translation invariance of the space O from (4.52), we obtain that

$$q\|y\|F_O(1) \le Me^{\omega}\|f\|_{O(\mathbb{R},X)}. \tag{4.53}$$

Since $q \ge \int_1^r \gamma(\tau)d\tau$, from relations (4.46), (4.48), (4.50) we deduce that

$$\|\Phi(\theta, r+1)y\| \ge e\|y\|. \tag{4.54}$$

Setting $h := r+1$ and since h does not depend on y or θ, we have that the relation (4.32) holds.

In conclusion, in both situations there is $h > 0$ such that

$$\|\Phi(\theta, h)y\| \ge e\|y\|, \quad \forall y \in \operatorname{Ker} P(\theta),\ \forall \theta \in \Theta. \tag{4.55}$$

Let $\nu = 1/h$ and let $K = \delta e$. Let $\theta \in \Theta$ and let $y \in \operatorname{Ker} P(\theta)$. Let $t > 0$. Then, there are $j \in \mathbb{N}$ and $s \in [0, h)$ such that $t = jh + s$. Using relations (4.31) and (4.32), we obtain that

$$\|\Phi(\theta, t)y\| \ge \frac{1}{\delta}\|\Phi(\theta, jh)y\| \ge \frac{1}{\delta}e^{j}\|y\| \ge \frac{1}{K}e^{\nu t}\|y\|. \tag{4.56}$$

□

According to the previous results we may formulate now a sufficient condition for the existence of the exponential dichotomy. Moreover, for the converse implication we will show that it sufficient to chose one of the spaces in the admissible pair from the class $\mathcal{R}(\mathbb{R})$. Thus, the main result of this section is as follows.

Theorem 4.7 (Necessary and sufficient condition for exponential dichotomy). *Let $\pi = (\Phi, \sigma)$ be a skew-product flow on $\mathcal{E} = X \times \Theta$ and let O, I be two Banach function spaces with $O, I \in \mathcal{T}(\mathbb{R})$ such that either $O \in \mathcal{Q}(\mathbb{R})$ or $I \in \mathcal{L}(\mathbb{R})$. The following assertions hold:*

(i) if the pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ is uniformly admissible for the system (E_π), then π is exponentially dichotomic.

(ii) if $I \subset O$ and one of the spaces I or O belongs to the class $\mathcal{R}(\mathbb{R})$, then π is exponentially dichotomic if and only if the pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ is uniformly admissible for the system (E_π).

Proof. (i) This follows from Theorem 3.13, Theorem 4.5, and Theorem 4.6.

(ii) Since $I \subset O$, it follows that there is $\alpha > 0$ such that

$$|u|_O \le \alpha |u|_I, \quad \forall u \in I. \tag{4.57}$$

Necessity. Suppose that π is exponentially dichotomic with respect to the family of projections $\{P(\theta)\}_{\theta\in\Theta}$ and let $K, \nu > 0$ be two constants given by Definition 4.1. According to Lemma 4.2, we have that $q := \sup_{\theta\in\Theta}\|P(\theta)\| < \infty$. For every $(\theta, t) \in \Theta \times \mathbb{R}_+$ we denote by $\Phi(\theta, t)_{|}^{-1}$ the inverse of the operator $\Phi(\theta, t)_{|} : \operatorname{Ker} P(\theta) \to \operatorname{Ker} P(\sigma(\theta, t))$.

Let $\theta \in \Theta$ and let $v \in C_{0c}(\mathbb{R}, X)$. We consider the function $f_v : \mathbb{R} \to X$ given by

$$f_v(t) = \int_{-\infty}^{t} \Phi(\sigma(\theta,\tau), t-\tau) P(\sigma(\theta,\tau)) v(\tau) d\tau \\ - \int_{t}^{\infty} \Phi(\sigma(\theta,t), \tau - t)_{|}^{-1} (I - P(\sigma(\theta,\tau)))\ v(\tau) d\tau. \tag{4.58}$$

We have that f_v is continuous, and a direct computation shows that the pair (f_v, v) satisfies (E_θ). In addition, we have that

$$\|f_v(t)\| \le qK \int_{-\infty}^{t} e^{-\nu(t-\tau)} \|v(\tau)\| d\tau \\ + (1+q)K \int_{t}^{\infty} e^{-\nu(\tau-t)} \|v(\tau)\| d\tau, \quad \forall t \in \mathbb{R}. \tag{4.59}$$

If $I \in \mathcal{R}(\mathbb{R})$, let $\gamma_{I,\nu} > 0$ be the constant given by Lemma 2.21. Then, from (4.59) and Lemma 2.21, it follows that $f_v \in I(\mathbb{R}, X)$ and

$$\|f_v\|_{I(\mathbb{R},X)} \le (1+2q) K \gamma_{I,\nu} \|v\|_{I(\mathbb{R},X)}. \tag{4.60}$$

Then, from (4.57) and (4.60), we deduce that $f_v \in O(\mathbb{R}, X)$ and

$$\|f_v\|_{O(\mathbb{R},X)} \le \alpha(1+2q) K \gamma_{I,\nu} \|v\|_{I(\mathbb{R},X)}. \tag{4.61}$$

If $O \in \mathcal{R}(\mathbb{R})$, let $\gamma_{O,\nu} > 0$ be the constant given by Lemma 2.21. Then, from (4.59), (4.57) and using Lemma 2.21, we successively obtain that $f_v \in O(\mathbb{R}, X)$ and

$$\|f_v\|_{O(\mathbb{R},X)} \le (1+2q) K \gamma_{O,\nu} \|v\|_{O(\mathbb{R},X)} \le \alpha(1+2q) K \gamma_{O,\nu} \|v\|_{I(\mathbb{R},X)}. \tag{4.62}$$

Let

$$\gamma := \begin{cases} \gamma_{I,\nu}, & \text{if } I \in \mathcal{R}(\mathbb{R}), \\ \gamma_{O,\nu}, & \text{if } I \notin \mathcal{R}(\mathbb{R}), \quad O \in \mathcal{R}(\mathbb{R}). \end{cases} \tag{4.63}$$

Then setting $L := \alpha(1+2q)K\gamma$ from relations (4.61) and (4.62), we have that

$$\|f_v\|_{O(\mathbb{R},X)} \le L \|v\|_{I(\mathbb{R},X)}. \tag{4.64}$$

Now let $v \in C_{0c}(\mathbb{R}, X)$ and $f \in O(\mathbb{R}, X)$ be such that the pair (f, v) satisfies (E_θ). We set $\varphi := f - f_v$, and we have that $\varphi \in O(\mathbb{R}, X)$ and

$$\varphi(t) = \Phi(\sigma(\theta,s), t-s)\varphi(s), \quad \forall t \ge s. \tag{4.65}$$

Let $\varphi_1(t) = P(\sigma(\theta,t))\varphi(t)$, for all $t \in \mathbb{R}$ and let $\varphi_2(t) = (I - P(\sigma(\theta,t)))\varphi(t)$, for all $t \in \mathbb{R}$. Then from (4.65), we obtain that

$$\varphi_k(t) = \Phi(\sigma(\theta,s),t-s)\varphi_k(s), \quad \forall t \geq s, \forall k \in \{1,2\}. \tag{4.66}$$

Let $t_0 \in \mathbb{R}$. From (4.66), it follows that

$$\|\varphi_1(t_0)\| \leq Ke^{-\nu(t_0-s)}\|\varphi_1(s)\| \leq qKe^{-\nu(t_0-s)}\|\varphi(s)\|, \ \forall s \leq t_0. \tag{4.67}$$

Since $\varphi \in O(\mathbb{R},X)$, from Remark 2.12 it follows that $\varphi \in M^1(\mathbb{R},X)$. Then, from (4.67), we have that

$$\begin{aligned}\|\varphi_1(t_0)\| &\leq qK\int_{s-1}^{s} e^{-\nu(t_0-\tau)}\|\varphi(\tau)\|d\tau \leq qKe^{-\nu(t_0-s)}\int_{s-1}^{s}\|\varphi(\tau)\|d\tau \\ &\leq qKe^{-\nu(t_0-s)}\|\varphi\|_{M^1(\mathbb{R},X)}, \quad \forall s \leq t_0.\end{aligned} \tag{4.68}$$

For $s \to -\infty$ in (4.68), it follows that $\varphi_1(t_0) = 0$. In addition, from (4.66) we have that

$$\frac{1}{K}e^{\nu(t-t_0)}\|\varphi_2(t_0)\| \leq \|\varphi_2(t)\| \leq (1+q)\|\varphi(t)\|, \quad \forall t \geq t_0. \tag{4.69}$$

This implies that

$$\frac{1}{K}e^{\nu(t-t_0)}\|\varphi_2(t_0)\| \leq (1+q)\int_t^{t+1}\|\varphi(\tau)\|d\tau \leq (1+q)\|\varphi\|_{M^1(\mathbb{R},X)}, \quad \forall t \geq t_0. \tag{4.70}$$

The relation (4.70) shows that

$$\|\varphi_2(t_0)\| \leq K(1+q)e^{-\nu(t-t_0)}\|\varphi\|_{M^1(\mathbb{R},X)}, \quad \forall t \geq t_0. \tag{4.71}$$

For $t \to \infty$ in (4.71), it follows that $\varphi_2(t_0) = 0$. This shows that $\varphi(t_0) = \varphi_1(t_0) + \varphi_2(t_0) = 0$. Since $t_0 \in \mathbb{R}$ was arbitrary, we deduce that $\varphi = 0$, so $f = f_v$. Then, from (4.64), we have that

$$\|f\|_{O(\mathbb{R},X)} \leq L\|v\|_{I(\mathbb{R},X)}. \tag{4.72}$$

Taking into account that L does not depend on $\theta \in \Theta$ or on $v \in C_{0c}(\mathbb{R},X)$, we finally conclude that the pair $(O(\mathbb{R},X), I(\mathbb{R},X))$ is uniformly admissible for the system (E_π).

Sufficiency follows from (i). □

Corollary 4.8. *Let $\pi = (\Phi,\sigma)$ be a skew-product flow on $\mathcal{E} = X \times \Theta$ and let V be a Banach function space with $V \in \mathcal{T}(\mathbb{R})$. Then, the following assertions hold:*

(i) *if the pair $(V(\mathbb{R},X), V(\mathbb{R},X))$ is uniformly admissible for the system (E_π), then, π is exponentially dichotomic;*

(ii) *if* $V \in \mathcal{R}(\mathbb{R})$, *then,* π *is exponentially dichotomic if and only if the pair* $(V(\mathbb{R}, X), V(\mathbb{R}, X))$ *is uniformly admissible for the system* (E_π).

Proof. We prove that either $V \in \mathcal{Q}(\mathbb{R})$ or $V \in \mathcal{L}(\mathbb{R})$. Indeed, suppose by contrary that $V \notin \mathcal{Q}(\mathbb{R})$ and $V \notin \mathcal{L}(\mathbb{R})$. Then, $M := \sup_{t>0} F_V(t) < \infty$ and $V \subset L^1(\mathbb{R}, \mathbb{R})$. From $V \subset L^1(\mathbb{R}, \mathbb{R})$, it follows that there is $\gamma > 0$ such that

$$\|v\|_1 \leq \gamma |v|_V, \quad \forall v \in V. \tag{4.73}$$

In particular, from $v = \chi_{[0,t)}$ in relation (4.73), we deduce that

$$t \leq \gamma \left|\chi_{[0,t)}\right|_V = \gamma F_V(t) \leq \gamma M, \quad \forall t > 0, \tag{4.74}$$

which is absurd. This shows that the assumption is false, which shows that either $V \in \mathcal{Q}(\mathbb{R})$ or $V \in \mathcal{L}(\mathbb{R})$. By applying Theorem 4.7, we obtain the conclusion. □

5. Applications and Conclusions

We have seen in the previous section that in the study of the exponential dichotomy of variational equations the classes $\mathcal{Q}(\mathbb{R})$ and, respectively, $\mathcal{L}(\mathbb{R})$ have a crucial role in the identification of the appropriate function spaces in the admissible pair. Moreover, it was also important to point out that it is sufficient to impose conditions either on the input space or on the output space. In this context, the natural question arises if these conditions are indeed necessary and whether our hypotheses may be dropped. The aim of this section is to answer this question. With this purpose, we will present an illustrative example of uniform admissibility, and we will discuss the concrete implications concerning the existence of the exponential dichotomy.

Let X be a Banach space. We denote by $C_0(\mathbb{R}, X)$ the space of all continuous functions $u : \mathbb{R} \to X$ with $\lim_{t\to\infty} u(t) = \lim_{t\to-\infty} u(t) = 0$, which is a Banach space with respect to the norm

$$|||u||| := \sup_{t\in\mathbb{R}} \|u(t)\|. \tag{5.1}$$

We start with a technical lemma.

Lemma 5.1. *If* O *is a Banach function space with* $O \in \mathcal{T}(\mathbb{R}) \setminus \mathcal{Q}(\mathbb{R})$, *then,* $C_0(\mathbb{R}, \mathbb{R}) \subset O$.

Proof. Let $c := \sup_{t>0} F_O(t)$. Let $u \in C_0(\mathbb{R}, \mathbb{R})$. Then, there is an unbounded increasing sequence $(t_n) \subset (0, \infty)$ such that $|u(t)| \leq 1/(n+1)$, for all $|t| \geq t_n$ and all $n \in \mathbb{N}$. Setting $u_n = u\chi_{[-t_n, t_n)}$ we have that

$$\left|u_{n+p} - u_n\right|_O \leq \frac{\left|\chi_{[-t_{n+p}, -t_n)}\right|_O}{n+1} + \frac{\left|\chi_{[t_n, t_{n+p})}\right|_O}{n+1} \leq \frac{2c}{n+1}, \quad \forall n \in \mathbb{N}, \ \forall p \in \mathbb{N}^*. \tag{5.2}$$

From relation (5.2), it follows that the sequence (u_n) is fundamental in O, so this is convergent, that is, there exists $v \in O$ such that $u_n \to v$ in O. According to Remark 2.4(ii), there exists a subsequence (u_{k_n}) such that $u_{k_n}(t) \to v(t)$ for a.e. $t \in \mathbb{R}$. This implies that $v(t) = u(t)$ for a.e. $t \in \mathbb{R}$, so $v = u$ in O. In conclusion, $u \in O$, and the proof is complete. $\square$

In what follows, we present a concrete situation which illustrates the relevance of the hypotheses on the underlying function spaces considered in the admissible pair, for the study of the dichotomous behavior of skew-product flows.

Example 5.2. Let $X = \mathbb{R} \times \mathbb{R}$ which is a Banach space with respect to the norm $\|(x_1, x_2)\| = |x_1| + |x_2|$. Let $\Theta = \mathbb{R}$ and let $\sigma : \Theta \times \mathbb{R} \to \Theta$, $\sigma(\theta, t) = \theta + t$. We have that σ is a flow on Θ. Let

$$\varphi : \mathbb{R} \longrightarrow (0, \infty), \qquad \varphi(t) = \begin{cases} \dfrac{2}{t+1}, & t \geq 0, \\ 1 + e^{-t}, & t < 0. \end{cases} \tag{5.3}$$

For every $(\theta, t) \in \Theta \times \mathbb{R}_+$, we consider the operator

$$\Phi(\theta, t) : X \longrightarrow X, \quad \Phi(\theta, t)(x_1, x_2) = \left(\frac{\varphi(\theta + t)}{\varphi(\theta)} x_1, e^t x_2 \right). \tag{5.4}$$

It is easy to see that $\pi = (\Phi, \sigma)$ is a skew-product flow.

Now, let $O,\ I$ be two Banach function spaces with $O, I \in \mathcal{T}(\mathbb{R})$ such that $O \notin \mathcal{Q}(\mathbb{R})$ and $I \notin \mathcal{L}(\mathbb{R})$. It follows that $I \subset L^1(\mathbb{R}, \mathbb{R})$, and, using Lemma 5.1, we obtain that $C_0(\mathbb{R}, \mathbb{R}) \subset O$. Then, there are $\alpha, \beta > 0$ such that

$$\begin{gathered} \|u\|_1 \leq \alpha |u|_I, \quad \forall u \in I, \\ \|u\|_O \leq \beta |||u|||, \quad \forall u \in C_0(\mathbb{R}, \mathbb{R}). \end{gathered} \tag{5.5}$$

Step 1. We prove that the pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ is uniformly admissible for the system (E_π).

Let $\theta \in \Theta$ and let $v = (v_1, v_2) \in C_{0c}(\mathbb{R}, X)$ and let $h > 0$ be such that $\operatorname{supp} v \subset (0, h)$. We consider the function $f : \mathbb{R} \to X$ where $f = (f_1, f_2)$ and

$$f_1(t) = \int_{-\infty}^{t} \frac{\varphi(\theta + t)}{\varphi(\theta + \tau)} v_1(\tau) d\tau, \qquad f_2(t) = -\int_{t}^{\infty} e^{-(\tau - t)} v_2(\tau) d\tau, \quad \forall t \in \mathbb{R}. \tag{5.6}$$

We have that f is continuous and an easy computation shows that the pair (f, v) satisfies (E_θ). Since $\operatorname{supp} v \subset (0, h)$, we obtain that $f_1(t) = 0$, for all $t \leq 0$ and $f_2(t) = 0$, for all $t \geq h$. From

$$f_1(t) = \varphi(\theta + t) \int_0^h \frac{v_1(\tau)}{\varphi(\theta + \tau)} d\tau, \qquad \forall t \geq h, \tag{5.7}$$

we have that $\lim_{t \to \infty} f_1(t) = 0$. In addition, from

$$f_2(t) = -e^t \int_0^h e^{-\tau} v_2(\tau) d\tau, \quad \forall t \leq 0, \tag{5.8}$$

we deduce that $\lim_{t\to-\infty} f_2(t) = 0$. Thus, we obtain that $f \in \mathcal{C}_0(\mathbb{R}, X)$ so $f \in O(\mathbb{R}, X)$. Moreover, from

$$\begin{aligned} |f_1(t)| &\leq \int_{-\infty}^{t} |v_1(\tau)| d\tau \leq \|v_1\|_{L^1(\mathbb{R},\mathbb{R})}, \quad \forall t \in \mathbb{R}, \\ |f_2(t)| &\leq \int_{t}^{\infty} |v_2(\tau)| d\tau \leq \|v_2\|_{L^1(\mathbb{R},\mathbb{R})}, \quad \forall t \in \mathbb{R}, \end{aligned} \tag{5.9}$$

it follows that

$$|||f||| \leq \|v\|_{L^1(\mathbb{R},X)}. \tag{5.10}$$

From relations (5.5) and (5.10), we obtain that

$$\|f\|_{O(\mathbb{R},X)} \leq \alpha\beta \|v\|_{I(\mathbb{R},X)}. \tag{5.11}$$

Let $\tilde{f} \in O(\mathbb{R}, X)$ be such that the pair $(\tilde{f}, v)$ satisfies (E_θ) and let $g = \tilde{f} - f$. Then, $g \in O(\mathbb{R}, X)$ and $g(t) = \Phi(\sigma(\theta, s), t - s) g(s)$, for all $t \geq s$. More exactly, if $g = (g_1, g_2)$, then we have that

$$g_1(t) = \frac{\varphi(\theta + t)}{\varphi(\theta + s)} g_1(s), \quad \forall t \geq s, \tag{5.12}$$

$$g_2(t) = e^{t-s} g_2(s), \quad \forall t \geq s. \tag{5.13}$$

Since $g \in O(\mathbb{R}, X)$ from Remark 2.12, it follows that $g \in M^1(\mathbb{R}, X)$, so $g_1, g_2 \in M^1(\mathbb{R}, \mathbb{R})$.

Let $t_0 \in \mathbb{R}$. For every $s \leq t_0$ from relation (5.12), we have that

$$\frac{|g_1(t_0)|}{\varphi(\theta + t_0)} = \int_{s-1}^{s} \frac{|g_1(\tau)|}{\varphi(\theta + \tau)} d\tau \leq \frac{1}{\varphi(\theta + s)} \int_{s-1}^{s} |g_1(\tau)| d\tau \leq \frac{\|g_1\|_{M^1(\mathbb{R},\mathbb{R})}}{\varphi(\theta + s)}. \tag{5.14}$$

Since $\varphi(r) \to \infty$ as $r \to -\infty$, for $s \to -\infty$ in (5.14), we obtain that $g_1(t_0) = 0$. In addition, for every $t \geq t_0$ from relation (5.13) we have that

$$e^{-t_0} |g_2(t_0)| = \int_{t}^{t+1} e^{-\tau} |g_2(\tau)| d\tau \leq e^{-t} \int_{t}^{t+1} |g_2(\tau)| d\tau \leq e^{-t} \|g_2\|_{M^1(\mathbb{R},\mathbb{R})}. \tag{5.15}$$

For $t \to \infty$ in (5.15) we deduce that $g_2(t_0) = 0$. So, we obtain that $g(t_0) = 0$. Taking into account that $t_0 \in \mathbb{R}$ was arbitrary it follows that $g = 0$. This implies that $\tilde{f} = f$. Then, from relation (5.11) we have that

$$\|\tilde{f}\|_{O(\mathbb{R},X)} \leq \alpha\beta \|v\|_{I(\mathbb{R},X)}. \tag{5.16}$$

We set $L = \alpha\beta$, and, taking into account that L does not depend on θ or v, we conclude that the pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ is uniformly admissible for the system (E_π).

Step 2. We prove that π is not exponentially dichotomic. Suppose by contrary that π is exponentially dichotomic with respect to the family of projections $\{P(\theta)\}_{\theta\in\Theta}$ and let $K, \nu > 0$ be two constants given by Definition 4.1. In this case, according to Proposition 2.1 from [18] we have that

$$\operatorname{Im} P(\theta) = \{x \in X : \Phi(\theta, t)x \longrightarrow 0 \text{ as } t \longrightarrow \infty\}, \quad \forall\theta \in \Theta. \tag{5.17}$$

This characterization implies that $\operatorname{Im} P(\theta) = \mathbb{R} \times \{0\}$, for all $\theta \in \Theta$. Then, from

$$\|\Phi(\theta, t)x\| \le Ke^{-\nu t}\|x\|, \quad \forall t \ge 0,\ \forall x \in \operatorname{Im} P(\theta),\ \forall\theta \in \Theta, \tag{5.18}$$

we obtain that

$$\frac{\varphi(\theta + t)}{\varphi(\theta)}|x_1| \le Ke^{-\nu t}|x_1|, \quad \forall x_1 \in \mathbb{R},\ \forall t \ge 0,\ \forall\theta \in \Theta, \tag{5.19}$$

which shows that

$$\frac{\varphi(\theta + t)}{\varphi(\theta)} \le Ke^{-\nu t}, \quad \forall t \ge 0,\ \forall\theta \in \Theta. \tag{5.20}$$

In particular, for $\theta = 0$, from (5.20), we have that

$$\frac{1}{t + 1} \le Ke^{-\nu t}, \quad \forall t \ge 0, \tag{5.21}$$

which is absurd. This shows that the assumption is false, so π is not exponentially dichotomic.

Remark 5.3. The above example shows that if I, O are two Banach function spaces from the class $\mathcal{T}(\mathbb{R})$ such that $O \notin \mathcal{Q}(\mathbb{R})$ and $I \notin \mathcal{L}(\mathbb{R})$, then the uniform admissibility of the pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ for the system (E_π) *does not imply* the existence of the exponential dichotomy of π. This shows that the hypotheses of the main result from the previous section are indeed necessary and emphasizes the fact that in the study of the exponential dichotomy in terms of the uniform admissibility at least one of the output space or the input space should belong to, respectively, $\mathcal{Q}(\mathbb{R})$ or $\mathcal{L}(\mathbb{R})$.

Finally, we complete our study with several consequences of the main result, which will point out some interesting conclusions for some usual classes of spaces often used in control-type problems arising in qualitative theory of dynamical systems. We will also show that, in our approach, the input space can be successively minimized, and we will discuss several optimization directions concerning the admissibility-type techniques.

Remark 5.4. The input-output characterizations for the asymptotic properties of systems have a wider applicability area if the input space is as small as possible and the output space is very general. In our main result, given by Theorem 4.7, the input functions belong to the space $\mathcal{C}_{0c}(\mathbb{R}, X)$ while the output space is a general Banach function space. By analyzing condition (ii) from Definition 3.6, we observe that the input-output characterization given by Theorem 4.7 becomes more flexible and provides a more competitive applicability spectrum when the norm on the input space is larger.

Another interesting aspect that must be noted is that the class $\mathfrak{T}(\mathbb{R})$ is closed to finite intersections. Indeed, if $I_1, \dots, I_n \in \mathfrak{T}(\mathbb{R})$, then we may define $I := I_1 \cap I_2 \cap \cdots I_n$ with respect to the norm

$$|u|_I := \max\{|u|_{I_1}, |u|_{I_2}, \dots, |u|_{I_n}\}, \tag{5.22}$$

which is a Banach function space which belongs to $\mathfrak{T}(\mathbb{R})$. So, taking as input space a Banach function space which is obtained as an intersection of Banach function spaces from the class $\mathfrak{T}(\mathbb{R})$ we will have a "larger" norm in our admissibility condition, and, thus the estimation will be more permissive and more general.

As a consequence of the aspects presented in the above remark we deduce the following corollaries.

Corollary 5.5. *Let $\pi = (\Phi, \sigma)$ be a skew-product flow on $X \times \Theta$. Let O_φ be an Orlicz space with $0 < \varphi(t) < \infty$, for all $t > 0$. Let $n \in \mathbb{N}^*$, let $O_{\varphi_1}, \dots, O_{\varphi_n}$ be Orlicz spaces such that $\varphi_k(1) < \infty$, for all $k \in \{1, \dots, n\}$ and let $I := O_{\varphi_1}(\mathbb{R}, \mathbb{R}) \cap \cdots \cap O_{\varphi_n}(\mathbb{R}, \mathbb{R}) \cap O_\varphi(\mathbb{R}, \mathbb{R})$. Then, π is exponentially dichotomic if and only if the pair $(O_\varphi(\mathbb{R}, X), I(\mathbb{R}, X))$ is uniformly admissible for the system (E_π).*

Proof. From Lemma 2.15 and Remark 2.20, it follows that $O_\varphi \in \mathcal{Q}(\mathbb{R}) \cap \mathcal{R}(\mathbb{R})$. By applying Theorem 4.7, the proof is complete. □

Corollary 5.6. *Let $\pi = (\Phi, \sigma)$ be a skew-product flow on $X \times \Theta$ and let $p \in [1, \infty)$. Let $n \in \mathbb{N}^*, q_1, \dots, q_n \in [1, \infty]$ and $I = L^{q_1}(\mathbb{R}, \mathbb{R}) \cap \cdots \cap L^{q_n}(\mathbb{R}, \mathbb{R}) \cap L^p(\mathbb{R}, \mathbb{R})$. Then, π is exponentially dichotomic if and only if the pair $(L^p(\mathbb{R}, X), I(\mathbb{R}, X))$ is admissible for the system (E_π).*

Proof. This follows from Corollary 5.5. □

Corollary 5.7. *Let $\pi = (\Phi, \sigma)$ be a skew-product flow on $X \times \Theta$ and let $p \in (1, \infty]$. Let $n \in \mathbb{N}^*, q_1, \dots, q_n \in (1, \infty]$ and $I = L^{q_1}(\mathbb{R}, \mathbb{R}) \cap \cdots \cap L^{q_n}(\mathbb{R}, \mathbb{R}) \cap L^p(\mathbb{R}, \mathbb{R})$. Then π is exponentially dichotomic if and only if the pair $(L^p(\mathbb{R}, X), I(\mathbb{R}, X))$ is uniformly admissible for the system (E_π).*

Proof. This follows from Theorem 4.7 by observing that $I \in \mathcal{L}(\mathbb{R})$. □

Remark 5.8. According to Remark 2.12, the largest space from the class $\mathfrak{T}(\mathbb{R})$ is $M^1(\mathbb{R}, \mathbb{R})$. Thus, considering the output space $M^1(\mathbb{R}, \mathbb{R})$, in order to obtain optimal input-output characterizations for exponential dichotomy in terms of admissibility, it is sufficient to work with smaller and smaller input spaces.

Corollary 5.9. *Let $\pi = (\Phi, \sigma)$ be a skew-product flow on $X \times \Theta$. Let $n \in \mathbb{N}^*$, $q_1, \dots, q_n \in (1, \infty]$ and $I = L^{q_1}(\mathbb{R}, \mathbb{R}) \cap \cdots \cap L^{q_n}(\mathbb{R}, \mathbb{R})$. Then, π is exponentially dichotomic if and only if the pair $(M^1(\mathbb{R}, X), I(\mathbb{R}, X))$ is uniformly admissible for the system (E_π).*

Proof. We observe that $I \in \mathcal{L}(\mathbb{R})$, and, from Remark 2.12, we have that $I \subset M^1(\mathbb{R}, \mathbb{R})$. By applying Theorem 4.7, we obtain the conclusion. □

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References

[1] E. Braverman and S. Zhukovskiy, "The problem of a lazy tester, or exponential dichotomy for impulsive differential equations revisited," *Nonlinear Analysis: Hybrid Systems*, vol. 2, no. 3, pp. 971–979, 2008.

[2] C. Cuevas and C. Vidal, "Discrete dichotomies and asymptotic behavior for abstract retarded functional difference equations in phase space," *Journal of Difference Equations and Applications*, vol. 8, no. 7, pp. 603–640, 2002.

[3] S.-N. Chow and H. Leiva, "Existence and roughness of the exponential dichotomy for skew-product semiflow in Banach spaces," *Journal of Differential Equations*, vol. 120, no. 2, pp. 429–477, 1995.

[4] S.-N. Chow and H. Leiva, "Two definitions of exponential dichotomy for skew-product semiflow in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 124, no. 4, pp. 1071–1081, 1996.

[5] S.-N. Chow and H. Leiva, "Unbounded perturbation of the exponential dichotomy for evolution equations," *Journal of Differential Equations*, vol. 129, no. 2, pp. 509–531, 1996.

[6] J. H. Liu, G. M. N'Guérékata, and N. Van Minh, *Topics on Stability and Periodicity in Abstract Differential Equations*, vol. 6 of *Series on Concrete and Applicable Mathematics*, World Scientific, Hackensack, NJ, USA, 2008.

[7] N. Van Minh, F. Räbiger, and R. Schnaubelt, "Exponential stability, exponential expansiveness, and exponential dichotomy of evolution equations on the half-line," *Integral Equations and Operator Theory*, vol. 32, no. 3, pp. 332–353, 1998.

[8] N. Van Minh, G. M. N'Guérékata, and R. Yuan, *Lectures on the Asymptotic Behavior of Solutions of Differential Equations*, Nova Science, New York, NY, USA, 2008.

[9] K. J. Palmer, "Exponential dichotomies and Fredholm operators," *Proceedings of the American Mathematical Society*, vol. 104, no. 1, pp. 149–156, 1988.

[10] K. J. Palmer, *Shadowing in Dynamical Systems*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.

[11] O. Perron, "Die stabilitätsfrage bei differentialgleichungen," *Mathematische Zeitschrift*, vol. 32, no. 1, pp. 703–728, 1930.

[12] M. Pituk, "A criterion for the exponential stability of linear difference equations," *Applied Mathematics Letters*, vol. 17, no. 7, pp. 779–783, 2004.

[13] H. M. Rodrigues and J. G. Ruas-Filho, "Evolution equations: dichotomies and the Fredholm alternative for bounded solutions," *Journal of Differential Equations*, vol. 119, no. 2, pp. 263–283, 1995.

[14] B. Sasu and A. L. Sasu, "Input-output conditions for the asymptotic behavior of linear skew-product flows and applications," *Communications on Pure and Applied Analysis*, vol. 5, no. 3, pp. 551–569, 2006.

[15] B. Sasu, "New criteria for exponential expansiveness of variational difference equations," *Journal of Mathematical Analysis and Applications*, vol. 327, no. 1, pp. 287–297, 2007.

[16] B. Sasu, "Robust stability and stability radius for variational control systems," *Abstract and Applied Analysis*, vol. 2008, Article ID 381791, 29 pages, 2008.

[17] B. Sasu, "On dichotomous behavior of variational difference equations and applications," *Discrete Dynamics in Nature and Society*, vol. 2009, Article ID 140369, 16 pages, 2009.

[18] B. Sasu, "Integral conditions for exponential dichotomy: a nonlinear approach," *Bulletin des Sciences Mathématiques*, vol. 134, no. 3, pp. 235–246, 2010.

[19] C. Foias, G. R. Sell, and R. Temam, "Inertial manifolds for nonlinear evolutionary equations," *Journal of Differential Equations*, vol. 73, no. 2, pp. 309–353, 1988.

[20] W. A. Coppel, *Dichotomies in Stability Theory*, vol. 629 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1978.

[21] J. L. Daleckii and M. G. Krein, *Stability of Solutions of Differential Equations in Banach Space*, American Mathematical Society, Providence, RI, USA, 1974.

[22] J. L. Massera and J. J. Schäffer, *Linear Differential Equations and Function Spaces*, vol. 21 of *Pure and Applied Mathematics*, Academic Press, New York, NY, USA, 1966.

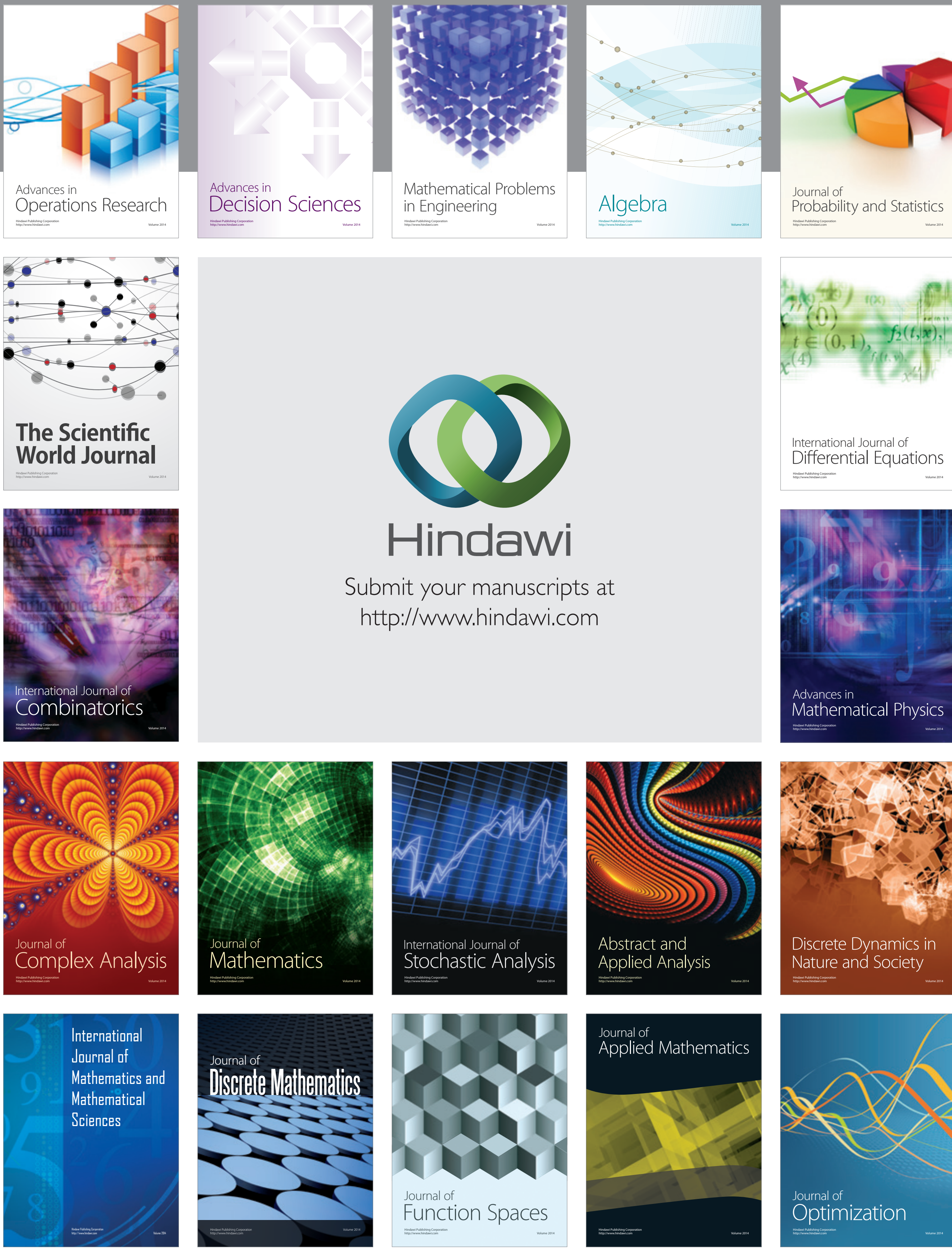
[23] L. Berezansky and E. Braverman, "On exponential dichotomy, Bohl-Perron type theorems and stability of difference equations," *Journal of Mathematical Analysis and Applications*, vol. 304, no. 2, pp. 511–530, 2005.

[24] A. D. Maĭzel', "On stability of solutions of systems of differential equations," *Trudy Ural'skogo Politekhnicheskogo Instituta*, vol. 51, pp. 20–50, 1954 (Russian).

[25] P. Meyer-Nieberg, *Banach Lattices*, Universitext, Springer, Berlin, Germany, 1991.

[26] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, NY, USA, 3rd edition, 1987.

[27] C. Bennett and R. Sharpley, *Interpolation of Operators*, vol. 129 of *Pure and Applied Mathematics*, Academic Press, Boston, Mass, USA, 1988.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