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Research Article

Principal Functions of Non-Selfadjoint Difference Operator with Spectral Parameter in Boundary Conditions

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We investigate the principal functions corresponding to the eigenvalues and the spectral singularities of the boundary value problem (BVP) $a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n$, $n \in \mathbb{N}$ and $(\gamma_0 + \gamma_1\lambda)y_1 + (\beta_0 + \beta_1\lambda)y_0 = 0$, where (a_n) and (b_n) are complex sequences, λ is an eigenparameter, and $\gamma_i, \beta_i \in \mathbb{C}$ for $i = 0, 1$.

1. Introduction

Let us consider the (BVP)

$$\begin{aligned} -y'' + q(x)y &= \lambda^2 y, \quad 0 \le x < \infty, \\ y'(0) - hy(0) &= 0 \end{aligned} \tag{1.1}$$

in $L^2(\mathbb{R}_+)$, where q is a complex-valued function and $\lambda \in \mathbb{C}$ is a spectral parameter and $h \in \mathbb{C}$. The spectral theory of the above BVP with continuous and point spectrum was investigated by Naĭmark [1]. He showed that the existence of the spectral singularities in the continuous spectrum of the BVP. He noted that the spectral singularities that belong to the continuous spectrum are the poles of the resolvents kernel but they are not the eigenvalues of the BVP. Also he showed that eigenfunctions and the associated functions (principal functions) corresponding to the spectral singularities are not the element of $L^2(\mathbb{R}_+)$. The spectral singularities in the spectral expansion of the BVP in terms of principal functions have been investigated in [2]. The spectral analysis of the quadratic pencil of Schrödinger, Dirac,

and Klein-Gordon operators with spectral singularities was studied in [3–8]. The spectral analysis of a non-selfadjoint difference equation with spectral parameter has been studied in [9]. In this paper, it is proved that the BVP

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n \in \mathbb{N}, \tag{1.2}$$

$$(\gamma_0 + \gamma_1\lambda)y_1 + (\beta_0 + \beta_1\lambda)y_0 = 0 \tag{1.3}$$

has a finite number of eigenvalues and spectral singularities with a finite multiplicities if

$$\sup_{n\in\mathbb{N}}\left[\exp\left(\varepsilon n^{\delta}\right)(|1 - a_n| + |b_n|)\right] < \infty \tag{1.4}$$

for some $\varepsilon > 0$ and $1/2 \le \delta \le 1$.

Let L denote difference operator of second order generated in $\ell_2(\mathbb{N})$ by

$$(\ell y)_n = a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1}, \quad n \in \mathbb{N} \tag{1.5}$$

and with boundary condition

$$(\gamma_0 + \gamma_1\lambda)y_1 + (\beta_0 + \beta_1\lambda)y_0 = 0, \quad \gamma_0\beta_1 - \gamma_1\beta_0 \neq 0, \quad \gamma_1 \neq a_0^{-1}\beta_0, \tag{1.6}$$

where $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ are complex sequences and $a_n \neq 0$ for all $n \in \mathbb{N} \cup \{0\}$ and $\gamma_i, \beta_i \in \mathbb{C}$ for $i = 0, 1$.

In this paper, which is extension of [9], we aim to investigate the properties of the principal functions corresponding to the eigenvalues and spectral singularities of the BVP (1.2)-(1.3).

2. Discrete Spectrum of (1.2)-(1.3)

Let

$$\sup_{n\in\mathbb{N}}\left[\exp\left(\varepsilon n^{\delta}\right)(|1 - a_n| + |b_n|)\right] < \infty \tag{2.1}$$

for some $\varepsilon > 0$ and $1/2 \le \delta \le 1$. The following result is obtained in [10, 11]: under the condition (2.1), equation (1.2) has the solution

$$e_n(z) = \alpha_n e^{inz}\left(1 + \sum_{m=1}^{\infty} A_{nm}e^{imz}\right), \quad n \in \mathbb{N} \cup \{0\} \tag{2.2}$$

for $\lambda = 2\cos z$, where $z \in \overline{\mathbb{C}}_+ := \{z : z \in \mathbb{C},\ \operatorname{Im} z \geq 0\}$ and α_n, A_{nm} are expressed in terms of (a_n) and (b_n) as

$$\begin{aligned}
\alpha_n &= \left(\prod_{k=n}^{\infty} a_k\right)^{-1}, \\
A_{n,1} &= -\sum_{k=n+1}^{\infty} b_k, \\
A_{n,2} &= -\sum_{k=n+1}^{\infty}\left(1-a_k^2\right) + \sum_{k=n+1}^{\infty} b_k \sum_{p=k+1}^{\infty} b_p, \\
A_{n,m+2} &= \sum_{k=n+1}^{\infty}\left(1-a_k^2\right)A_{k+1,m} \sum_{k=n+1}^{\infty} b_k A_{k,m+1} + A_{n+1,m}.
\end{aligned} \tag{2.3}$$

Moreover, A_{nm} satisfies

$$|A_{nm}| \leq C \sum_{k=n+\lceil m/2 \rceil}^{\infty} (|1-a_k| + |b_k|), \tag{2.4}$$

where $\lceil m/2 \rceil$ is the integer part of $m/2$ and $C > 0$ is a constant. So $e(z) = \{e_n(z)\}$ is continuous in $\operatorname{Im} z = 0$ and analytic in $\mathbb{C}_+ := \{z : z \in \mathbb{C},\ \operatorname{Im} z > 0\}$ with respect to z.

Let us define $f(z)$ using (2.2) and the boundary condition (1.3) as

$$f(z) = \left(\gamma_0 + 2\gamma_1 \cos z\right)e_1(z) + \left(\beta_0 + 2\beta_1 \cos z\right)e_0(z). \tag{2.5}$$

The function f is analytic in $\mathbb{C}_+$, continuous in $\overline{\mathbb{C}}_+$, and $f(z) = f(z + 2\pi)$.

We denote the set of eigenvalues and spectral singularities of L by $\sigma_d(L)$ and $\sigma_{ss}(L)$, respectively. From the definition of the eigenvalues and spectral singularities, we have [12]

$$\begin{aligned}
\sigma_d(L) &= \{\lambda : \lambda = 2\cos z,\ z \in P_0,\ F(z) = 0\}, \\
\sigma_{ss}(L) &= \left\{\lambda : \lambda = 2\cos z,\ z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right],\ F(z) = 0\right\} \setminus \{0\}.
\end{aligned} \tag{2.6}$$

From (2.2) and (2.5), we get

$$\begin{aligned} f(z) &= \left[\gamma_0 + \gamma_1\left(e^{iz} + e^{-iz}\right)\right]\left[\alpha_1 e^{iz}\left(1 + \sum_{m=1}^{\infty} A_{1m} e^{imz}\right)\right] \\ &\quad + \left[\beta_0 + \beta_1\left(e^{iz} + e^{-iz}\right)\right]\left[\alpha_0\left(1 + \sum_{m=1}^{\infty} A_{0m} e^{imz}\right)\right] \\ &= \alpha_0\beta_1 e^{-iz} + \gamma_1\alpha_1 + \alpha_0\beta_0 + \left(\gamma_0\alpha_1 + \alpha_0\beta_1\right)e^{iz} + \gamma_1\alpha_1 e^{i2z} \\ &\quad + \sum_{m=1}^{\infty} \alpha_0\beta_1 A_{0m} e^{i(m-1)z} + \sum_{m=1}^{\infty}\left(\gamma_1\alpha_1 A_{1m} + \alpha_0\beta_0 A_{0m}\right)e^{imz} \\ &\quad + \sum_{m=1}^{\infty}\left(\gamma_0\alpha_1 A_{1m} + \alpha_0\beta_1 A_{0m}\right)e^{i(m+1)z} + \sum_{m=1}^{\infty}\gamma_1\alpha_1 A_{1m} e^{i(m+2)z}. \end{aligned} \tag{2.7}$$

Let

$$\begin{aligned} F(z) = f(z)e^{iz} &= \alpha_0\beta_1 + \left(\gamma_1\alpha_1 + \alpha_0\beta_0\right)e^{iz} + \left(\gamma_0\alpha_1 + \alpha_0\beta_1\right)e^{2iz} + \gamma_1\alpha_1 e^{3iz} \\ &\quad + \sum_{m=1}^{\infty}\alpha_0\beta_1 A_{0m} e^{imz} + \sum_{m=1}^{\infty}\left(\gamma_1\alpha_1 A_{1m} + \alpha_0\beta_0 A_{0m}\right)e^{i(m+1)z} \\ &\quad + \sum_{m=1}^{\infty}\left(\gamma_0\alpha_1 A_{1m} + \alpha_0\beta_1 A_{0m}\right)e^{i(m+2)z} + \sum_{m=1}^{\infty}\gamma_1\alpha_1 A_{1m} e^{i(m+3)z}; \end{aligned} \tag{2.8}$$

then the function F is analytic in $\mathbb{C}_+$, continuous in $\overline{\mathbb{C}}_+$, and $F(z) = F(z + 2\pi)$. It follows from (2.6) and (2.8) that

$$\begin{aligned} \sigma_d(L) &= \{\lambda : \lambda = 2\cos z,\ z \in P_0,\ F(z) = 0\}, \\ \sigma_{ss}(L) &= \left\{\lambda : \lambda = 2\cos z,\ z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right],\ F(z) = 0\right\} \setminus \{0\}. \end{aligned} \tag{2.9}$$

Definition 2.1. The multiplicity of a zero of F in P is called the multiplicity of the corresponding eigenvalue or spectral singularity of the BVP (1.2) and (1.3).

3. Principal Functions

Let $\lambda_1, \lambda_2, \ldots, \lambda_p$ and $\lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_q$ denote the zeros of F in $P_0 := \{z : z \in \mathbb{C}, z = x + iy,\ -\pi/2 \le x \le 3\pi/2,\ y > 0\}$ and $[-\pi/2, 3\pi/2]$ with multiplicities $m_1, m_2, \ldots, m_p$ and $m_{p+1}, m_{p+2}, \ldots, m_q$, respectively.

Definition 3.1. Let $\lambda = \lambda_0$ be an eigenvalue of L. If the vectors $y^{(0)}, y^{(1)}, \ldots, y^{(s)}$; $y^{(k)} = \{y_n^{(k)}\}_{n\in\mathbb{N}}$ $k = 0, 1, \ldots, s$ satisfy the equations

$$\begin{aligned} \left(ly^{(0)}\right)_n - \lambda_0 y_n^{(0)} &= 0, \\ \left(ly^{(k)}\right)_n - \lambda_0 y_n^{(k)} - y_n^{(k-1)} &= 0, \quad k = 1, 2, ..., s;\ n \in \mathbb{N}, \end{aligned} \tag{3.1}$$

then vector $y^{(0)}$ is called the eigenvector corresponding to the eigenvalue $\lambda = \lambda_0$ of L. The vectors $y^{(1)}, \ldots, y^{(s)}$ are called the associated vectors corresponding to $\lambda = \lambda_0$. The eigenvector and the associated vectors corresponding to $\lambda = \lambda_0$ are called the principal vectors of the eigenvalue $\lambda = \lambda_0$.

The principal vectors of the spectral singularities of L are defined similarly. We define the vectors

$$\begin{aligned} V_n^{(k)}(\lambda_j) &= \frac{1}{k!}\left\{\frac{d^k}{d\lambda^k}E_n(\lambda)\right\}\Bigg|_{\lambda=\lambda_j}, \quad k = 0, 1, \ldots, m_j - 1;\ j = 1, 2, \ldots, p, \\ V_n^{(k)}(\lambda_j) &= \frac{1}{k!}\left\{\frac{d^k}{d\lambda^k}E_n(\lambda)\right\}\Bigg|_{\lambda=\lambda_j}, \quad k = 0, 1, \ldots, m_j - 1;\ j = p+1, p+2, \ldots, q, \end{aligned} \tag{3.2}$$

where $\lambda = 2\cos z$, $z \in P_0$, and

$$\{E_n(\lambda)\} := \left\{e_n\left(\arccos\frac{\lambda}{2}\right)\right\}, \quad n \in \mathbb{N}. \tag{3.3}$$

Moreover, if $y(\lambda) = \{y_n(\lambda)\}_{n\in\mathbb{N}}$ is a solution of (1.2), then $(d^k/d\lambda^k)y(\lambda) = \{(d^k/d\lambda^k)y_n(\lambda)\}_{n\in\mathbb{N}}$ satisfies

$$a_{n-1}\frac{d^k}{d\lambda^k}y_{n-1}(\lambda) + b_n\frac{d^k}{d\lambda^k}y_n(\lambda) + a_n\frac{d^k}{d\lambda^k}y_{n+1}(\lambda) = \lambda\frac{d^k}{d\lambda^k}y_n(\lambda) + k\frac{d^{k-1}}{d\lambda^{k-1}}y_n(\lambda). \tag{3.4}$$

From (3.2) and (3.4), we get that

$$\begin{aligned} \left(\ell V^{(0)}(\lambda_j)\right)_n - \lambda_j V_n^{(0)}(\lambda_j) &= 0, \\ \left(\ell V^{(k)}(\lambda_j)\right)_n - \lambda_j V_n^{(k)}(\lambda_j) - V_n^{(k-1)}(\lambda_j) &= 0, \quad k = 1, 2, \ldots, m_j - 1;\ j = 1, 2, \ldots, q. \end{aligned} \tag{3.5}$$

Consequently, the vectors $V_n^{(k)}(\lambda_j)$; $k = 0, 1, \ldots, m_j - 1$, $j = 1, 2, \ldots, p$ and $V_n^{(k)}(\lambda_j)$; $k = 0, 1, \ldots, m_j - 1$, $j = p+1, p+2, \ldots, q$ are the principal vectors of eigenvalues and spectral singularities of L, respectively.

Theorem 3.2.

$$\begin{aligned} &V_n^{(k)}(\lambda_j) \in \ell_2(\mathbb{N}); \quad k = 0, 1, \ldots, m_j - 1, \ j = 1, 2, \ldots, p, \\ &V_n^{(k)}(\lambda_j) \notin \ell_2(\mathbb{N}); \quad k = 0, 1, \ldots, m_j - 1, \ j = p+1, \ldots, q. \end{aligned} \tag{3.6}$$

Proof. Using $E_n(\lambda) = e_n(\arccos(\lambda/2))$, we obtain that

$$\left\{ \frac{d^k}{d\lambda^k} E_n(\lambda) \right\} \Big|_{\lambda=\lambda_j} = \sum_{\nu=0}^{k} C_\nu \left\{ \frac{d^\nu}{d\lambda^\nu} e_n(z) \right\}_{z=z_j}, \quad n \in \mathbb{N}, \tag{3.7}$$

where $\lambda_j = 2\cos z_j$; $z_j \in P = P_0 \cup [-\pi/2, 3\pi/2]$, $j = 1, 2, \ldots, q$; C_ν is a constant depending on λ_j.

From (2.2), we find that

$$\begin{aligned} \left\{ \frac{d^\nu}{dz^\nu} e_n(z) \right\}_{z=z_j} &= \alpha_n e^{inz_j} \left\{ (in)^\nu + \sum_{m=1}^{\infty} [i(n+m)]^\nu A_{nm} e^{imz_j} \right\} \\ &= \alpha_n e^{inz_j} (in)^\nu + \alpha_n e^{inz_j} \sum_{m=1}^{\infty} [i(n+m)]^\nu A_{nm} e^{imz_j}. \end{aligned} \tag{3.8}$$

For the principal vectors $V_n^{(k)}(\lambda_j) = \{V^{(k)}(\lambda_j)\}_{n\in\mathbb{N}}$, $k = 0, 1, \ldots, m_j - 1$, $j = 1, 2, \ldots, p$, corresponding to the eigenvalues $\lambda_j = 2\cos z_j$, $j = 1, 2, \ldots, p$, of L, we get

$$\left\{ \frac{d^k}{d\lambda^k} E_n(\lambda) \right\} \Big|_{\lambda=\lambda_j} = \sum_{\nu=0}^{k} C_\nu \left\{ \alpha_n e^{inz_j} (in)^\nu + \alpha_n e^{inz_j} \sum_{m=1}^{\infty} [i(n+m)]^\nu A_{nm} e^{imz_j} \right\}; \tag{3.9}$$

then

$$V_n^{(k)}(\lambda_j) = \frac{1}{k!} \left\{ \sum_{\nu=0}^{k} C_\nu \left[\alpha_n e^{inz_j} (in)^\nu + \alpha_n e^{inz_j} \sum_{m=1}^{\infty} [i(n+m)]^\nu A_{nm} e^{imz_j} \right] \right\} \tag{3.10}$$

for $k = 0, 1, \ldots, m_j - 1$, $j = 1, 2, \ldots, p$.

Since $\operatorname{Im}\lambda_j > 0$, $j = 1, 2, \ldots, p$ from (3.10) we obtain that

$$\begin{aligned}\sum_{n=1}^{\infty}\left|\frac{1}{k!}\sum_{\nu=0}^{k}C_\nu\alpha_n e^{inz_j}(in)^\nu\right|^2 &\le \frac{1}{(k!)^2}\left[\sum_{n=1}^{\infty}\sum_{\nu=0}^{k}|C_\nu||\alpha_n|e^{-n\operatorname{Im}z_j}|n^\nu|\right]^2\\ &\le \frac{A}{(k!)^2}\left[\sum_{n=1}^{\infty}e^{-n\operatorname{Im}z_j}\left(1+n+n^2+\cdots+n^k\right)\right]^2\\ &\le \frac{A}{(k!)^2}(k+1)^2\left(\sum_{n=1}^{\infty}e^{-n\operatorname{Im}z_j}n^k\right)^2\\ &< \infty,\end{aligned} \tag{3.11}$$

where A is a constant. Now we define the function

$$g_n(z) = \frac{1}{k!}\sum_{\nu=0}^{k}\alpha_n e^{inz_j}\sum_{m=1}^{\infty}[i(n+m)]^\nu A_{nm}e^{imz_j}, \quad j = 1, 2, \ldots, p. \tag{3.12}$$

From (2.4), we obtain that

$$\begin{aligned}\left|g_n(z)\right| &\le \sum_{\nu=0}^{k}|\alpha_n|e^{-n\operatorname{Im}z_j}\sum_{m=1}^{\infty}|n+m|^\nu|A_{nm}|e^{-m\operatorname{Im}z_j}\\ &\le |\alpha_n|e^{-n\operatorname{Im}z_j}\left[\sum_{m=1}^{\infty}|A_{nm}|e^{-m\operatorname{Im}z_j}+\sum_{m=1}^{\infty}(n+m)|A_{nm}|e^{-m\operatorname{Im}z_j}\right.\\ &\qquad\left.+\cdots+\sum_{m=1}^{\infty}(n+m)^k|A_{nm}|e^{-m\operatorname{Im}z_j}\right]\\ &< Be^{-n\operatorname{Im}z_j},\end{aligned} \tag{3.13}$$

where $B = |\alpha_n|\sum_{m=1}^{\infty}\sum_{\nu=0}^{k}|A_{nm}|e^{-m\operatorname{Im}z_j}(n+m)^\nu$. Therefore, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\left|g_n(z)\right|^2 &\le \sum_{n=1}^{\infty}B^2e^{-2n\operatorname{Im}z_j}, \quad j = 1, 2, \ldots, p\\ &< \infty.\end{aligned} \tag{3.14}$$

It follows from (3.11) and (3.14) that $V_n^{(k)}(\lambda_j) \in \ell_2(\mathbb{N})$, $k = 0, 1, \ldots, m_j - 1$, $j = 1, 2, \ldots, p$.

If we consider (3.10) for the principal vectors corresponding to the spectral singularities $\lambda_j = 2\cos z_j$, $j = p+1,\ p+2,\ldots,q$, of L and consider that $\operatorname{Im} z_j = 0$ for the spectral singularities, then we have

$$V_n^{(k)}(\lambda_j) = \frac{1}{k!}\left\{\sum_{\nu=0}^{k} C_\nu \alpha_n e^{inz_j}(in)^\nu + \alpha_n e^{inz_j}\sum_{\nu=0}^{k}\sum_{m=1}^{\infty}[i(n+m)]^\nu A_{nm}e^{imz_j}\right\} \tag{3.15}$$

for $k = 0,1,\ldots,m_j-1$, $j = p+1, p+2,\ldots,q$.

Since $\operatorname{Im}\lambda_j = 0$, $j = p+1,\ldots,q$ from (3.15) we find that

$$\frac{1}{k!}\sum_{n=1}^{\infty}\left|\sum_{\nu=0}^{k} C_\nu \alpha_n e^{inz_j}(in)^\nu\right|^2 = \infty. \tag{3.16}$$

Now we define $t_n(z) = \sum_{\nu=0}^{k}\sum_{m=1}^{\infty}[i(n+m)]^\nu A_{nm}e^{imz_j}$, and using (2.4) we get

$$\begin{aligned}
|t_n(z)| &\le \sum_{\nu=0}^{k}\sum_{m=1}^{\infty}\left|(n+m)^\nu\right||A_{nm}| \\
&\le \sum_{\nu=0}^{k}\sum_{m=1}^{\infty}(n+m)^\nu C \sum_{k=n+\lceil m/2\rceil}^{\infty}(|1-a_k|+|b_k|) \\
&\le C\sum_{\nu=0}^{k}\sum_{m=1}^{\infty}(n+m)^\nu \sum_{k=n+\lceil m/2\rceil}^{\infty}\exp(-\varepsilon k)\exp(\varepsilon k)(|1-a_k|+|b_k|) \\
&\le C\sum_{\nu=0}^{k}\sum_{m=1}^{\infty}(n+m)^\nu \exp\left[\frac{-\varepsilon}{4}(n+m)\right]\sum_{k=n+\lceil m/2\rceil}^{\infty}\exp(\varepsilon k)(|1-a_k|+|b_k|) \\
&\le C_1\sum_{\nu=0}^{k}\sum_{m=1}^{\infty}(n+m)^\nu \exp\left[\frac{-\varepsilon}{4}(n+m)\right] \\
&= C_1 e^{(-\varepsilon/4)n}\sum_{m=1}^{\infty}\sum_{\nu=0}^{k}(n+m)^\nu \exp\left(\frac{-\varepsilon}{4}m\right) \\
&= Ae^{(-\varepsilon/4)n},
\end{aligned} \tag{3.17}$$

where

$$A = C_1\sum_{m=1}^{\infty}\sum_{\nu=0}^{k}(n+m)^\nu \exp\left(\frac{-\varepsilon}{4}m\right). \tag{3.18}$$

If we use (3.17), we obtain that

$$\frac{1}{k!}\sum_{n=1}^{\infty}\left|\alpha_n e^{inz_j}\sum_{\nu=0}^{k}\sum_{m=1}^{\infty}[i(n+m)]^{\nu}A_{nm}e^{imz_j}\right|^2 \le \frac{1}{k!}\sum_{n=1}^{\infty}\alpha_n^2 A^2 e^{-\varepsilon n/2} < \infty. \tag{3.19}$$

So $V_n^{(k)} \notin \ell_2(\mathbb{N})$, $k = 0,1,\ldots,m_j-1$, $j = p+1,p+2,\ldots q$. □

Let us introduce Hilbert spaces

$$\begin{aligned} H_k(\mathbb{N}) &= \left\{ y = \{y_n\}_{n\in\mathbb{N}} : \sum_{n\in\mathbb{N}}(1+|n|)^{2k}\left|y_n\right|^2 < \infty \right\}, \\ H_{-k}(\mathbb{N}) &= \left\{ u = \{u_n\}_{n\in\mathbb{N}} : \sum_{n\in\mathbb{N}}(1+|n|)^{-2k}|u_n|^2 < \infty \right\}, \quad k = 0,1,2,\ldots, \end{aligned} \tag{3.20}$$

with $\|y\|_k^2 = \sum_{n\in\mathbb{N}}(1+|n|)^{2k}|y_n|^2$, $\|u\|_{-k}^2 = \sum_{n\in\mathbb{N}}(1+|n|)^{-2k}|u_n|^2$, respectively. It is obvious that $H_0(\mathbb{N}) = \ell_2(\mathbb{N})$ and

$$H_{k+1}(\mathbb{N}) \subsetneqq H_k(\mathbb{N}) \subsetneqq l_2(\mathbb{N}) \subsetneqq H_{-k}(\mathbb{N}) \subsetneqq H_{-(k+1)}(\mathbb{N}), \quad k = 1,2,\ldots \tag{3.21}$$

Theorem 3.3. $V_n^{(k)}(\lambda_j) \in H_{-(k+1)}(\mathbb{N})$, $k = 0,1,\ldots,m_j-1$, $j = p+1,\ldots,q$.

Proof. From (3.15), we have

$$\begin{aligned} &\sum_{n=1}^{\infty}(1+|n|)^{-2(k+1)}\left|\frac{1}{k!}\sum_{\nu=0}^{k}C_\nu\alpha_n e^{inz_j}(in)^{\nu}\right|^2 < \infty, \\ &\sum_{n=1}^{\infty}(1+|n|)^{-2(k+1)}\left|\frac{1}{k!}\sum_{\nu=0}^{k}\alpha_n e^{inz_j}\sum_{m=1}^{\infty}[i(n+m)]^{\nu}A_{nm}e^{imz_j}\right|^2 < \infty \end{aligned} \tag{3.22}$$

for $k = 0,1,\ldots,m_j-1$, $j = p+1,p+2,\ldots,q$. Therefore, we obtain that $V_n^{(k)}(\lambda_j) \in H_{-(k+1)}(\mathbb{N})$, $k = 0,1,\ldots,m_j-1$, $j = p+1,p+2,\ldots,q$. □

Let us choose $m_0 = \max\{m_{p+1}, m_{p+2},\ldots,m_q\}$. By Theorem 3.2 and (3.21), we get the following.

Theorem 3.4. $V_n^{(k)}(\lambda_j) \in H_{-m_0}(\mathbb{N})$, $k = 0,1,\ldots,m_j-1$, $j = p+1,p+2,\ldots,q$.

Proof. The proof of theorem is trivial. □

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