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Research Article

Completely Dissipative Maps and Stinespring's Dilation-Type Theorem on σ-C^*-Algebras

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The notion of the completely dissipative maps on σ-C^*-algebras is introduced. We show that a completely dissipative map induces a representation of a σ-C^*-algebra. The classical Stinespring's dilation-type theorem is extended to a more general setting.

1. Introduction

There has been increased interest [1–12] in topological *-algebras that are inverse limits of C^*-algebras, called Pro-C^*-algebras. These algebras were introduced in [5] as a generalization of C^*-algebras and were called locally C^*-algebras. The same objects have been studied by other authors, for instance in [2, 13]. It is shown in [7, 14] that they arise naturally in the study of certain aspects of C^*-algebras such as the tangent algebras of C^*-algebras, multipliers of Pedersen's ideal, noncommutative analogues of classical Lie groups, and K-theory.

Motivated mainly by the works [11, 12, 15–18], in this paper, we shall introduce the notion of the completely dissipative maps on σ-C^*-algebras consider an abstract version of Stinespring's dilation-type theorem, and explore the natural relation completely dissipative maps to σ-C^*-algebras. We shall show that a completely positive definite map induces a representation of a σ-C^*-algebra. We also get a classification of completely positive definite maps on σ-C^*-algebras. Our approach is motivated by the setting of Banach algebras and C^*-algebras. This paper is organized as follows. In Section 1, introduction and some preliminaries are given. In Section 2, we present some definitions, results and notation which will be used throughout the paper. In Section 3, we shall prove the main result of this paper.

All the vector spaces and algebras considered throughout this paper are over the field $\mathbb{C}$ of complexes, and the topological spaces are supposed to be Hausdorff.

A locally m-convex (lmc-) algebra is a topological algebra A whose topology is defined by a family (p_α), $\alpha \in I$ (I a directed index set), of submultiplicative seminorms. A is called an lmc^*-algebra if, in addition, has an involution $*$ such that $p_\alpha(x^*) = p_\alpha(x)$, for any $\alpha \in I$, $x \in A$. If moreover, each p_α, $\alpha \in I$, is a C^*-seminorm, that is, $p_\alpha(x^*x) = p_\alpha(x)^2$ for any $\alpha \in I$, $x \in A$, A is called to be an lmc-C^*-algebra. Given an lmc-algebra A denote by (A_α) the inverse system of Banach algebras corresponding to A, that is, A_α is the completion of the normed algebra A/N_α, $N_\alpha = \mathrm{Ker}(p_\alpha)$, $\alpha \in I$, under norm $\|\cdot\|_\alpha$, with $\|x_\alpha\|_\alpha = p_\alpha(x)$, $x_\alpha = x + N_\alpha$, $x \in A$. A Pro-C^*-algebra A is a complete lmc-C^*-algebra. In this setting, (p_α) can be replaced with the collection $S(A)$, the set of all continuous C^*-seminorms on A. For a Pro-C^*-algebra A, and for a $p \in S(A)$, the normed *-algebra $(A/N_\alpha, \|\cdot\|_\alpha)$ is automatically complete (see [13]), so that $A_\alpha = A/N_\alpha$ is a C^*-algebra. Thus, A is an inverse limit of C^*-algebras, that is, $A = \varprojlim A_\alpha$ (see [6]). A metrizable Pro-C^*-algebra is called a σ-C^*-algebra with its topology determined by a countable subfamily (p_n) of $S(A)$, $n \in \mathbb{N}$.

2. Preliminaries

In this section, we present some definitions and results used in this paper. Our first goal is to define completely dissipative maps on σ-C^*-algebras. For this, we need to recall the following theorem in [15, Theorem 3.1.2 and Corollary 3.2.1].

Theorem 2.1 (see [15]). *Let A be a C^*-algebra with unit e, $L : A \to A$ a bounded *-map and $L(e) = 0$. Let $\phi_t = \exp(tL)$. Then $\phi_t (t \in \mathbb{R}^+)$ is completely positive on A if and only if L is completely dissipative, that is, $D(L_n; X, X) \geq 0$ for all $X \in M_n(A)$, where $D(L_n; X, Y) = L_n(X^*Y) - L_n(X^*)Y - X^*L_n(Y)$, for all $X, Y \in M_n(A)$, and $L_n = L \otimes I_n$.*

We can now introduce completely dissipative maps on σ-C^*-algebras as follows.

Definition 2.2. Let A be a σ-C^*-algebra, H a Hilbert space, $\phi : A \to B(H)$ a continuous map. Set $D : A \times A \to B(H)$ as $D(a, b) = \phi(ab) - \phi(a) - \phi(b)$. If the following conditions are satisfied:

(i)

$$\left[D\left(a_i^*, a_j\right)\right]_{i,j=1}^n \in M_n^+(B(H)); \tag{2.1}$$

(ii) there exists a function $\rho : A \to \mathbb{R}^+$ such that

$$\begin{aligned}&\sum_{i=1}^n \sum_{j=1}^n \left\langle \left[D\left(b_i^* a_i^* a^* a a_j, b_j\right) - D\left(a_i^* a^* a a_j, b_j\right)\right] x_j, x_i \right\rangle_H \\ &\quad \leq \rho(a^*a) \sum_{i=1}^n \sum_{j=1}^n \left\langle \left[D\left(b_i^* a_i^* a_j, b_j\right) - D\left(a_i^* a_j, b_j\right)\right] x_j, x_i \right\rangle_H\end{aligned} \tag{2.2}$$

for every $a, a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in A$, $x_1, x_2, \ldots, x_n \in H$, and $n \in \mathbb{N}$, then ϕ is called a completely dissipative map.

From the corresponding facts in the Aronszajn-Kolmogorov theorem of [17], we have the following result.

Lemma 2.3 (see [17]). *Let A be a σ-C^*-algebra, H a Hilbert space, and a map $T : A \times A \to B(H)$ be given such that, for all $a_1, a_2, \ldots, a_n \in A$,*

$$\left[T\left(a_i^*, a_j\right)\right]_{i,j=1}^n \in M_n^+(B(H)). \tag{2.3}$$

Then there exists a Hilbert space K and a map $V : A \to B(H, K)$ such that

$$T(a,b) = V(a^*)^* V(b), \quad \forall a, b \in A. \tag{2.4}$$

Definition 2.4. Let A be a σ-C^*-algebra, H a Hilbert space, and a map $\phi : A \times A \times A \to B(H)$. We say that ϕ is completely positive definite if for any $a_i, b_i \in A$, $i = 1, 2, \ldots, n \in \mathbb{N}$, we have

$$\left[\phi\left(b_i^*, a_i^* a_j, b_j\right)\right]_{i,j=1}^n \in M_n^+(B(H)). \tag{2.5}$$

If there exists a function $\rho : A \to \mathbb{R}^+$ such that the map $\phi : A \times A \times A \to B(H)$ satisfies the following additional condition:

$$\sum_{i=1}^n \sum_{j=1}^n \left\langle \phi\left(b_i^*, a_i^* a^* a a_j, b_j\right) x_j, x_i \right\rangle \le \rho(a^* a) \sum_{i=1}^n \sum_{j=1}^n \left\langle \phi\left(b_i^*, a_i^* a_j, b_j\right) x_j, x_i \right\rangle, \tag{2.6}$$

for every $a, a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in A$, $x_1, x_2, \ldots, x_n \in H$, and $n \in \mathbb{N}$, then we say, ϕ is relatively bounded.

3. The Stinespring's Dilation-Type Theorem

To prove the main theorem, the following things have to be elucidated. If X and Y are vector spaces, we denote by $X \otimes Y$ their algebraic tensor product. This is linearly spanned by elements $x \otimes y$ ($x \in X$, $y \in Y$). If $\sigma : X \times Y \to Z$ is a bilinear map, where X, Y, and Z are vector spaces, then there is a unique linear map $\sigma' : X \otimes Y \to Z$ such that $\sigma'(x \otimes y) = \sigma(x, y)$ for all $x \in X$ and $y \in Y$.

If τ, μ are linear functionals on the vector spaces X, Y, respectively, then there is a unique linear functional $\tau \otimes \mu$ on $X \otimes Y$ such that

$$(\tau \otimes \mu)(x \otimes y) = \tau(x)\mu(y), \quad \forall x \in X, \ y \in Y, \tag{3.1}$$

since the function

$$X \times Y \longrightarrow \mathbb{C}, \qquad (x, y) \longrightarrow \tau(x)\mu(y) \tag{3.2}$$

is bilinear.

Suppose that $\sum_{j=1}^{n} x_j \otimes y_j = 0$, where $x_j \in X$ and $y_j \in Y$. If $y_1, \ldots, y_n$ are linearly independent, then $x_1 = \cdots = x_n = 0$. For, in this case, there exist linear functionals $\mu_j : Y \to \mathbb{C}$ such that $\mu_j(y_i) = \delta_{ij}$ (Kronecker delta). If $\tau : X \to \mathbb{C}$ is linear, we have

$$0 = (\tau \otimes \mu_j)\left(\sum_{i=1}^{n} x_i \otimes y_i\right) = \sum_{i=1}^{n} \tau(x_i)\rho_j(y_i) = \sum_{i=1}^{n} \tau(x_i)\delta_{ij} = \tau(x_j). \tag{3.3}$$

Thus, $\tau(x_j) = 0$ for arbitrary τ, and this shows that $x_1 = \cdots = x_n = 0$.

Similarly, if $\sum_{j=1}^{n} x_j \otimes y_j = 0$, and $x_1, \ldots, x_n$ are linearly independent, then $y_1 = \cdots = y_n = 0$.

Theorem 3.1. *Let A be a σ-C^*-algebra with unit e, H a Hilbert space, $\phi : A \to B(H)$ be a completely dissipative map, and $\widehat{D} : A \times A \times A \to B(H)$ defined by*

$$\widehat{D}(a, b, c) = D(ab, c) - D(b, c), \tag{3.4}$$

where $D(a, b) = \phi(ab) - \phi(a) - \phi(b)$, then

(i) *$\widehat{D}$ is a relatively bounded completely positive definite map,*

(ii) *there exists a Hilbert space X, a representation φ of A on X, and a map $V : A \to B(H, X)$ such that*

$$\widehat{D}(a, b, c) = V(a^*)^* \varphi(b) V(c), \quad \forall a, b, c \in A, \tag{3.5}$$

(iii) *$\{\varphi(a)V(b)x : a, b \in A, x \in H\}$ spans a dense subspace of X,*

(iv) *the representation (φ, X, V) associated with ϕ is unique up to a unitary equivalence,*

(v) *the map $V : A \to B(H, X)$ is a 1-cocycle with respect to the representation φ, that is, $\Delta_{\varphi}^{1} V = \varphi(a)V(b) - V(ab) + V(a) = 0$.*

Proof. (i) By the definition, for any $a_i, b_i \in A$, $i = 1, 2, \ldots, n$, $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\widehat{D}(b_i^*, a_i^* a_j, b_j) \\
&\quad = D(b_i^* a_i^* a_j, b_j) - D(a_i^* a_j, b_j) \\
&\quad = \phi(b_i^* a_i^* a_j b_j) - \phi(b_i^* a_i^* a_j) - \phi(b_j) - [\phi(a_i^* a_j b_j) - \phi(a_i^* a_j) - \phi(b_j)] \\
&\quad = \phi(b_i^* a_i^* a_j b_j) - \phi(b_i^* a_i^* a_j) - \phi(a_i^* a_j b_j) + \phi(a_i^* a_j), \\
&D(b_i^* a_i^*, a_j b_j) - D(b_i^* a_i^*, a_j) + D(a_i^*, a_j) - D(a_i^*, a_j b_j) \\
&\quad = \phi(b_i^* a_i^* a_j b_j) - \phi(b_i^* a_i^*) - \phi(a_j b_j) - [\phi(b_i^* a_i^* a_j) - \phi(b_i^* a_i^*) - \phi(a_j)] \\
&\qquad + [\phi(a_i^* a_j) - \phi(a_i^*) - \phi(a_j)] - [\phi(a_i^* a_j b_j) - \phi(a_i^*) - \phi(a_j b_j)] \\
&\quad = \phi(b_i^* a_i^* a_j b_j) - \phi(b_i^* a_i^* a_j) + \phi(a_i^* a_j) - \phi(a_i^* a_j b_j).
\end{aligned} \tag{3.6}$$

Thus,

$$\widehat{D}(b_i^*, a_i^* a_j, b_j) = D(b_i^* a_i^*, a_j b_j) - D(b_i^* a_i^*, a_j) + D(a_i^*, a_j) - D(a_i^*, a_j b_j). \tag{3.7}$$

From Lemma 2.3, there exists a Hilbert space K, and a map $V : A \to B(H, K)$ such that

$$D(a,b) = V(a^*)^* V(b), \quad \forall a, b \in A. \tag{3.8}$$

It follows that

$$\begin{aligned}
\widehat{D}(b_i^*, a_i^* a_j, b_j) &= V(a_i b_i)^* V(a_j b_j) - V(a_i b_i)^* V(a_j) + V(a_i)^* V(a_j) - V(a_i)^* V(a_j b_j) \\
&= V(a_i b_i)^* \left[V(a_j b_j) - V(a_j)\right] + V(a_i)^* \left[V(a_j) - V(a_j b_j)\right] \\
&= \left[V(a_i b_i)^* - V(a_i)^*\right] \left[V(a_j b_j) - V(a_j)\right] \\
&= \left[V(a_i b_i) - V(a_i)\right]^* \left[V(a_j b_j) - V(a_j)\right].
\end{aligned} \tag{3.9}$$

Therefore, we obtain

$$\left[\widehat{D}(b_i^*, a_i^* a_j, b_j)\right]_{i,j=1}^n \in M_n^+(B(H)). \tag{3.10}$$

Since $\phi : A \to B(H)$ be a completely dissipative map, then by Definition 2.2(ii), there exists a function $\rho : A \to \mathbb{R}^+$ such that

$$\begin{aligned}
&\sum_{i=1}^n \sum_{j=1}^n \left\langle \left[D(b_i^* a_i^* a^* a a_j, b_j) - D(a_i^* a^* a a_j, b_j)\right] x_j, x_i \right\rangle_H \\
&\quad \le \rho(a^* a) \sum_{i=1}^n \sum_{j=1}^n \left\langle \left[D(b_i^* a_i^* a_j, b_j) - D(a_i^* a_j, b_j)\right] x_j, x_i \right\rangle_H,
\end{aligned} \tag{3.11}$$

where $D(a,b) = \phi(ab) - \phi(a) - \phi(b)$, and $a, a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in A$, $x_1, x_2, \ldots, x_n \in H$. Thus, we have

$$\begin{aligned}
&\sum_{i=1}^n \sum_{j=1}^n \left\langle \widehat{D}(b_i^*, a_i^* a^* a a_j, b_j) x_j, x_i \right\rangle_H \\
&\quad = \sum_{i=1}^n \sum_{j=1}^n \left\langle \left[D(b_i^* a_i^* a^* a a_j, b_j) - D(a_i^* a^* a a_j, b_j)\right] x_j, x_i \right\rangle_H \\
&\quad \le \rho(a^* a) \sum_{i=1}^n \sum_{j=1}^n \left\langle \left[D(b_i^* a_i^* a_j, b_j) - D(a_i^* a_j, b_j)\right] x_j, x_i \right\rangle_H \\
&\quad = \rho(a^* a) \sum_{i=1}^n \sum_{j=1}^n \left\langle \widehat{D}(b_i^*, a_i^* a_j, b_j) x_j, x_i \right\rangle_H.
\end{aligned} \tag{3.12}$$

It follows that $\widehat{D}$ is relatively bounded.

(ii) Let $A\otimes A\otimes H$ be the algebraic tensor product of A, A, and H. The algebraic tensor product of A, A, and H consists of elements of the form $\sum_{i=1}^{n} a_i\otimes b_i\otimes x_i$ for all a_i, b_i in A, x_i in H, $i=1,2,\ldots,n$ and n in $\mathbb{N}$. A map $\langle\cdot,\cdot\rangle : (A\otimes A\otimes H)\times(A\otimes A\otimes H)\to\mathbb{C}$ is defined on elementary terms by

$$\left\langle a\otimes b\otimes x, c\otimes d\otimes y\right\rangle = \left\langle \widehat{D}(d^*, c^*a, b)x, y\right\rangle_H, \tag{3.13}$$

where $\langle\cdot,\cdot\rangle_H$ refers to the Hilbert inner product on H, and extending by linearity. Hence, we have

$$\langle\xi,\eta\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m}\left\langle \widehat{D}\left(d_j^*, c_j^*a_i, b_i\right)x_i, y_j\right\rangle_H, \tag{3.14}$$

for $\xi=\sum_{i=1}^{n} a_i\otimes b_i\otimes x_i$, and $\eta=\sum_{j=1}^{m} c_j\otimes d_j\otimes y_j$ in $A\otimes A\otimes H$.

It is clear that $\langle\cdot,\cdot\rangle$ is linear in the first variable and conjugate linear in the second. For such a form, the polarisation identity

$$\langle\xi,\eta\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k\left\langle \xi+i^k\eta, \xi+i^k\eta\right\rangle \tag{3.15}$$

holds. The existence of such sesquilinear form is easily seen if we show that

$$\sum_{j=1}^{n} a_j\otimes b_j\otimes x_j = 0 \Longrightarrow \left\langle \sum_{j=1}^{n} a_j\otimes b_j\otimes x_j, \sum_{j=1}^{n} a_j\otimes b_j\otimes x_j\right\rangle = 0. \tag{3.16}$$

Choose linearly independent elements $e_1,\ldots,e_m$ in H having the same linear span as $x_1,\ldots,x_n$. Then $x_j=\sum_{i=1}^{m}\lambda_{ij}\ e_i$ for unique scalars λ_{ij}. Since $\sum_{j=1}^{n} a_j\otimes b_j\otimes x_j=0$, we have

$$\sum_{j=1}^{n} a_j\otimes b_j\otimes\left(\sum_{i=1}^{m}\lambda_{ij}e_i\right) = \sum_{i,j}\lambda_{ij}a_j\otimes b_j\otimes e_i = 0, \tag{3.17}$$

and, therefore, $\sum_{j=1}^{n}\lambda_{ij}a_j\otimes b_j = 0$, for $i=1,2,\ldots,m$, because $e_1,\ldots,e_m$ are linearly independent.

Similarly, choose linearly independent elements $\varepsilon_1,\ldots,\varepsilon_p$ in A having the same linear span as $b_1,\ldots,b_n$. Then $b_j=\sum_{k=1}^{p}\gamma_{kj}\varepsilon_k$ for unique scalars γ_{kj}. Since $\sum_{j=1}^{n}\lambda_{ij}a_j\otimes b_j=0$, for $i=1,2,\ldots,m$, we have

$$\sum_{j=1}^{n}\lambda_{ij}a_j\otimes b_j = \sum_{j=1}^{n}\lambda_{ij}a_j\otimes\left(\sum_{k=1}^{p}\gamma_{kj}\varepsilon_k\right) = \sum_{j,k}\lambda_{ij}\gamma_{kj}a_j\otimes\varepsilon_k = 0, \tag{3.18}$$

and hence $\sum_{j=1}^{n}\lambda_{ij}\gamma_{kj}a_j = 0$, for $i = 1,\dots,m$, $k = 1,\dots,p$, because $\varepsilon_1,\dots,\varepsilon_p$ are linearly independent. Hence,

$$\begin{aligned}
&\left\langle \sum_{j=1}^{n} a_j \otimes b_j \otimes x_j, \sum_{j=1}^{n} a_j \otimes b_j \otimes x_j \right\rangle \\
&\quad = \left\langle \sum_{i,j,k} \lambda_{ij}\ \gamma_{kj}\ a_j \otimes \varepsilon_k \otimes e_i, \sum_{i,j,k} \lambda_{ij}\ \gamma_{kj}\ a_j \otimes \varepsilon_k \otimes e_i \right\rangle \\
&\quad = \left\langle \sum_{i,k} 0 \otimes \varepsilon_k \otimes e_i, \sum_{i,k} 0 \otimes \varepsilon_k \otimes e_i \right\rangle = \sum_{i,k,s,l} \left\langle \widehat{D}(\varepsilon_s^*, 0, \varepsilon_k^*) e_i, e_l \right\rangle = \sum_{i,k,s,l} \langle 0\ e_i, e_l \rangle = 0.
\end{aligned} \tag{3.19}$$

It follows from the polarisation identity that a sesquilinear form $\langle\cdot,\cdot\rangle$ is hermitian if and only if $\langle\xi,\xi\rangle \in \mathbb{R}(\xi \in A \otimes A \otimes H)$. Thus, positive sesquilinear forms are hermitian.

Let $\xi \in A \otimes A \otimes H$, and suppose that $\xi = \sum_{i=1}^{n} a_i \otimes b_i \otimes x_i$, then we have the formula

$$\langle\xi,\xi\rangle = \sum_{i,j=1}^{n} \left\langle \widehat{D}\left(b_j^*, a_j^* a_i, b_i\right) x_i, y_j \right\rangle_H. \tag{3.20}$$

Since $\widehat{D} : A \otimes A \otimes A \rightarrow B(H)$ is relatively bounded completely positive definite map from (i), it follows that

$$\langle\xi,\xi\rangle \geq 0, \quad \forall \xi \in A \otimes A \otimes H. \tag{3.21}$$

Then, the sesquilinear form $\langle\cdot,\cdot\rangle$ is positive semidefinite. Setting

$$N = \{\xi \in A \otimes A \otimes H : \langle\xi,\xi\rangle = 0\}, \tag{3.22}$$

it follows from the Cauchy-Schwartz inequality that the set N is a subspace of $A \otimes A \otimes H$. The induced form

$$\langle \xi + N, \eta + N \rangle = \langle \xi, \eta \rangle, \quad \forall \xi + N,\ \eta + N \in \frac{(A \otimes A \otimes H)}{N} \tag{3.23}$$

on the quotient space $(A \otimes A \otimes H)/N$ is thus positive definite, and it is an inner product on $(A \otimes A \otimes H)/N$. Thus, the quotient space $(A \otimes A \otimes H)/N$ is a pre-Hilbert space under this inner product. By completing it with respect to the induced inner product, we get a Hilbert space X.

For $a \in A$, define a linear map $\varphi(a)$ on $A \otimes A \otimes H$ by $\varphi(a)(c \otimes d \otimes x) = (ac) \otimes d \otimes x$, where $c, d \in A$, $x \in H$, and extending by linearity. Hence for each $\xi = \sum_{i=1}^{n} a_i \otimes b_i \otimes x_i$ in $A \otimes A \otimes H$, we have

$$\varphi(a)\left(\sum_{i=1}^{n} a_i \otimes b_i \otimes x_i\right) = \sum_{i=1}^{n} (a a_i) \otimes b_i \otimes x_i. \tag{3.24}$$

Since $\widehat{D} : A \otimes A \otimes A \to B(H)$ is relatively bounded completely positive definite map from (i); hence, there exists a function $\rho : A \to \mathbb{R}^+$ such that

$$\sum_{i,j=1}^{n} \left\langle \widehat{D}(b_i^*, a_i^* a^* a a_j, b_j) x_j, x_i \right\rangle_H \leq \rho(a^* a) \sum_{i,j=1}^{n} \left\langle \widehat{D}(b_i^*, a_i^* a_j, b_j) x_j, x_i \right\rangle_H, \tag{3.25}$$

for every $a, a_1, a_2, \ldots, a_n \in A$, $x_1, x_2, \ldots, x_n \in H$, and $n \in \mathbb{N}$. Thus, for any $\xi = \sum_{i=1}^{n} a_i \otimes b_i \otimes x_i \in A \otimes A \otimes H$, by an easy calculation, we have

$$\begin{aligned} \langle \varphi(a)\xi, \varphi(a)\ \xi \rangle &= \sum_{i,j=1}^{n} \left\langle \widehat{D}\Big(b_j^*, a_j^* a^* a a_i, b_i\Big) x_i, x_j \right\rangle_H \\ &\leq \rho(a^* a) \sum_{i,j=1}^{n} \left\langle \widehat{D}\Big(b_j^*, a_j^* a_i, b_i\Big) x_i, x_j \right\rangle_H \\ &= \rho(a^* a) \left\langle \sum_{i=1}^{n} a_i \otimes b_i \otimes x_i, \sum_{j=1}^{n} a_j \otimes b_j \otimes x_j \right\rangle = \rho(a^* a) \langle \xi, \xi \rangle. \end{aligned} \tag{3.26}$$

So $\varphi(A)$ leaves N invariant, and the induced map

$$\begin{aligned} \varphi(a)(\xi + N) = \varphi(a)\left(\sum_{i=1}^{n} a_i \otimes b_i \otimes x_i + N \right) &= \varphi(a)(\xi) + N \\ &= \sum_{i=1}^{n} (a a_i) \otimes b_i \otimes x_i + N \end{aligned} \tag{3.27}$$

defines a linear operator on the quotient space $(A \otimes A \otimes H)/N$. Since $\|\varphi(a)\| \leq \rho(a^* a)^{1/2}$ for all $a \in A$, therefore, $\varphi(a)$ extends to a bounded linear operator from X to X by continuity, which will be also denoted by $\varphi(a)$, thus $\varphi(a)$ in $B(X)$. It is easily to see that φ is representation, and that φ is a *-representation, that is, $\varphi(a)^* = \varphi(a^*)$, for all $a \in A$. Indeed, for $\xi = \sum_{i=1}^{n} a_i \otimes b_i \otimes x_i$ and $\eta = \sum_{j=1}^{m} c_j \otimes d_j \otimes y_j$ in $A \otimes A \otimes H$, it follows that

$$\begin{aligned} \langle\ \xi, \varphi(a)^* \eta \rangle &= \langle \varphi(a)\xi, \eta \rangle \\ &= \left\langle \varphi(a)\left(\sum_{i=1}^{n} a_i \otimes b_1 \otimes x_i \right), \sum_{j=1}^{m} c_j \otimes d_j \otimes y_j \right\rangle \\ &= \left\langle \sum_{i=1}^{n} (a a_i) \otimes b_i \otimes x_i, \sum_{j=1}^{m} c_j \otimes d_j \otimes y_j \right\rangle \\ &= \sum_{i,j} \left\langle \widehat{D}\Big(d_j^*, c_j^* a a_i, b_i\Big) x_i, y_j \right\rangle_H \\ &= \sum_{i,j} \left\langle \widehat{D}(d_j^*, (a^* c_j)^* a_i, b_i) x_i, y_j \right\rangle_H \end{aligned}$$

$$\begin{aligned}
&= \left\langle \sum_{i=1}^{n} a_i \otimes b_i \otimes x_i, \sum_{j=1}^{m} a^* c_j \otimes d_j \otimes y_j \right\rangle \\
&= \left\langle \sum_{i=1}^{n} a_i \otimes b_i \otimes x_i, \varphi(a^*) \left(\sum_{j=1}^{m} c_j \otimes d_j \otimes y_j \right) \right\rangle \\
&= \langle \xi, \varphi(a^*)\eta \rangle.
\end{aligned} \tag{3.28}$$

For $a \in A$ and $x \in H$, we define $V : A \to B(H, X)$ as follows:

$$V(a)x = e \otimes a \otimes x + N. \tag{3.29}$$

It follows that

$$\langle V(a)x, V(a)x \rangle = \left\langle \widehat{D}(a^*, e, a)x, x \right\rangle_H \leq \left\| \widehat{D}(a^*, e, a) \right\| \cdot \|x\|^2. \tag{3.30}$$

So

$$\|V(a)\| \leq \left\| \widehat{D}(a^*, e, a) \right\|^{1/2}, \tag{3.31}$$

and for any $a, b, c \in A$ and $x, y \in H$, we obtain

$$\left\langle V(a^*)^* \varphi(b) V(c)x, y \right\rangle_H = \langle \varphi(b) V(c)x, V(a^*)y \rangle_H = \left\langle \widehat{D}(a, b, c)x, y \right\rangle_H. \tag{3.32}$$

Hence,

$$\widehat{D}(a, b, c) = V(a^*)^* \varphi(b) V(c), \quad \forall a, b, c \in A. \tag{3.33}$$

(iii) For every a, b in A and x in H, we have

$$\varphi(a)V(b)(x) = \varphi(a)(e \otimes b \otimes x + N) = a \otimes b \otimes x + N. \tag{3.34}$$

So, the linear span of the set $\{\varphi(a)V(b)x : a, b \in A, x \in H\}$ is precisely $(A \otimes A \otimes H)/N$ since every element of $(A \otimes A \otimes H)/N$ is the finite sum of $a \otimes b \otimes x + N$ with a, b in A and $x \in H$. Thus, it follows that $\{\varphi(a)V(b)x : a, b \in A, x \in H\}$ spans a dense subspace of X.

(iv) Let (φ, X, V) and (ψ, Y, W) be the representations associated with ϕ. For each $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in A$, and $x_1, x_2, \ldots, x_n \in H$, we have

$$\begin{aligned}
\left\| \sum_{i=1}^{n} \varphi(a_i) V(b_i) x_i \right\|_X^2 &= \left\langle \sum_{i=1}^{n} \varphi(a_i) V(b_i) x_i, \sum_{i=1}^{n} \varphi(a_i) V(b_i) x_i \right\rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \left\langle V(b_j)^* \varphi(a_j^* a_i) V(b_i) x_i, x_j \right\rangle_H \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \left\langle \widehat{D}\left(b_j^*, a_j^* a_i, b_i\right) x_i, x_j \right\rangle_H \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \left\langle W(b_j)^* \psi(a_j^* a_i) W(b_i) x_i, x_j \right\rangle_H \\
&= \left\langle \sum_{i=1}^{n} \psi(a_i) W(b_i) x_i, \sum_{i=1}^{n} \psi(a_i) W(b_i) x_i \right\rangle \\
&= \left\| \sum_{i=1}^{n} \psi(a_i) W x_i \right\|_Y^2
\end{aligned} \tag{3.35}$$

and so the linear map defined by

$$U_0\left(\sum_{i=1}^{n} \varphi(a_i) V(b_i) x_i \right) = \sum_{i=1}^{n} \psi(a_i) W(b_i) x_i \tag{3.36}$$

extends to an isometry from X to Y, which will be denoted by U. Since the range of U_0 is dense in Y; thus, U is a unitary operator of X onto Y.

From the relation $\widehat{D}(a,b,c) = V(a^*)^* \varphi(b) V(c) = W(a^*)^* \psi(b) W(c)$, $a, b, c \in A$, for all $a_i, b_i, c_j, d_j \in A$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$ and $x_1, \ldots, x_n, y_1, \ldots, y_m \in H$, we get

$$\begin{aligned}
\left\langle U\left(\sum_{i=1}^{n} \varphi(a_i) V(b_i) x_i \right), \sum_{j=1}^{m} \psi(c_j) W(d_j) y_j \right\rangle &= \left\langle \sum_{i=1}^{n} \psi(a_i) W(b_i) x_i, \sum_{j=1}^{m} \psi(c_j) W(d_j) y_j \right\rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \left\langle W(d_j)^* \psi(c_j^* a_i) W(b_i) x_i, y_j \right\rangle_H \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \left\langle \widehat{D}\left(d_j^*, c_j^* a_i, b_i\right) x_i, y_j \right\rangle_H \\
&= \left\langle \sum_{i=1}^{n} \varphi(a_i) V(b_i) x_i, \sum_{j=1}^{m} \varphi(c_j) V(d_j) y_j \right\rangle.
\end{aligned} \tag{3.37}$$

Therefore, $U^*(\sum_{j=1}^m \varphi(c_j)W(d_j)y_j) = \sum_{j=1}^m \varphi(c_j)V(d_j)y_j$. For each $a, a_i, b_i \in A$, $i = 1, 2, \ldots, n$ and $x_1, \ldots, x_n \in H$, we have

$$U\varphi(a)U^*\left(\sum_{i=1}^n \psi(a_i)W(b_i)x_i\right) = U\left(\sum_{i=1}^n \varphi(aa_i)V(b_i)x_i\right) = \psi(a)\left(\sum_{i=1}^n \psi(a_i)W(b_i)x_i\right). \quad (3.38)$$

Since $U\varphi(a)U^*$ and $\psi(a)$ are bounded, and $\{\psi(a)W(b)x : a, b \in A, x \in H\}$ spans a dense subspace of Y, we then have $U\varphi(a)U^* = \psi(a)$ for each $a \in A$.

(v) For all $a, b \in A$, we let $T = \varphi(a)V(b) - V(ab) + V(a)$. Then, $T \in B(H, X)$. Thus,

$$\begin{aligned}
T^*T &= [\varphi(a)V(b) - V(ab) + V(a)]^*[\varphi(a)V(b) - V(ab) + V(a)] \\
&= [V(b)^*\varphi(a)^* - V(ab)^* + V(a)^*][\varphi(a)V(b) - V(ab) + V(a)] \\
&= V(b)^*\varphi(a^*a)V(b) - V(b)^*\varphi(a)^*V(ab) + V(b)^*\varphi(a^*a) - V(ab)^*\varphi(a)V(b) \\
&\quad + V(ab)^*V(ab) - V(ab)^*V(a) + V(a)^*\varphi(a)V(b) - V(a)^*V(ab) + V(a)^*V(a) \\
&= \widehat{D}(b^*, a^*a, b) - \widehat{D}(b^*, a^*, ab) + \widehat{D}(b^*, a^*, a) - \widehat{D}(b^*a^*, a, b) \\
&\quad + \widehat{D}(a^*, a, b) + \widehat{D}(b^*a^*, e, ab) - \widehat{D}(b^*a^*, e, a) - \widehat{D}(a^*, e, ab) + \widehat{D}(a^*, e, a) \\
&= D(b^*a^*a, b) - D(a^*a, b) - D(b^*a^*, ab) + D(a^*, ab) \\
&\quad + D(b^*a^*, a) - D(a^*, a) - D(b^*a^*a, b) + D(a, b) \\
&\quad + D(a^*a, b) - D(a, b) + D(b^*a^*, ab) - D(e, ab) \\
&\quad - D(b^*a^*, a) + D(e, a) - D(a^*, ab) + D(e, ab) + D(a^*, a) - D(e, a) \\
&= 0.
\end{aligned} \quad (3.39)$$

We, therefore, get $T = 0$, that is, $\Delta_\varphi^1 V = \varphi(a)V(b) - V(ab) + V(a) = 0$. This completes the proof. □

Finally, as in cohomology theory of Banach algebras [16], we introduce the notion of n-cocycles.

Definition 3.2. Let A be a σ-C^*-algebra, and φ a representation of A in a Hilbert space H. Let K be another Hilbert space. For $n \geq 1$, the group (under pointwise addition) of all n-cochains on A with values in $B(K, H)$ is denoted by $C^n(A; B(K, H))$ consisting of all continuous n-linear maps of A^n into $B(K,H)$. We set $C^n(A;B(K,H)) = 0$ for $n \leq 0$. We define a group homomorphism (coboundary operator) $\Delta_\varphi^n : C^n(A;B(K,H)) \to C^{n+1}(A;B(K,H))$ with respect to φ by $\Delta_\varphi^n = 0$ for $n \leq 0$ and, $n \geq 1$,

$$\begin{aligned}\Delta_\varphi^n f(a_1, a_2, \ldots, a_{n+1}) = \varphi(a_1) f(a_2, a_3, \ldots, a_{n+1}) \\ + \sum_{i=1}^{n} (-1)^i f(a_1, a_2, a_{i-1}, a_i a_{i+1}, a_{i+2}, \ldots, a_{n+1}) \\ + (-1)^{n+1} f(a_1, a_2, \ldots, a_n),\end{aligned} \tag{3.40}$$

where $a_1, a_2, \ldots, a_{n+1} \in A$.

We set

$$Z_\varphi^n(A;B(K,H)) = \left\{ f \in C^n(A;B(K,H)) : \Delta_\varphi^n f = 0 \right\}. \tag{3.41}$$

Elements of $Z_\varphi^n(A;B(K,H))$ are called n-cocycles with respect to the representation φ of A. We see that a continuous linear map $f : A \to B(K,H)$ is called 1-cocycle with respect to the representation φ of A if it satisfied $\Delta_\varphi^1 f = \varphi(a) f(b) - f(ab) + f(a) = 0$. The 1-cocycle appears at (v) in Theorem 3.1. For further, results will report in another paper.

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