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Research Article

On the Completeness of the System $\{z^{\lambda_n}\log^j z\}$ in L_a^2

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Let $L_a^2(\Omega)$ denote a Hilbert space consisting of analytic functions on an unbounded domain Ω located outside an angle domain with vertex at the origin. We obtain a completeness theorem for the system $M_\Lambda = \{z^{\lambda_n}\log^j z, j = 0, 1, \ldots, m_n - 1\}_{n=1}^{\infty}$, in $L_a^2(\Omega)$.

1. Introduction

Let Ω denote a domain in the complex z plane. Let $L_a^2(\Omega)$ denote the space consisting of all functions f analytic in Ω with

$$\iint_\Omega |f(z)|^2 \mathrm{d}m < \infty, \quad z = x + iy, \tag{1.1}$$

where $\mathrm{d}m$ is the area element in the z plane (i.e., $\mathrm{d}m = \mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$ for $z = re^{i\theta}$). It is well known that, with the inner product

$$\langle f, g\rangle = \iint_\Omega f(z)\overline{g(z)}\mathrm{d}m, \tag{1.2}$$

and the norm $\|f\| = \langle f, f\rangle^{1/2}$, $L_a^2(\Omega)$ is a Hilbert space (see, e.g., [1, Chapter 1]).

We say a system $\{h_n\} \subset L_a^2(\Omega)$ is *complete* in $L_a^2(\Omega)$ if its linear span is dense in $L_a^2(\Omega)$ (see, e.g., [2–7]). The following lemma provides an elementary fact on completeness (see, e.g., [1, Chapter 1] and [3, Lemma 1.1]).

Lemma 1.1. *A necessary and sufficient condition for the system* $\{h_n\}$ *to be complete in* $L_a^2(\Omega)$ *is that for any* $f \in L_a^2(\Omega)$, *if* $\langle f, h_n \rangle = 0$ *for all* h_n, *then* $f(z) \equiv 0$.

A *Dzhrbasian domain* (see, e.g., [8]) is an unbounded simply connected domain satisfying the following conditions:

Condition Ω(I). For $r > 0$, let $\sigma(r)$ denote the linear measure of the intersection of the circle $|z| = r$ and Ω. There exists $r_0 > 0$ such that, for $r > r_0$,

$$\sigma(r) \leq e^{-\alpha(r)}, \tag{1.3}$$

where $\alpha(r) > 0$ satisfies

$$\alpha(r) = \alpha(r_0) + \int_{r_0}^{r} \frac{\varphi(t)}{t} \mathrm{d}t \tag{1.4}$$

with $\varphi(r) \geq 0$ and $\varphi(r) \uparrow \infty$ as $r \to \infty$.

Condition Ω(II). The complement of Ω consists of m unbounded simply connected domains Ω_j $(j = 1, 2, \ldots, m)$, each containing an angle domain Δ_j with opening π/β_j, $\beta_j > 1/2$.

For a Dzhrbasian domain satisfying Condition Ω(I) and Condition Ω(II), Dzhrbasian proved that if

$$\int^{\infty} \frac{\alpha(r)}{r^{1+\vartheta}} \mathrm{d}r = +\infty \tag{1.5}$$

(here $\int^{\infty}$ means that the lower limit of the integral is sufficiently large), where

$$\vartheta = \max\{\beta_1, \ldots, \beta_m\}, \tag{1.6}$$

then the polynomial system $\{1, z, z^2, \ldots\}$ is complete in $L_a^2(\Omega)$.

Motivated by the result of Dzhrbasian, Shen [4–7] studied the completeness of the system $\{z^{\lambda_n}\}$ in $L_a^2(\Omega)$, where $\{\lambda_n\}$ is a sequence of complex numbers satisfying

$$\text{the } \lambda_n \text{ are all distinct and } \lim_{n \to \infty} |\lambda_n| = \infty, \tag{1.7}$$

$$\lim_{n \to \infty} \frac{n}{|\lambda_n|} = D, \quad 0 < D < \infty, \tag{1.8}$$

$$\operatorname{Re} \lambda_n > 0, \qquad |\operatorname{Im} \lambda_n| \leq A \tag{1.9}$$

for some constant A. Shen also supposed that Ω is a Dzhrbasian domain with the vertex of Δ_1 at the origin (hence 0 is outside of Ω). The following result was obtained in [4, 5].

Theorem A. *Assume that the sequence* $\{\lambda_n\}$ *satisfy* (1.7), (1.8), *and* (1.9), *and* Ω *is a Dzhrbasian domain which satisfies* $\Omega(I)$, $\Omega(II)$. *If*

$$2\beta_1(1 - D) < 1, \tag{1.10}$$

and, for some ε_0,

$$\int^{\infty} \frac{\alpha(r)}{r^{1+\eta}} \mathrm{d}r = \infty, \tag{1.11}$$

where

$$\eta = \max\left\{ \vartheta, \frac{1}{1/\beta_1 - 2(1-D)} + \varepsilon_0 \right\}, \tag{1.12}$$

then the system $\{z^{\lambda_n}\}$ *is complete in* $L_a^2(\Omega)$.

An improved version of Shen's result is given in [3]. Let Ω be a Dzhrbasian domain with the added requirement of

$$\Delta_1 = \left\{ z : \left|\arg(z) - \pi\right| < \frac{\pi}{2\varpi} \right\}, \tag{1.13}$$

where $\varpi > 1/2$ is some constant. In [3], for such a domain, results on the completeness on $\{z^{\lambda_n}\}$ in $L_a^2(\Omega)$ were obtained, assuming $\{z^{\lambda_n}\}$ is a sequence of complex numbers satisfying (1.7) and (1.8), but (1.9) is replaced by the more general condition

$$\left|\arg(\lambda_n)\right| < \beta < \frac{\pi}{2}, \tag{1.14}$$

thus allowing $\Im(\lambda_n) \to 0$ as $n \to \infty$.

More accurately, the main result in [3] is described as follows.

Theorem B. *Assume that the sequence* $\{\lambda_n\}$ *satisfy* (1.7), (1.8), *and* (1.14), *and* Ω *is a Dzhrbasian domain which satisfies* $\Omega(I)$, $\Omega(II)$, *and* (1.13). *Moreover, assume that*

$$2\varpi(1 - D\cos\beta) < 1. \tag{1.15}$$

Let

$$\eta = \max\left\{ \vartheta, \frac{1}{h} + \varepsilon_0 \right\}, \tag{1.16}$$

where ϑ *is defined in* (1.6), ε_0 *is some positive number, and*

$$h = \max_{0<\delta<D\cos\beta - 1 + 1/2\varpi} \frac{2\delta}{\sqrt{D^2\sin^2\beta + \delta^2}} \left(D\cos\beta - 1 + \frac{1}{2\varpi} - \delta \right). \tag{1.17}$$

If

$$\int^{\infty} \frac{\alpha(r)}{r^{1+\eta}} \mathrm{d}r = +\infty, \tag{1.18}$$

then the system $\{z^{\lambda_n}\}$ *is complete in* $L_a^2(\Omega)$.

Remark 1.2. The h in (1.17) is well defined, for reference we refer to [3, Remark 4].

In this paper, motivated by the work in [3–7], we will investigate the completeness of the system $M_\Lambda = \{z^{\lambda_n}\log^j z,\ j = 0, 1, \ldots, m_n - 1\}_{n=1}^{\infty}$ in $L_a^2(\Omega)$, where Ω is a Dzhrbasian domain with the added requirement of Δ_1 satisfying (1.13). The system M_Λ is associated with the multiplicity sequence $\Lambda = \{\lambda_n, m_n\}_{n=1}^{\infty}$, that is, a sequence where $\{\lambda_n\}$ are complex numbers with $\lambda_n \neq \lambda_m$ wherever $n \neq m$, and each λ_n having multiplicity equal to m_n. The sequence Λ satisfies (1.7), (1.14) and also

$$\lim \frac{n_\Lambda(t)}{t} = D, \quad 0 < D < \infty, \tag{1.19}$$

where $n_\Lambda(t) = \sum_{|\lambda_n| \le t} m_n$ is the so-called counting function of the sequence Λ. We note that when $m_n = 1$ for all λ_n, the above relation is equivalent to (1.8). To describe even further the sequence Λ, thus the system M_Λ as well, we need some definitions from [9]. We denote by $\mathbf{L}(\mathbf{c}, \mathbf{D})$ the class of all complex sequence $\mathbf{A} = \{a_n\}$, $|a_n| \le |a_{n+1}|$ satisfying the following properties: (1)$n/|a_n| \to D \ge 0$,(2) for $n \neq k$ one has that $|a_n - a_k| \ge c|n - k|$ for some constant c and (3) $\sup|\arg(a_n)| < \pi/2$. The following definition is from [9].

Definition 1.3. Let the sequence $\mathbf{A} \in \mathbf{L}(\mathbf{c}, \mathbf{D})$ and a, b be real positive numbers such that $a + b < 1$. We say that a sequence $\mathbf{B} = \{b_n\}_{n=1}^{\infty}$ belongs to the class $\mathbf{A}_{a,b}$ if for all $n \in \mathbb{N}$ we have

$$b_n \in \{z : |z - a_n| \le a_n^a\}, \tag{1.20}$$

and for all $k \neq n$ one of the following holds:

(i) $b_k = b_n$,

(ii) $|b_k - b_n| \ge \max\{e^{-|a_k|^b}, e^{-|a_n|^b}\}$.

We may write **B** in the form of a multiplicity sequence $\Lambda = \{\lambda_n, m_n\}_{n=1}^{\infty}$, by grouping together all those terms that have the same modulus and ordering them so that $|\lambda_n| < |\lambda_{n+1}|$. This form of **B** is called as $\{\lambda, m\}$ *reordering* (see [9]).

We prove the elementary fact.

Lemma 1.4. *Suppose Ω is a Dzhrbasian domain such that Conditions Ω (I), Condition Ω (II), and* (1.13) *are satisfied. Moreover, suppose* $\Lambda = \{\lambda_n, m_n\}_{n=1}^{\infty}$ *is a sequence of complex numbers which is a* $\{\lambda, m\}$ *reordering of* $\mathbf{B} = \{b_n\} \in \mathbf{A}_{a,b}$ *of a sequence* $\mathbf{A} = \{a_n\} \in \mathbf{L}(\mathbf{c}, \mathbf{D})$ *such that* $\arg(a_n) \to 0$ *as* $n \to \infty$, *satisfying* (1.14). *Then* $z^{\lambda_n}\log^{m_n} z \in L_a^2(\Omega)$.

Proof. Due to the definition of the domain Δ_1, the principal branch of $\log z$, that is, $\log z$, is well defined on Ω. Thus, $(\log^j z)(z^{\lambda_n}) = \log^j z \exp\{\lambda_n \log z\}$ is an analytic function in Ω. Let

$z = re^{i\theta}$ and $\lambda_n = |\lambda_n|e^{i\theta_n}$. Considering $|\theta_n| < \beta < \pi/2$ and $|\theta| < \pi - \pi/2\varpi$ whenever $z \in \Omega$, there exists some positive constant A such that for $z \in \Omega$.

$$\left|z^{\lambda_n}\log^{m_n} z\right| < (Ar)^{|\lambda_n|+m_n}. \tag{1.21}$$

Since $\varphi(r) \uparrow \infty$ as $r \to \infty$, for r sufficiently large which is denoted by r_1, we have $\varphi(r) > 2(|\lambda_n| + m_n) + 2$. Without loss of generality, we can suppose $r_1 > r_0$. By (1.4), we have

$$e^{-\alpha(r)}r^{2(|\lambda_n|+m_n)} < exp\{-(2(|\lambda_n| + m_n) + 2)\}r^{2(|\lambda_n|+m_n)} = \frac{r_1^{2(|\lambda_n|+m_n)}}{r_4^2}. \tag{1.22}$$

Thus, we have

$$\begin{aligned}\iint_\Omega \left|z^{\lambda_n}\log^{m_n} z\right|^2 \mathrm{d}x\,\mathrm{d}y &\le \int_0^{r_0} 2\pi r(Ar)^{2(|\lambda_n|+m_n)}\mathrm{d}r + \int_{r_0}^{\infty} \sigma(r)(Ar)^{2(|\lambda_n|+m_n)}\mathrm{d}r \\ &\le \frac{2\pi A^{2(|\lambda_n|+m_n)} r_0^{2(|\lambda_n|+m_n)+2}}{2(|\lambda_n| + m_n) + 2} + \int_{r_0}^{\infty} e^{-\alpha(r)}(Ar)^{2(|\lambda_n|+m_n)}\mathrm{d}r \\ &< \infty.\end{aligned} \tag{1.23}$$

□

The main result of this paper is as follows.

Theorem 1.5. *Suppose that* Ω *is a Dzhrbasian domain such that Conditions* Ω *(I),* Ω *(II), and* (1.13) *are satisfied. Moreover, suppose that* $\Lambda = \{\lambda_n, m_n\}_{n=1}^{\infty}$ *is a sequence of complex numbers which is a* $\{\lambda, m\}$ *reordering of* $\mathbf{B} = \{b_n\} \in \mathbf{A}_{a,b}$ *of a sequence* $\mathbf{A} = \{a_n\} \in \mathbf{L}(\mathbf{c}, \mathbf{D})$ *such that* $\arg(a_n) \to 0$ *as* $n \to \infty$*, satisfying* (1.14)*. If*

$$\int^{\infty} \frac{\alpha(r)}{r^{1+\eta}}\mathrm{d}r = +\infty, \tag{1.24}$$

where η *is defined in* (1.16)*,* ϑ *is defined in* (1.6)*, and* h *is defined in* (1.17)*, then the system* M_Λ *is complete in* $L_a^2(\Omega)$*.*

The paper is organized as follows. In Section 2, crucial lemmas in proving Theorem 1.5 will be presented. In Section 3, the completeness theorem above will be proved.

2. Preliminary Lemmas

We consider the function

$$G(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\lambda_n^2}\right)^{\mu_n}, \tag{2.1}$$

where μ_n denotes the multiplicity of the term $1 - z^2/\lambda_n^2$ and the integral

$$K(s) = -\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{e^{-iys}}{G(iy)} \mathrm{d}y, \quad s = u + iv. \tag{2.2}$$

For sufficiently small $\delta > 0$, denote

$$B_\delta = \{ s = u + iv : |v| \leq \pi D \cos\beta - \delta\pi \}. \tag{2.3}$$

Under the Conditions Ω (I), Ω (II), and Definition 1.3, by [10], we can get the following estimates which will play an important role in the proof of Theorem 1.5.

Lemma 2.1. *Given $\varepsilon > 0$,*

$$\frac{1}{|K(iy)|} \leq A(\varepsilon) e^{(-\pi D\cos\alpha + \varepsilon)|y|}, \tag{2.4}$$

where $A(\varepsilon)$ is a constant which depends only on ε.

Lemma 2.2. *There exists a sequence $\{t_k\}$ with $k \geq t_k \geq (1-\lambda)k$ (λ is some sufficiently small positive number) such that, for $s = u + iv \in B_\delta, \operatorname{Re} s = u \geq 0$,*

$$\left| K(s) - \sum_{|\lambda_n| < t_k} \sum_{m=0}^{m_n - 1} a_{n,m} s^m e^{-\lambda_n s} \right| \leq A^{t_k} e^{-u t_k \sin(\mu\pi)}, \tag{2.5}$$

and, for $s = u + iv \in B_\delta, \operatorname{Re} s = u \leq 0$,

$$\left| K(s) - \sum_{|\lambda_n| < t_k} \sum_{m=0}^{m_n - 1} a_{n,m} s^m e^{-\lambda_n s} \right| \leq A^{t_k} e^{u t_k}, \tag{2.6}$$

where A is a constant independent of s and t_k, while μ is a small positive number satisfying

$$\tan(\mu\pi) < \frac{\delta}{D\sin\beta}. \tag{2.7}$$

Let $z = e^{\xi}$, $\xi = \xi_1 + i\xi_2$ and denote the image of Ω in the ξ plane by Ω'. It follows from Condition Ω(II) and (1.13) that Ω' must be located inside the strip

$$B_\xi = \left\{ \xi = \xi_1 + i\xi_2 : |\xi_2| < \pi\left(1 - \frac{1}{2\varpi}\right) \right\}. \tag{2.8}$$

Denote

$$B_{\varpi} = \left\{ s = u + iv : |v| < \pi D \cos\beta - \pi\left(1 - \frac{1}{2\varpi}\right) \right\},$$
$$B_{\varpi}^{\delta} = \left\{ s = u + iv : |v| \le \pi D \cos\beta - \delta\pi - \pi\left(1 - \frac{1}{2\varpi}\right) \right\}. \tag{2.9}$$

Suppose that

$$2\varpi(1 - D\cos\beta) < 1 \tag{2.10}$$

from which $\pi D \cos\beta - \pi(1 - 1/2\varpi) > 0$ follows, choosing δ sufficiently small so that

$$0 < \delta < D\cos\beta - 1 + \frac{1}{2\varpi}. \tag{2.11}$$

It is obvious that if $s \in B_{\varpi}^{\delta}$ and $\xi \in \Omega'$, then $|\operatorname{Im}(s - \xi)| < \pi D \cos\beta - \delta\pi$, that is, $s - \xi \in B_{\delta}$ in (2.3). Thus for any $f(z) \in L_a^2(\Omega)$, we can define a function for $s \in B_{\varpi}^{\delta}$ by

$$F(s) = \int_{\Omega'} \overline{f(e^{\xi})} \left| e^{\xi} \right|^2 K(s - \xi) \mathrm{d}\xi_1 \, \mathrm{d}\xi_2, \quad \xi = \xi_1 + i\xi_2. \tag{2.12}$$

Remark 2.3. By Lemma 2.6 in [10], when $\xi \in \Omega'$ is fixed $K(s - \xi)$ is analytic for $s \in B_{\varpi}^{\delta}$; when $s \in B_{\varpi}^{\delta}$ is fixed, $K(s - \xi)$ is both measurable and bounded for $\xi \in \Omega'$. Thus, it is not hard to prove that $F(s)$ is analytic and bounded in B_{ϖ}^{δ} (see [11, Chapter 10, Exercise 16; 1, Section 3] and [3, page 8]).

The following lemma will be crucial in our proof of Theorem 1.5.

Lemma 2.4. *If for $s \in B_{\varpi}^{\delta}$, $F(s) \equiv 0$ where $F(s)$ is defined by* (2.12), *then*

$$\iint_{\Omega} \overline{f(z)} z^n \mathrm{d}x \, \mathrm{d}y = 0, \quad n = 0, 1, 2, \ldots. \tag{2.13}$$

Proof. See [3, Lemma 2.4]. □

We end this section by presenting two more lemmas. The first one is the so-called Carleman's Theorem (see [12, page 103]).

Lemma 2.5. *Let* $\log^{-} r = \max\{-\log r, 0\}$. *If $g(w)$ is analytic and bounded in the half-plane* $\operatorname{Im}(w) \ge 0$ *and*

$$\int_{-\infty}^{+\infty} \frac{\log^{-}|g(t)|}{1 + t^2} \mathrm{d}t = \infty, \tag{2.14}$$

then $g(w) \equiv 0$.

We also need a result of M. M. Dzhrbasian (see [13, Section 10, Lemma 1]).

Lemma 2.6. *Suppose* $\alpha(r)$ *be given as in* (1.4), *let*

$$M_n = \int_{r_0}^{\infty} e^{-\alpha(r)} r^n \mathrm{d}r,$$
$$\Phi(r) = \sup_{n\geq 1} \frac{r^n}{\sqrt{M_{2n}}}. \tag{2.15}$$

Then there exists some constant $A > 0$ *such that for* r *sufficiently large*

$$\log \Phi(r) \geq A\alpha(r). \tag{2.16}$$

3. Proof of Theorem 1.5

Proof. Let us fix some notations. Throughout this section, A will denote positive constants, and it may be different at each occurrence.

To prove Theorem 1.5, it suffices to show that if $f \in L_a^2(\Omega)$ and

$$\left\langle f(z), z^{\lambda_n} \log^j z \right\rangle = 0, \quad j = 0, 1, 2, \ldots, m_n - 1,\ n = 1, 2, \ldots, \tag{3.1}$$

then $f(z) \equiv 0$. We claim that, by letting $F(s)$ be the function as in (2.12), we only need to prove $F(s) \equiv 0$ for $s \in B_{\varpi}^{\delta}$. Indeed by Lemma 2.4, it follows that (2.13) is satisfied, that is $\langle f(z), z^n \rangle = 0$, $n = 0, 1, \ldots$. Since (1.24) holds, by Dzhrbasian's result the system $\{z^n\}$ is complete in $L_a^2(\Omega)$ which means $f(z) \equiv 0$. Our claim is now justified.

For $s \in B_\delta$, let $\{t_k\}$ be the sequence defined in Lemma 2.2, with $k \geq t_k \geq (1-\lambda)k$ where λ is a sufficiently small positive number. Then

$$\begin{aligned}
F(s) &= \iint_{\Omega'} \overline{f(e^{\xi})} \left| e^{\xi} \right|^2 K(s-\xi) \mathrm{d}\xi_1\, \mathrm{d}\xi_2, \quad \xi = \xi_1 + i\xi_2 \\
&= \iint_{\Omega'} \overline{f(e^{\xi})} \left| e^{\xi} \right|^2 \left[K(s-\xi) - \sum_{|\lambda_n|<t_k} \sum_{m=0}^{m_n-1} a_{n,m} (s-\xi)^m e^{-\lambda_n (s-\xi)} \right] \mathrm{d}\xi_1\, \mathrm{d}\xi_2 \\
&\quad + \iint_{\Omega'} \overline{f(e^{\xi})} \left| e^{\xi} \right|^2 \left(\sum_{|\lambda_n|<t_k} \sum_{m=0}^{m_n-1} a_{n,m} (s-\xi)^m e^{-\lambda_n (s-\xi)} \right) \mathrm{d}\xi_1\, \mathrm{d}\xi_2 \\
&=: F_{1,k}(s) + F_{2,k}(s).
\end{aligned} \tag{3.2}$$

Since

$$\left\langle f(z), z^{\lambda_n} \log^j z \right\rangle = 0, \quad j = 0, 1, 2, \ldots, m_n - 1,\ n = 1, 2, \ldots, \tag{3.3}$$

we have

$$F_{2,k}(s) = 0. \tag{3.4}$$

Hence, for $s = u + iv \in B_\delta$, $F(s) = F_{1,k}(s)$. By (2.5) and (2.6) in Lemma 2.2, we have

$$\begin{aligned} |F(s)| &= |F_{1,k}(s)| \\ &\le A^{t_k}\Bigg(e^{-ut_k \sin(\mu\pi)} \iint_{\Omega' \bigcap \{\operatorname{Re}(s-\xi)\ge 0\}} \left|\overline{f(e^\xi)}\right| \left|e^\xi\right|^2 \left|e^\xi\right|^{t_k \sin(\mu\pi)} \mathrm{d}\xi_1\, \mathrm{d}\xi_2 \\ &\qquad + e^{-ut_k} \iint_{\Omega' \bigcap \{\operatorname{Re}(s-\xi)\le 0\}} \left|\overline{f(e^\xi)}\right| \left|e^\xi\right|^2 \left|e^\xi\right|^{t_k} \mathrm{d}\xi_1\, \mathrm{d}\xi_2 \Bigg), \end{aligned} \tag{3.5}$$

where A is a constant independent of k and s. Hence, for $\operatorname{Re} s = u \ge 0$,

$$\begin{aligned} |F(s)| &\le A^{t_k}\left(\frac{\iint_\Omega |f(z)||z|^{t_k} \mathrm{d}x\, \mathrm{d}y}{|e^s|^{t_k \sin(\mu\pi)}} + \frac{\iint_\Omega |f(z)||z|^{t_k} \mathrm{d}x\, \mathrm{d}y}{|e^s|^{t_k}} \right) \\ &\le A^{t_k} \frac{\iint_\Omega |f(z)||z|^{t_k} \mathrm{d}x\, \mathrm{d}y}{|e^s|^{t_k \sin(\mu\pi)}}. \end{aligned} \tag{3.6}$$

By Schwarz'z inequality

$$|F(s)| \le A^{t_k} \frac{\left(\iint_\Omega |f(z)|^2 \mathrm{d}x\, \mathrm{d}y\right)^{1/2} \left(\iint_\Omega |z|^{2t_k} \mathrm{d}x\, \mathrm{d}y\right)^{1/2}}{|e^s|^{t_k \sin(\mu\pi)}}, \tag{3.7}$$

and, by Condition Ω(I), we have the estimate

$$\iint_\Omega |z|^{2t_k} \mathrm{d}x\, \mathrm{d}y \le A^{t_k} \int_{r_0}^\infty e^{-\alpha(r)} r^{2t_k} \mathrm{d}r, \tag{3.8}$$

where A is some positive constant independent of k and s. Thus, by $k \ge t_k \ge (1-\lambda)k$, we have

$$\begin{aligned} |F(s)| &\le A^{t_k} \frac{\left(\int_{r_0}^\infty e^{-\alpha(r)} r^{2t_k} \mathrm{d}r\right)^{1/2}}{|e^s|^{t_k \sin(\mu\pi)}} \\ &\le A^k \frac{\left(\int_{r_0}^\infty e^{-\alpha(r)} r^{2k} \mathrm{d}r\right)^{1/2}}{|e^s|^{(1-\lambda)k \sin(\mu\pi)}}, \end{aligned} \tag{3.9}$$

for every $k = 1, 2, \ldots$. Hence,

$$|F(s)| \leq \inf_{k \geq 1} \left\{ A^k \frac{\left(\int_{r_0}^{\infty} e^{-\alpha(r)} r^{2k} \mathrm{d}r \right)^{1/2}}{|e^s|^{(1-\lambda)k \sin(\mu\pi)}} \right\}$$
$$= \inf_{k \geq 1} \left\{ \frac{\left(\int_{r_0}^{\infty} e^{-\alpha(r)} r^{2k} \mathrm{d}r \right)^{1/2}}{\left((1/A)|e^s|^{(1-\lambda)\sin(\mu\pi)} \right)^k} \right\}. \tag{3.10}$$

Let

$$t = \frac{1}{A} |e^s|^{(1-\lambda)\sin(\mu\pi)},$$
$$M_n = \int_{r_0}^{\infty} e^{-\alpha(r)} r^n \mathrm{d}r. \tag{3.11}$$

Then

$$|F(s)| \leq \inf_{n \geq 1} \left\{ \frac{\sqrt{M_{2n}}}{t^n} \right\}. \tag{3.12}$$

If we let

$$\Phi(t) = \sup_{n \geq 1} \frac{t^n}{\sqrt{M_{2n}}}, \tag{3.13}$$

then it follows from Lemma 2.6 that there is some constant $q > 0$ so that

$$\Phi(t) > e^{q\alpha(t)}. \tag{3.14}$$

Combining (3.12) and (3.14) shows that for $\Re s \geq 0$

$$|F(s)| \leq e^{-A\alpha(\overline{r})}, \quad \overline{r} = \frac{1}{A} |e^s|^{(1-\lambda)\sin(\mu\pi)}. \tag{3.15}$$

In order to use Lemma 2.5, we transform the domain B_δ into the upper half-plane $\operatorname{Im} z \geq 0$.

(i) First, let $z_1 = e^s$, B_δ is then transformed into an angle $|\arg z_1| \leq m\pi$, where

$$m = D\cos\beta - \delta - 1 + \frac{1}{2\varpi}. \tag{3.16}$$

(ii) Let $z_2 = z_1^{1/2m}$. The above angle domain is transformed into the right half-plane $\operatorname{Re} z_2 \geq 0$.

(iii) Finally, let $z = iz_2$; the right half-plane is then transformed into the upper half-plane $\operatorname{Im} z \geq 0$.

More accurately, we have

$$|e^s| = |z_1| = \left|z_2^{2m}\right| = \left|(-iz)^{2m}\right| = \left|z^{2m}\right|,$$
$$F(s) = F(\log z_1) = F\left(\log z_2^{2m}\right) = F\left(\log(-iz)^{2m}\right). \tag{3.17}$$

Define $g(z) = F(\log(-iz)^{2m})$; it is obvious that $g(z)$ is analytic and bounded in the upper half-plane $\operatorname{Im} z \geq 0$. By (3.15), for $\operatorname{Im} z \geq 0$ and $|z|$ sufficiently large, we have

$$\left|g(z)\right| \leq e^{-A\alpha(A|z|^{2m(1-\lambda)\sin(\mu\pi)})} = e^{-A\alpha(A|z|^{m'})}, \tag{3.18}$$

where A is some positive constant independent of z, m is given by (3.16), and

$$m' = 2m(1-\lambda)\sin(\mu\pi) = 2\left(D\cos\beta - \delta - 1 + \frac{1}{2\varpi}\right)(1-\lambda)\sin(\mu\pi). \tag{3.19}$$

Let $\tan(\mu\pi) \to \delta/(D\sin\beta)$ in (2.7), then

$$\sin(\mu\pi) \longrightarrow \frac{\delta}{\sqrt{D^2\sin^2\beta + \delta^2}}. \tag{3.20}$$

Denote

$$m'' = \frac{2\delta}{\sqrt{D^2\sin^2\beta + \delta^2}}\left(D\cos\beta - \delta - 1 + \frac{1}{2\varpi}\right)(1-\lambda). \tag{3.21}$$

By (3.18), for $\operatorname{Im} z \geq 0$ and $|z|$ sufficiently large, we have

$$\left|g(z)\right| \leq e^{-A_2\alpha(A_3|z|^{m''})}. \tag{3.22}$$

It is obvious that δ can be chosen such that $0 < \delta < D\cos\beta - 1 + 1/2\varpi$.

Denote

$$h' = \max_{0<\delta<D\cos\beta - 1 + 1/2\varpi} m''. \tag{3.23}$$

By (3.22), for $\operatorname{Im} z \geq 0$ and $|z|$ sufficiently large, we have

$$\left|g(z)\right| \leq e^{-A_2\alpha(A_3|z|^{h'})}. \tag{3.24}$$

Since $h' = h(1-\lambda)$, choosing λ sufficiently small yields

$$\frac{1}{h'} < \frac{1}{h} + \varepsilon_0, \tag{3.25}$$

where ε_0 is defined in (1.16). Thus, by (3.24),

$$\int^{\infty} \frac{\log|g(t)|}{t^2} \mathrm{d}t \le -A \int^{\infty} \frac{\alpha(w)}{w^{1+1/h'}} \mathrm{d}w, \tag{3.26}$$

where A is some positive constant independent of $w = ct^{h'}$. Thus, by (1.24), we have

$$\int^{\infty} \frac{\log|g(t)|}{t^2} \mathrm{d}t = -\infty. \tag{3.27}$$

Hence

$$\int^{\infty} \frac{\log|g(t)|}{1+t^2} \mathrm{d}t = -\infty. \tag{3.28}$$

Let $\int_{-\infty}$ mean that the upper limit of the integral is a negative number with sufficiently large magnitude. Similarly, we have

$$\int_{-\infty} \frac{\log|g(t)|}{t^2} \mathrm{d}t \le \int_{-\infty} \frac{-A_2\alpha\left(A_3|t|^{h'}\right)}{t^2} \mathrm{d}t = \int^{\infty} \frac{-A_2\alpha\left(A_3 t^{h'}\right)}{t^2} \mathrm{d}t = -\infty. \tag{3.29}$$

Hence

$$\int_{-\infty} \frac{\log|g(t)|}{1+t^2} \mathrm{d}t = -\infty. \tag{3.30}$$

By Remark 2.3, we know that

$$\int_a^b \frac{\log|g(t)|}{1+t^2} \mathrm{d}t < +\infty \tag{3.31}$$

for every finite closed interval $[a, b]$, thus

$$\int_{-\infty}^{\infty} \frac{\log|g(t)|}{1+t^2} \mathrm{d}t = -\infty, \tag{3.32}$$

and, by Lemma 2.5, $g(z) \equiv 0$. □

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