Research Article

Spatiality of Derivations of Operator Algebras in Banach Spaces

Quanyuan Chen[1, 2] and Xiaochun Fang[1]

[1] *Department of Mathematics, Tongji University, Shanghai 200092, China*
[2] *Department of Information Engineering, Jingdezhen Ceramic Institute, Jingdezhen, Jiangxi 333403, China*

Correspondence should be addressed to Quanyuan Chen, cqy0798@163.com

Suppose that $\mathcal{A}$ is a transitive subalgebra of $B(X)$ and its norm closure $\overline{\mathcal{A}}$ contains a nonzero minimal left ideal $\mathcal{J}$. It is shown that if δ is a bounded reflexive transitive derivation from $\mathcal{A}$ into $B(X)$, then δ is spatial and implemented uniquely; that is, there exists $T \in B(X)$ such that $\delta(A) = TA - AT$ for each $A \in \mathcal{A}$, and the implementation T of δ is unique only up to an additive constant. This extends a result of E. Kissin that "if $\overline{\mathcal{A}}$ contains the ideal $C(H)$ of all compact operators in $B(H)$, then a bounded reflexive transitive derivation from $\mathcal{A}$ into $B(H)$ is spatial and implemented uniquely." in an algebraic direction and provides an alternative proof of it. It is also shown that a bounded reflexive transitive derivation from $\mathcal{A}$ into $B(X)$ is spatial and implemented uniquely, if X is a reflexive Banach space and $\overline{\mathcal{A}}$ contains a nonzero minimal right ideal $\mathcal{J}$.

1. Introduction

Throughout this paper, X is a Banach space (and X will be replaced by H if it is a Hilbert space) and $\mathcal{A}$ is a subalgebra of $B(X)$, the Banach algebra of all bounded operators on X. Suppose that $\mathcal{A} \subseteq B(X)$ and $\mathcal{M} \subseteq B(X)$ is an $\mathcal{A}$-bimodule. A linear map δ from $\mathcal{A}$ into $\mathcal{M}$ is called a *derivation* if

$$\delta(AB) = \delta(A)B + A\delta(B) \quad \text{for any } A, B \in \mathcal{A}. \tag{1.1}$$

Then $\mathcal{A}$ is called the *domain* of δ and denoted by $\mathrm{Dom}(\delta)$. The derivation δ is called *inner* (resp., *spatial*) if there exists an operator $T \in \mathcal{M}$ (resp., $T \in B(X)$) such that

$$\delta(A) = TA - AT \quad \text{for any } A \in \mathcal{A}. \tag{1.2}$$

If the operator T is not bounded, then δ is said to be *quasispatial*. More precisely, if there exists a densely defined, closed operator $T : \mathrm{Dom}(T) \to X$ such that

$$A(\mathrm{Dom}(T)) \subseteq \mathrm{Dom}(T), \quad \delta(A)x = (TA - AT)x \quad \text{for any } A \in \mathcal{A},\ x \in \mathrm{Dom}(T), \qquad (1.3)$$

then the derivation δ is called *quasispatial*, and the operator T is an *implementation* of δ. Compared to the spatiality, the quasispatiality is a slightly weaker notion.

Given a bounded derivation δ on an operator algebra, the natural question is whether the derivation δ is inner (or spatial). The spatiality of derivations is a classical problem when formulated for self-adjoint algebras and non-self-adjoint reflexive operator algebras. And it has been extensively studied in the literature in a large variety of situations, and some interesting results have been obtained [1–13]. For example, every derivation of a C^*-algebra is spatial [12], every derivation of a von Neumann algebra is inner [13], and so is the derivation of a nest algebra [14]. Every derivation from an atom Boolean subspace lattice algebra into its ideal is quasispatial [7]. A necessary and sufficient condition is given for a derivation on CDC algebras to be quasispatial [6]. In [10], the quasispatiality of derivations on CSL algebras is studied.

As to general operator algebras, it is well known that every derivation of $B(X)$ is inner [9] and that every derivation of a standard operator subalgebra on a normed space X is spatial [4]. Since these operator algebras are transitive, the above question for a reflexive transitive derivation in a Banach space X is raised naturally as follows.

Problem 1. Suppose that $\mathcal{A}$ is a transitive subalgebra of $B(X)$. Let δ be a bounded reflexive transitive derivation from $\mathcal{A}$ into $B(X)$. Does there always exist $T \in B(X)$ such that $\delta(A) = TA - AT$ for each $A \in \mathcal{A}$? Is the implementation T unique only up to an additive constant if there is any?

In the case when X is a Hilbert space, it is Problem 2.12 of [5]. Although the problem is still open, some strong conditions have been found by Kissin to imply that such derivations are spatial and implemented uniquely (Proposition 2.11, [5]). In particular, Kissin has proved that the answer to Problem 1 is affirmative under the conditions that $\mathcal{A}$ is a transitive subalgebra of $B(H)$, and $\overline{\mathcal{A}}$ contains the ideal $C(H)$ of all compact operators in $B(H)$. As far as we know, there are no other solution to Problem 1.

The purpose of this paper is to investigate the quasispatiality of derivations and to address the above question in a Banach space X. The paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we investigate the quasispatiality and spatiality of derivations. The main result (Theorem 3.1) shows that if $\mathcal{A}$ is a transitive operator algebra on a Banach space X and $\overline{\mathcal{A}}$ contains a nonzero minimal left ideal $\mathcal{O}$, then a bounded reflexive transitive derivation δ from $\mathcal{A}$ into $B(X)$ is spatial and implemented uniquely. As an application, the quasispatiality of the adjoint of a derivation is discussed in Section 4. The main result (Theorem 4.2) in Section 4 shows that under the conditions that X is a reflexive Banach space and $\overline{\mathcal{A}}$ contains a nonzero minimal right ideal $\mathcal{O}$, δ is also spatial and implemented uniquely if δ is a bounded reflexive transitive derivation from $\mathcal{A}$ into $B(X)$. As another application, Proposition 2.11 (ii) of [5] can be proved by using Theorem 3.1 of this paper, which is Corollary 4.3. Since Theorem 2.5 and Proposition 2.11 of [5] hold in a Hilbert space and they are not valid in a Banach space without the approximation property, Theorems 3.1 and 4.2 of this paper extend the result of Kissin to Banach spaces in an algebraic direction.

The role of "compact operators" is replaced by that of "minimal one-sided ideals". The proof of our results relies on the quasispatiality of the derivation and Banach algebra techniques.

This paper is a continuation of [5]. Some definitions and notations can be found in [5].

2. Preliminaries

Throughout this paper, X is a complex Banach space, and X^* is the topological dual space of X, the Banach space of all continuous linear functionals on X. We denote by $\mathcal{F}(X)$ the algebra of all finite-rank operators on X. If a subalgebra $\mathcal{A}$ contains $\mathcal{F}(X)$, then $\mathcal{A}$ is called a *standard operator algebra*. For a bounded operator A on X, denote by Lat A the lattice of all closed invariant subspaces of A and A^* the adjoint operator of A. For a subalgebra $\mathcal{A}$ of $B(X)$, denote by Lat($\mathcal{A}$) the lattice of all closed subspaces invariant under every operator in $\mathcal{A}$. For a set $\mathcal{L}$ of subspaces of X, denote by Alg $\mathcal{L}$ the algebra of all operators in $B(X)$ which leave all subspaces in $\mathcal{L}$ invariant. An operator algebra $\mathcal{A}$ is *transitive* if Lat $\mathcal{A} = \{\{0\}, X\}$; $\mathcal{A}$ is *reflexive* if

$$\mathcal{A} = \text{Alg Lat } \mathcal{A}, \quad \text{where Alg Lat } \mathcal{A} = \{T \in B(X) : \text{Lat } \mathcal{A} \subset \text{Lat } T\}. \tag{2.1}$$

For $0 \neq x \in X$ and $0 \neq f \in X^*$, the rank-one operator $x \otimes f$ acts on X by $(x \otimes f)y = f(y)x$ for $y \in X$. Let A be an operator on X with $\text{Dom}(A) \subseteq X$. If $x \in \text{Dom}(A)$ and $f \in \text{Dom}(A^*)$, then $A(x \otimes f) = Ax \otimes f$ and $(x \otimes f)A = x \otimes (A^*f)$. Let M be a nonempty subset of X and N a nonempty subset of X^*. The annihilator $M^\perp$ of M and the preannihilator $^\perp N$ of N are defined as follows [15]: $M^\perp = \{f \in X^* : f(x) = 0 \text{ for all } x \in M\}$, $^\perp N = \{x \in X : f(x) = 0 \text{ for all } f \in N\}$. It is obvious that $M^\perp$ is a weak*-closed subspace of X^* and $^\perp N$ is a norm-closed subspace of X. For a subalgebra $\mathcal{A}$ and a closed, densely defined operator T with domain $\text{Dom}(T)$, we say that T *commutes with* $\mathcal{A}$, if $A(\text{Dom}(T)) \subseteq \text{Dom}(T)$ and $TA\xi = AT\xi$ for any $A \in \mathcal{A}$ and any $\xi \in \text{Dom}(T)$. For a subalgebra $\mathcal{A}$ of $B(X)$, let $\mathcal{A}^* = \{A^* : A \in \mathcal{A}\}$ in notation.

A subset $\mathcal{I}$ of an algebra $\mathcal{A}$ is a *left ideal* of $\mathcal{A}$ if $\mathcal{A}\mathcal{I} \subseteq \mathcal{I}$, a *right ideal* if $\mathcal{I}\mathcal{A} \subseteq \mathcal{I}$, and a *two-sided ideal* if it is both a left and a right ideal. A left ideal $\mathcal{I}$ of $\mathcal{A}$ is *minimal* if every left ideal of $\mathcal{A}$ included in $\mathcal{I}$ is either $\mathcal{I}$ or $\{0\}$, similarly for *minimal* right ideals.

A derivation δ is *bounded* (resp., *closed*) if the map $\text{Dom}(\delta) \ni A \mapsto \delta(A) \in B(X)$ is bounded (resp., closed) in the operator norm topology. The derivation δ is *transitive* if its domain $\text{Dom}(\delta)$ is a transitive operator algebra; δ is *reflexive* if

$$\mathcal{A}_\delta = \left\{ \widehat{A} = \begin{pmatrix} A & \delta(A) \\ 0 & A \end{pmatrix} : A \in \text{Dom}(\delta) \right\} \tag{2.2}$$

is a reflexive operator algebra on $X \oplus X$. Denote by $\text{Imp}(\delta)$ the set of all closed, densely defined operators implementing the derivation δ as in (1.3). For a densely defined, closed operator T with domain $\text{Dom}(T)$, we can define the derivation Δ_T with domain

$$\text{Dom}(\Delta_T) = \{A \in B(X) : A(\text{Dom}(T)) \subseteq \text{Dom}(T), \ TA - AT \text{ is bounded on } \text{Dom}(T)\} \tag{2.3}$$

by

$$\Delta_T(A) = \text{Closure}(TA - AT), \quad \text{for any } A \in \text{Dom}(\Delta_T). \tag{2.4}$$

If a densely defined, closed operator T implements a derivation δ, then the derivation Δ_T is an extension of the derivation δ. In fact, $A \in \text{Dom}(\delta)$ implies $A \in \text{Dom}(\Delta_T)$ and $\Delta_T(A) = \delta(A)$ for any $A \in \text{Dom}(\delta)$ by (1.3) and (2.4). For any set Λ of derivations δ_λ ($\lambda \in \Lambda$), define the derivation $\delta_\Lambda = \bigcap_{\lambda \in \Lambda} \delta_\lambda$ by

$$\text{Dom}(\delta_\Lambda) = \left\{ A \in \bigcap_{\lambda \in \Lambda} \text{Dom}(\delta_\lambda) : \delta_\lambda(A) \text{ coincide for any } \lambda \in \Lambda \right\},$$
$$\delta_\Lambda(A) = \delta_\lambda(A) \quad \text{for any } A \in \text{Dom}(\delta_\Lambda) \text{ and any } \lambda \in \Lambda. \tag{2.5}$$

In particular,

$$\delta_{\text{Imp}(\delta)} = \bigcap_{T \in \text{Imp}(\delta)} \Delta_T. \tag{2.6}$$

Remark 2.1. Δ_T is a reflexive transitive derivation for any densely defined, closed operator T.

Indeed, $\text{Dom}(\Delta_T)$ is a subalgebra of $B(X)$, and Δ_T is a derivation.

If $x \in \text{Dom}(T)$ and $h \in \text{Dom}(T^*)$, $(x \otimes h)z = h(z)x \in \text{Dom}(T)$ for any $z \in \text{Dom}(T)$. So $(x \otimes h)(\text{Dom}(T)) \subseteq \text{Dom}(T)$. Since $(T(x \otimes h) - (x \otimes h)T)(z) = (Tx \otimes h)z - (x \otimes T^*h)z$, $\|T(x \otimes h) - (x \otimes h)T\| \leq \|Tx\|\|h\| + \|x\|\|T^*h\|$ so that $T(x \otimes h) - (x \otimes h)T$ is bounded on $\text{Dom}(T)$. It follows that $x \otimes h \in \text{Dom}(\Delta_T)$ for any $x \in \text{Dom}(T)$ and $h \in \text{Dom}(T^*)$.

Let $M \in \text{Lat}(\text{Dom}(\Delta_T))$ with $0 \neq x_0 \in M$. Then $(x \otimes h)(x_0) = h(x_0)x \in M$ for any $x \in \text{Dom}(T)$ and $h \in \text{Dom}(T^*)$. Since $\text{Dom}(T^*)$ is dense, there exists $h_0 \in \text{Dom}(T^*)$ such that $h_0(x_0) \neq 0$ by Hahn-Banach Theorem. It follows that $x \in M$ for any $x \in \text{Dom}(T)$ so that $\text{Dom}(T) \subseteq M$. Thus $X = \overline{\text{Dom}(T)} \subseteq \overline{M} = M$. Therefore $\text{Dom}(\Delta_T)$ is transitive.

For the derivation Δ_T, $\mathcal{A}_{\Delta_T} = \{ \begin{pmatrix} A & \Delta_T(A) \\ 0 & A \end{pmatrix} : A \in \text{Dom}(\Delta_T) \}$. It is easy to see that $X \oplus \{0\} \in \text{Lat}(\mathcal{A}_{\Delta_T})$ and $G(T + tI) = \{ \begin{pmatrix} T(x)+tx \\ x \end{pmatrix} : x \in \text{Dom}(T) \} \in \text{Lat}(\mathcal{A}_{\Delta_T})$ for any $t \in \mathbb{C}$. Let $B_2 = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \in B(X \oplus X)$ and $B_2 \in \text{Alg Lat}(\mathcal{A}_{\Delta_T})$. Then $\text{Lat}(\mathcal{A}_{\Delta_T}) \subseteq \text{Lat}(B_2)$. Since $X \oplus \{0\} \in \text{Lat}(\mathcal{A}_{\Delta_T}) \subseteq \text{Lat}(B_2)$, $\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} B_{11}x \\ B_{21}x \end{pmatrix} \in X \oplus \{0\}$ for any $x \in X$. It follows that $B_{21}X = \{0\}$. Thus $B_{21} = 0$ and $B_2 = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}$. Since $G(T + tI) \in \text{Lat}(\mathcal{A}_{\Delta_T}) \subseteq \text{Lat}(B_2)$, $\begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}\begin{pmatrix} T(x)+tx \\ x \end{pmatrix} = \begin{pmatrix} B_{11}Tx+tB_{11}x+B_{12}x \\ B_{22}x \end{pmatrix} \in G(T + tI)$ for any $x \in \text{Dom}(T)$ and any $t \in \mathbb{C}$. Then $B_{22}x \in \text{Dom}(T)$ and $B_{11}Tx + tB_{11}x + B_{12}x = (T + tI)B_{22}x$ for any $x \in \text{Dom}(T)$ and any $t \in \mathbb{C}$. It follows that $B_{11}x = B_{22}x \in \text{Dom}(T)$ and $B_{11}Tx + B_{12}x = TB_{22}x$ for all $x \in \text{Dom}(T)$. Since $\text{Dom}(T)$ is dense in X and $B_{11}, B_{22} \in B(X)$, $B_{11} = B_{22}$. So $B_{11}(\text{Dom}(T)) \subseteq \text{Dom}(T)$ and $B_{12} = TB_{11} - B_{11}T$ on $\text{Dom}(T)$. Therefore, $B_{12} = TB_{11} - B_{11}T$ is bounded on $\text{Dom}(T)$. It follows that $B_{11} = B_{22} \in \text{Dom}(\Delta_T)$ and $B_{12} = \Delta_T(B_{11})$. Thus $B_2 = \begin{pmatrix} B_{11} & \Delta_T(B_{11}) \\ 0 & B_{11} \end{pmatrix} \in \mathcal{A}_{\Delta_T}$. It follows that $\mathcal{A}_{\Delta_T}$ is a reflexive algebra. Therefore Δ_T is a reflexive transitive derivation.

3. Main Results

In this section, we discuss the quasispatiality of bounded transitive derivations on operator algebras in a Banach space. The main result is as follows.

Theorem 3.1. *Suppose that $\mathcal{A}$ is a transitive subalgebra of $B(X)$ and $\overline{\mathcal{A}}$, the norm closure of $\mathcal{A}$, contains a nonzero minimal left ideal $\mathfrak{I}$. If δ is a bounded reflexive transitive derivation from $\mathcal{A}$ into $B(X)$, then δ is spatial and implemented uniquely, or, more precisely, there exists $T \in B(X)$ such that $\delta(A) = TA - AT$ for each $A \in \mathcal{A}$ and the implementation T is unique only up to an additive constant.*

The proof of Theorem 3.1 will proceed through several lemmas, in each of which we maintain the same notation. The results similar to the following two lemmas (Lemmas 3.2 and 3.3) can be found in [5]. For the sake of completeness, we outline the proof.

Lemma 3.2. *Let δ be a reflexive transitive derivation from $\mathrm{Dom}(\delta)(\mathrm{Dom}(\delta) = \mathcal{A} \subseteq B(X))$ into $B(X)$. Then δ is quasispatial.*

Proof. If $\mathrm{Imp}(\delta)$ is nonempty, then the lemma is trivially true. Therefore, for the rest of the argument, we assume that $\mathrm{Imp}(\delta)$ is empty.

First, we have that if δ is a transitive derivation from $\mathrm{Dom}(\delta)$ into $B(X)$, then

$$\mathrm{Lat}(\mathcal{A}_\delta) = \{\{0\} \oplus \{0\}, X \oplus \{0\}, X \oplus X, G(T) : T \in \mathrm{Imp}(\delta)\}, \tag{3.1}$$

where $G(T) = \{\left(\begin{smallmatrix} T(x) \\ x \end{smallmatrix}\right) : x \in \mathrm{Dom}(T)\}$ is the graph of T.

Indeed, it is easy to see that $\{0\} \oplus \{0\}$, $X \oplus \{0\}$, $X \oplus X$, and all $G(T)$ are invariant subspaces of $\mathcal{A}_\delta$. For the converse, suppose that $M \in \mathrm{Lat}\,\mathcal{A}_\delta$ such that $M \neq \{0\} \oplus \{0\}$, $M \neq X \oplus \{0\}$, and $M \neq X \oplus X$. If there is a vector of form $\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) \in M$ with $x \neq 0$, then $\left(\begin{smallmatrix} Ax \\ 0 \end{smallmatrix}\right) \in M$ for any $A \in \mathcal{A}$. It follows that $X \oplus \{0\} \subseteq M$ by the transitivity of $\mathrm{Dom}(\delta)$. If M contains no vector of form $\left(\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right)$ with $x \neq 0$, then $M = X \oplus \{0\}$; otherwise, there is a vector of form $\left(\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right) \in M$ with $x \neq 0$, then we have that $M = X \oplus X$. Therefore $\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) \in M$ implies that $x = 0$. It follows that there is a closed operator T such that

$$M = G(T) = \left\{ \begin{pmatrix} T(x) \\ x \end{pmatrix} : x \in \mathrm{Dom}(T) \right\}, \quad T \in \mathrm{Imp}(\delta). \tag{3.2}$$

As $\mathrm{Imp}(\delta) = \emptyset$, by (3.1),

$$\mathrm{Alg}\,\mathrm{Lat}\,\mathcal{A}_\delta = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} : A, B, C \in B(X) \right\}$$
$$\neq \mathcal{A}_\delta = \left\{ \begin{pmatrix} A & \delta(A) \\ 0 & A \end{pmatrix} : A \in \mathrm{Dom}(\delta) \right\}. \tag{3.3}$$

It is a contradiction to the assumption that δ is reflexive, which shows that

$$\mathrm{Imp}(\delta) \neq \emptyset. \tag{3.4}$$

Therefore δ is quasispatial. $\qquad\square$

Lemma 3.3. *Let δ be a reflexive transitive derivation from* $\mathrm{Dom}(\delta)$ *into* $B(X)$. *Then* $\delta = \delta_{\mathrm{Imp}(\delta)}$.

Proof. By (3.4), $\mathrm{Imp}(\delta) \neq \emptyset$. And by (2.4) of [5],

$$\mathrm{Imp}(\delta) = \mathrm{Imp}\big(\delta_{\mathrm{Imp}(\delta)}\big). \tag{3.5}$$

As both δ and $\delta_{\mathrm{Imp}(\delta)}$ are transitive derivations, $\mathrm{Lat}\,\mathcal{A}_{\delta} = \mathrm{Lat}\,\mathcal{A}_{\delta_{\mathrm{Imp}(\delta)}}$ by (3.1). Since both of these derivations are reflexive,

$$\mathcal{A}_{\delta} = \mathrm{Alg}\,\mathrm{Lat}\,\mathcal{A}_{\delta} = \mathrm{Alg}\,\mathrm{Lat}\,\mathcal{A}_{\delta_{\mathrm{Imp}(\delta)}} = \mathcal{A}_{\delta_{\mathrm{Imp}(\delta)}}. \tag{3.6}$$

Hence

$$\delta = \delta_{\mathrm{Imp}(\delta)}. \tag{3.7}$$

$\square$

Lemma 3.4. *Let* $\mathcal{A}$ *be a transitive subalgebra of* $B(X)$. *If* $\overline{\mathcal{A}}$ *contains a nonzero minimal left ideal* $\mathcal{I}$, *then* $\overline{\mathcal{A}} \cap \mathcal{F}(X)$ *is also a transitive subalgebra of* $B(X)$.

Proof. We complete the proof step by step.

Step 1. We have that

$$\mathcal{I}^2 = \{B_1 B_2 : B_1, B_2 \in \mathcal{I}\} \neq \{0\}. \tag{3.8}$$

Indeed, suppose that $\mathcal{I}^2 = \{0\}$. Let $0 \neq A_0 \in \mathcal{I}$. Then $\overline{\mathcal{A}} A_0 \subseteq \mathcal{I}$. It follows that $(\overline{\mathcal{A}} A_0)^2 \subseteq \mathcal{I}^2$ and $(\overline{\mathcal{A}} A_0)^2 = \{0\}$. Set $X_0 = \{x \in X : A(x) = 0 \text{ for all } A \in \overline{\mathcal{A}}\}$. It is obvious that X_0 is an $\overline{\mathcal{A}}$-invariant closed subspace of X. Hence $X_0 = \{0\}$ since $\mathcal{A}$ is a transitive operator algebra. Since $(\overline{\mathcal{A}} A_0)^2 = \{0\}$, $A_0 \overline{\mathcal{A}} A_0(X) \subseteq X_0$. It follows that $A_0 \overline{\mathcal{A}} A_0(X) = \{0\}$ so that $A_0 \overline{\mathcal{A}} A_0 = \{0\}$. We can choose $x_0 \in X$ ($x_0 \neq 0$) such that $A_0(x_0) \neq 0$. It is obvious that $\overline{\mathcal{A}} A_0(x_0)$ is an $\mathcal{A}$-invariant linear manifold of X and $\overline{\mathcal{A}} A_0(x_0) \neq \{0\}$. It follows that $\overline{\mathcal{A}} A_0(x_0)$ is dense in X. However, $\overline{\mathcal{A}} A_0(x_0)$ is contained in the null space of A_0, $A_0 = 0$, which is a contradiction.

Step 2. Since $\mathcal{I}^2 \neq \{0\}$, it follows from Lemma 2.1.5 and Corollary 2.1.6 of [16] that there exists an idempotent P in $\overline{\mathcal{A}}$ such that $\mathcal{I} = \overline{\mathcal{A}} P$ and $P \overline{\mathcal{A}} P$ is a division algebra consisting of scalar multiples of P (with identity P), that is,

$$P \overline{\mathcal{A}} P = \{\mu P : \mu \text{ is a complex scalar}\}. \tag{3.9}$$

Then P is a rank-one operator; that is, there exist $x_1 \in X$ and $f \in X^*$ such that

$$P = x_1 \otimes f \in \overline{\mathcal{A}}, \quad f(x_1) = 1. \tag{3.10}$$

Indeed, let $x_1 : 0 \neq x_1 \in X$ such that $Px_1 = x_1$ and let $M_1 = \overline{\mathcal{A}}(x_1) = \{A(x_1) : A \in \overline{\mathcal{A}}\}$. It is obvious that $M_1 \neq \{0\}$ and M_1 is an $\overline{\mathcal{A}}$-invariant linear manifold of X. Hence M_1 is dense in X. Since $PA(x_1) = PAP(x_1) = \mu P(x_1) = \mu x_1$ for any $A \in \overline{\mathcal{A}}$, the restriction $P|_{M_1}$ of P on M_1 has one-dimensional range. As P is bounded, it also has one-dimensional range. Hence $P = x_1 \otimes f$, where f is a continuous functional.

Step 3. We have $P = x_1 \otimes f \in \mathcal{O} \subseteq \overline{\mathcal{A}}$. As $\overline{\mathcal{A}}$ is an algebra, the set of rank-one operators

$$\mathcal{S}_P = \{APB : A, B \in \mathcal{A}\} = \{Ax_1 \otimes B^* f : A, B \in \mathcal{A}\} \tag{3.11}$$

lies in $\overline{\mathcal{A}} \cap \mathcal{F}(X)$. For each $0 \neq y \in X$, the linear manifold

$$M_y = \left\{ \sum_{i=1}^{n} S_i y : S_i \in \mathcal{S}_P, \ i = 1, 2, \dots, n, \ n = 1, 2, \dots \right\} \tag{3.12}$$

is dense in X. Indeed, if M_y is not dense in X, by Hahn-Banach theorem, there exists $0 \neq g \in X^*$ such that $g(z) = 0$ for all $z \in M_y$. Hence

$$0 = g\big((Ax_1 \otimes B^* f)y\big) = g\big((B^* f(y))Ax_1\big) = f(By)g(Ax_1) \quad \text{for any } A, B \in \mathcal{A}. \tag{3.13}$$

If $f(By) = 0$ for all $B \in \mathcal{A}$, then f vanishes on the linear manifold $\mathcal{A}y = \{By : B \in \mathcal{A}\}$. However, as $\mathcal{A}$ is transitive, the manifold $\mathcal{A}y$ is dense in X, so that $f = 0$. This contradiction shows that there is $B_1 \in \mathcal{A}$ such that $f(B_1 y) \neq 0$. Then (3.13) implies that $g(Ax_1) = 0$ for all $A \in \mathcal{A}$. Repeating the above argument, we obtain that $g = 0$. This contradiction shows that any manifold M_y is dense in X. Therefore the algebra $\overline{\mathcal{A}} \cap \mathcal{F}(X)$ is transitive. □

Lemma 3.5. *Suppose that $\mathcal{A}$ is a transitive subalgebra of $B(X)$ and $\overline{\mathcal{A}}$ contains a nonzero minimal left ideal $\mathcal{O}$. Then the operators commuting with $\overline{\mathcal{A}}$ are scalars. More precisely, if T is a closed, densely defined operator such that $A(\mathrm{Dom}(T)) \subseteq \mathrm{Dom}(T)$ and $AT\xi = TA\xi$ for any $A \in \overline{\mathcal{A}}$ and any $\xi \in \mathrm{Dom}(T)$, then $T = \mu I$ for some complex scalar μ.*

Proof. For any $y \in X$, there exists a net of operators $\{A_\lambda\} \subseteq \mathcal{A}$ such that $\lim_\lambda A_\lambda(x_1) = y$ by the transitivity of $\mathcal{A}$. Then $\overline{\mathcal{A}} \ni A_\lambda(x_1 \otimes f) = A_\lambda(x_1) \otimes f \xrightarrow{\|\cdot\|} y \otimes f$, since $P = x_1 \otimes f \in \overline{\mathcal{A}}$ as in (3.10). It follows that

$$y \otimes f \in \overline{\mathcal{A}} \quad \text{for any } y \in X. \tag{3.14}$$

Suppose that T is a closed, densely defined operator commuting with $\overline{\mathcal{A}}$. As $S = (y \otimes f) \in \overline{\mathcal{A}}$ for any $y \in X$, then $(y \otimes f)(\xi) \in \mathrm{Dom}(T)$ and $(y \otimes f)T\xi = T(y \otimes f)\xi$ for any $\xi \in \mathrm{Dom}(T)$. Since $\mathrm{Dom}(T)$ is dense in X and $f \neq 0$, there is $\xi_0 \in \mathrm{Dom}(T)$ such that $f(\xi_0) \neq 0$. So

$$(y \otimes f)(\xi_0) = f(\xi_0)y \in \mathrm{Dom}(T), \qquad f(T\xi_0)y = f(\xi_0)T(y), \quad \text{for any } y \in X. \tag{3.15}$$

It follows that $\mathrm{Dom}(T) = X$. Set $\mu = f(T\xi_0)/f(\xi_0)$. Then $T = \mu I$, as required. □

Lemma 3.6. *Suppose that $\mathscr{A}$ is a subalgebra of $B(X)$ and $\overline{\mathscr{A}}$ is its norm closure. Then* $\mathrm{Lat}(\mathscr{A}) = \mathrm{Lat}(\overline{\mathscr{A}})$.

Proof. Clearly, $\mathrm{Lat}(\overline{\mathscr{A}}) \subseteq \mathrm{Lat}(\mathscr{A})$. Conversely, let $M \in \mathrm{Lat}(\mathscr{A})$. For any $B \in \overline{\mathscr{A}}$, there exists a net of operators $\{A_\lambda\} \subseteq \mathscr{A}$ such that $A_\lambda \xrightarrow{\|\cdot\|} B$. Then $A_\lambda x \in M$ for any $x \in M$. Since $A_\lambda x \to Bx$ and M is a closed subspace of X, $Bx \in M$. It follows that $M \in \mathrm{Lat}(B)$ so that $M \in \mathrm{Lat}(\overline{\mathscr{A}})$. Therefore $\mathrm{Lat}(\mathscr{A}) \subseteq \mathrm{Lat}(\overline{\mathscr{A}})$. The Lemma follows. $\qquad\square$

Let $\mathscr{A}$ be a subalgebra of $B(X)$ and δ be a bounded derivation from $\mathscr{A}$ ($\mathscr{A} = \mathrm{Dom}(\delta)$) into $B(X)$. For any $B \in \overline{\mathscr{A}}$, there exists a net of operators $\{A_\lambda\} \subseteq \mathscr{A}$ such that $A_\lambda \xrightarrow{\|\cdot\|} B$. If $\{A_\lambda\}$ is a uniformly convergent net in $\mathscr{A}$; then $\{\delta(A_\lambda)\}$ is also a uniformly convergent net in $B(X)$ since $\|\delta(A_{\lambda_1}) - \delta(A_{\lambda_2})\| \le \|\delta\|\|A_{\lambda_1} - A_{\lambda_2}\|$ for any indexes λ_1, λ_2. Also, if two nets of operators $\{A_\lambda^{(1)}\}, \{A_\lambda^{(2)}\}$ in $\mathscr{A}$ converge uniformly to the same limit, then $\{\delta(A_\lambda^{(1)})\}, \{\delta(A_\lambda^{(2)})\}$ converge uniformly to the same limit, since $\|\delta(A_\lambda^{(1)}) - \delta(A_\lambda^{(2)})\| \le \|\delta\|\|A_\lambda^{(1)} - A_\lambda^{(2)}\|$. Therefore a linear map $\overline{\delta}$ can be unambiguously defined by

$$\mathrm{Dom}\left(\overline{\delta}\right) = \overline{\mathrm{Dom}(\delta)} = \overline{\mathscr{A}}, \qquad \overline{\delta}(B) = \lim_\lambda \delta(A_\lambda) \tag{3.16}$$

for any $B \in \overline{\mathscr{A}}$ and any net $\{A_\lambda\}$ in $\mathscr{A}$ such that $A_\lambda \xrightarrow{\|\cdot\|} B$, where $\overline{\delta}(B)$ is the limit of $\delta(A_\lambda)$ in the operator norm topology. It is obvious that $\|\overline{\delta}\| = \|\delta\|$ if δ is bounded.

Proposition 3.7. *Let δ be a bounded transitive derivation from $\mathscr{A}(\mathscr{A} = \mathrm{Dom}(\delta))$ into $B(X)$. Then $\overline{\delta}$ is a bounded transitive derivation from $\overline{\mathscr{A}}$ into $B(X)$ and*

$$\mathrm{Imp}\left(\overline{\delta}\right) = \mathrm{Imp}(\delta). \tag{3.17}$$

Proof. Since $\mathscr{A}$ is transitive, $\overline{\mathscr{A}}$ is also transitive. It is obvious that $\overline{\delta}$ is a linear map. For any $B_1, B_2 \in \overline{\mathscr{A}}$, there are two nets of operators $\{A_\lambda^{(1)}\}, \{A_\lambda^{(2)}\}$ in $\mathscr{A}$ such that $A_\lambda^{(i)} \xrightarrow{\|\cdot\|} B_i (i = 1, 2)$. Then $A_\lambda^{(1)} A_\lambda^{(2)} \xrightarrow{\|\cdot\|} B_1 B_2$ and

$$\overline{\delta}(B_1 B_2) = \lim_\lambda \delta\left(A_\lambda^{(1)} A_\lambda^{(2)}\right) = \lim_\lambda \delta\left(A_\lambda^{(1)}\right) A_\lambda^{(2)} + \lim_\lambda A_\lambda^{(1)} \delta\left(A_\lambda^{(2)}\right) = \overline{\delta}(B_1) B_2 + B_1 \overline{\delta}(B_2). \tag{3.18}$$

It follows that the linear map $\overline{\delta}$ is a transitive derivation from $\overline{\mathscr{A}}$ into $B(X)$.

By (2.2), $\mathscr{A}_\delta = \left\{\begin{pmatrix} A & \delta(A) \\ 0 & A \end{pmatrix} : A \in \mathscr{A}\right\}$ and $\mathscr{A}_{\overline{\delta}} = \left\{\begin{pmatrix} B & \overline{\delta}(B) \\ 0 & B \end{pmatrix} : B \in \overline{\mathscr{A}}\right\}$. So that $\mathscr{A}_{\overline{\delta}} = \overline{\mathscr{A}_\delta}$ by (3.16) and $\mathrm{Lat}(\mathscr{A}_{\overline{\delta}}) = \mathrm{Lat}(\mathscr{A}_\delta)$ by Lemma 3.6. By (3.1),

$$\mathrm{Lat}(\mathscr{A}_\delta) = \left\{\{0\} \oplus \{0\}, X \oplus \{0\}, X \oplus X, G(F) : F \in \mathrm{Imp}(\delta)\right\},$$

$$\mathrm{Lat}(\mathscr{A}_{\overline{\delta}}) = \left\{\{0\} \oplus \{0\}, X \oplus \{0\}, X \oplus X, G(F_1) : F_1 \in \mathrm{Imp}\left(\overline{\delta}\right)\right\}. \tag{3.19}$$

It follows that $\mathrm{Imp}(\overline{\delta}) = \mathrm{Imp}(\delta)$. $\qquad\square$

Proof of Theorem 3.1. By (3.4), $\text{Imp}(\delta) \neq \emptyset$. Suppose that $T \in \text{Imp}(\delta)$. Then T is a closed, densely defined operator on X and

$$\{(T + \mu I) : \mu \text{ is a complex scalar}\} \subseteq \text{Imp}(\delta). \tag{3.20}$$

Let $\delta_0 = 0$ be the derivation (with the same domain $\mathscr{A}$ of δ) from $\mathscr{A}$ into $\underline{B}(X)$, that is, $\delta_0(A) = 0$ for any $A \in \mathscr{A}$. If $T_0 \in \text{Imp}(\delta_0)$ with domain $\text{Dom}(T_0)$, then $T_0 \in \text{Imp}(\overline{\delta_0})$ by (3.17), and

$$A(\text{Dom}(T_0)) \subseteq \text{Dom}(T_0), \quad \overline{\delta_0}(A)x = (T_0 A - AT_0)x \quad \text{for any } A \in \overline{\mathscr{A}}, \ x \in \text{Dom}(T_0), \tag{3.21}$$

that is, $T_0 A x = A T_0 x$ for any $x \in \text{Dom}(T_0)$. Then $T_0 = \mu I$ for some scalar μ by Lemma 3.5. It follows that

$$\text{Imp}(\delta_0) = \{\mu I : \mu \text{ is a complex scalar}\}. \tag{3.22}$$

Let $T_1 \in \text{Imp}(\delta)$ be any closed, densely defined operators implementing δ. Then $T_1 \in \text{Imp}(\overline{\delta})$ by (3.17). Then $\text{Dom}(T_1)$ is a nonzero $\text{Dom}(\overline{\delta})$-invariant linear manifold of X by (1.3); that is, $\text{Dom}(T_1)$ is a nonzero $\overline{\mathscr{A}}$-invariant linear manifold of X. Therefore $A(\text{Dom}(T_1)) \subseteq \text{Dom}(T_1)$ for any $A \in \overline{\mathscr{A}}$. If $y \in X$, $y \otimes f \in \overline{\mathscr{A}}$ by (3.14) and $(y \otimes f)\xi = f(\xi)y \in \text{Dom}(T_1)$ for any $\xi \in \text{Dom}(T_1)$. Since $\text{Dom}(T_1)$ is dense in X and $f \neq 0$, there is $\xi_0 \in \text{Dom}(T_1)$ such that $f(\xi_0) \neq 0$. So $y \in \text{Dom}(T_1)$. Since y is arbitrary chosen, it follows that $\text{Dom}(T_1) = X$, the whole space. Then $\text{Dom}(T) = X$ also holds since T_1 is an arbitrary implementation of δ. It is obvious that $\text{Dom}(T_1 - T) = (\text{Dom}(T) \cap \text{Dom}(T_1)) = X$. Therefore the operator $T_1 - T$ is everywhere defined.

Set $T_2 = T_1 - T$. Then T_2 is closable.

Indeed, suppose that $\xi_n \in X$, $\xi_n \to 0$ and $T_2(\xi_n) \to \eta$. Since $T, T_1 \in \text{Imp}(\overline{\delta})$, $\overline{\delta}(A)x = (T_1 A - AT_1)x$ and $\overline{\delta}(A)x = (TA - AT)x$ for any $A \in \text{Dom}(\overline{\delta})$ and any $x \in X$. So $T_2 A \xi_n = A T_2 \xi_n$ for any $A \in \overline{\mathscr{A}}$. As $P = x_1 \otimes f \in \overline{\mathscr{A}}$ by (3.10) and $\overline{\mathscr{A}}$ is an algebra, $T_2 P A \xi_n = P A T_2 \xi_n$ for any $A \in \overline{\mathscr{A}}$. Then

$$\begin{aligned} f(A\eta)x_1 &= (x_1 \otimes f)A\eta = PA\eta = \lim_{n \to \infty} PAT_2\xi_n = \lim_{n \to \infty} T_2 PA\xi_n \\ &= \lim_{n \to \infty} T_2(x_1 \otimes f)A\xi_n = \lim_{n \to \infty} f(A\xi_n)T_2 x_1 = 0 \end{aligned} \tag{3.23}$$

for any $A \in \overline{\mathscr{A}}$. Since $x_1 \neq 0$, $f(A\eta) = 0$ for any $A \in \overline{\mathscr{A}}$. Thus f vanishes on the linear manifold $\overline{\mathscr{A}}\eta = \{A\eta : A \in \overline{\mathscr{A}}\}$. The transitivity of $\overline{\mathscr{A}}$ assures that $\overline{\mathscr{A}}\eta = \{0\}$ or $\overline{\mathscr{A}}\eta$ is dense in X. Since $f \neq 0$, $\overline{\mathscr{A}}\eta = \{0\}$ and $\eta = 0$.

It follows that T_2 is a closable operator with domain $\text{Dom}(T_2) = X$, the whole space. So that T_2 is a closed operator. Clearly, $T_2 = T_1 - T \in \text{Imp}(\delta_0)$. Then $T_1 - T = \mu I$ for some scalar μ by (3.22), that is, $T_1 = T + \mu I$. It follows that

$$\text{Imp}(\delta) = \{(T + \mu I) : \mu \text{ is a complex scalar}\}. \tag{3.24}$$

Since $\Delta_T = \Delta_{T+\mu I}$, by (3.7),

$$\delta = \delta_{\mathrm{Imp}(\delta)} = \Delta_T. \tag{3.25}$$

Since T is closed and $\mathrm{Dom}(T) = X$, T is bounded by the closed graph theorem. It follows from (2.4), (3.24), and (3.25) that $\delta(A) = TA - AT$ for each $A \in \mathcal{A}$ and the implementation T is unique only up to an additive constant. $\quad\square$

4. Applications

Suppose that $\mathcal{A}$ is a subalgebra of $B(X)$ and δ is a derivation from $\mathcal{A}$ ($\mathcal{A} = \mathrm{Dom}(\delta)$) into $B(X)$. We can define the adjoint δ^* of δ by

$$\mathrm{Dom}(\delta^*) = (\mathcal{A})^* = \{A^* : A \in \mathcal{A}\}, \quad \delta^*(A^*) = (\delta(A))^* \quad \text{for any } A \in \mathcal{A}. \tag{4.1}$$

Lemma 4.1. *Let δ be a derivation from $\mathcal{A}$($\mathcal{A} = \mathrm{Dom}(\delta)$) into $B(X)$.*

   (1) *The adjoint δ^* of δ is a derivation from $\mathcal{A}^*$ into $B(X^*)$.*

   (2) *Furthermore, suppose that X is a reflexive Banach space and $\mathcal{A}$ is a subalgebra of $B(X)$. Then δ^* is a reflexive (resp., transitive) derivation from $\mathcal{A}^*$ into $B(X^*)$ if and only if δ is a reflexive (resp., transitive) derivation from $\mathcal{A}$ into $B(X)$. And $\mathrm{Imp}(\delta^*) = \{-T^* : T \in \mathrm{Imp}(\delta)\}$.*

Proof. (1) It is obvious that δ^* is a linear map from $\mathcal{A}^*$ into $B(X^*)$. If $A_1^*, A_2^* \in \mathrm{Dom}(\delta^*)$, then

$$\begin{aligned} \delta^*\left(A_1^* A_2^*\right) &= \delta^*\left((A_2 A_1)^*\right) = (\delta(A_2 A_1))^* = (\delta(A_2)A_1 + A_2\delta(A_1))^* \\ &= A_1^*(\delta(A_2))^* + (\delta(A_1))^* A_2^* = A_1^*(\delta^*(A_2^*)) + (\delta^*(A_1^*))A_2^*. \end{aligned} \tag{4.2}$$

It follows that δ^* is a derivation.

   (2) Suppose that δ is transitive. Then $\mathcal{A}$ is transitive and $\mathrm{Lat}\,\mathcal{A} = \{X, \{0\}\}$. Since $M \in \mathrm{Lat}\,A$ if and only if $M^\perp \in \mathrm{Lat}(A^*)$ for any $A \in B(X)$,

$$\mathrm{Lat}(\mathrm{Dom}(\delta^*)) = \mathrm{Lat}(\mathcal{A}^*) = \left\{M^\perp : M \in \mathrm{Lat}\,\mathcal{A}\right\} = \left\{\{0\}^\perp, X^\perp\right\} = \{X^*, \{0\}\}. \tag{4.3}$$

We have that $\mathrm{Dom}(\delta^*)$ is transitive, that is, δ^* is transitive.

   Conversely, suppose that δ^* is transitive. We obtain, as above, that δ^{**} is transitive. Since X is a reflexive Banach space, $A^{**} = A$ for any bounded operator $A \in B(X)$. It follows from (4.1) that $\delta = \delta^{**}$ is transitive by the definition of δ^*.

Set $J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, where I is the identity operator on X^* and 0 is the zero operator on X^*. Since

$$\mathcal{A}_{\delta^*} = \left\{ \begin{pmatrix} A^* & \delta^*(A^*) \\ 0 & A^* \end{pmatrix} : A \in \mathcal{A} \right\} = \left\{ J \begin{pmatrix} A^* & 0 \\ \delta(A)^* & A^* \end{pmatrix} J : A \in \mathcal{A} \right\}$$

$$= \left\{ J \begin{pmatrix} A & \delta(A) \\ 0 & A \end{pmatrix}^* J : A \in \mathcal{A} \right\} = J(\mathcal{A}_\delta)^* J = J^{-1}(\mathcal{A}_\delta)^* J,$$

(4.4)

the two operator algebras $\mathcal{A}_{\delta^*}$ and $(\mathcal{A}_\delta)^*$ are similar. We have that

$$\mathrm{Lat}(\mathcal{A}_{\delta^*}) = J\big(\mathrm{Lat}((\mathcal{A}_\delta)^*)\big) = J(\mathrm{Lat}(\mathcal{A}_\delta))^{\perp}.$$

(4.5)

It is obvious that

$$(X \oplus X)^{\perp} = \{0\} \oplus \{0\}, \qquad (\{0\} \oplus \{0\})^{\perp} = X^* \oplus X^*,$$

$$(X \oplus \{0\})^{\perp} = \{0\} \oplus X^*, \qquad J(\{0\} \oplus X^*) = X^* \oplus \{0\}.$$

(4.6)

If $T \in \mathrm{Imp}(\delta)$, $\begin{pmatrix} g \\ -T^*(g) \end{pmatrix} \in (G(T))^{\perp}$ for any $g \in \mathrm{Dom}(T^*)$, since

$$G(T) = \left\{ \begin{pmatrix} T(x) \\ x \end{pmatrix} : x \in \mathrm{Dom}(T) \right\}, \qquad g(T(x)) + (-T^*(g))(x) = 0 \quad \text{for any } x \in \mathrm{Dom}(T).$$

(4.7)

Conversely, if $g, h \in X^*$ such that $\begin{pmatrix} g \\ h \end{pmatrix} \in (G(T))^{\perp}$, then

$$g(Tx) + h(x) = 0 \quad \text{for any } x \in \mathrm{Dom}(T).$$

(4.8)

So that the functional $x \mapsto g(Tx)(= -h(x))$ is a bounded functional on $\mathrm{Dom}(T)$. Therefore $g \in \mathrm{Dom}(T^*)$ and $h = (-T^*(g))$ holds on $\mathrm{Dom}(T)$ by (4.8). As $(-T^*(g))$ and h are bounded functionals on X and $\mathrm{Dom}(T)$ is dense in X, $h = (-T^*(g))$ on X. It follows that

$$(G(T))^{\perp} = \left\{ \begin{pmatrix} g \\ -T^*(g) \end{pmatrix} : g \in \mathrm{Dom}(T^*) \right\},$$

$$J(G(T))^{\perp} = \left\{ \begin{pmatrix} -T^*(g) \\ g \end{pmatrix} : g \in \mathrm{Dom}(T^*) \right\} = G(-T^*).$$

(4.9)

Hence

$$\mathrm{Lat}(\mathcal{A}_{\delta^*}) = \big\{ \{0\} \oplus \{0\}, X^* \oplus \{0\}, X^* \oplus X^*, G(-T^*) : T \in \mathrm{Imp}(\delta) \big\}.$$

(4.10)

It follows from (3.4) that

$$\operatorname{Imp}(\delta^*) = \{-T^* : T \in \operatorname{Imp}(\delta)\}. \tag{4.11}$$

Suppose that δ is reflexive. Then $\mathcal{A}_\delta$ is a reflexive operator algebra on $X \oplus X$. Since $\mathcal{A}_{\delta^*} = J^{-1}(\mathcal{A}_\delta)^* J$, the two operator algebras $\mathcal{A}_{\delta^*}$ and $(\mathcal{A}_\delta)^*$ are similar. It follows that $\mathcal{A}_{\delta^*}$ is reflexive if and only if $(\mathcal{A}_\delta)^*$ is reflexive. Since $\mathcal{A}_\delta$ is reflexive, $(\mathcal{A}_\delta)^*$ is also reflexive by the reflexivity of the Banach space $X \oplus X$. Thus $\mathcal{A}_{\delta^*}$ is reflexive, that is, δ^* is reflexive.

Conversely, suppose that δ^* is reflexive. We obtain, as above, that δ^{**} is reflexive. It follows that $\delta = \delta^{**}$ is reflexive. □

Theorem 4.2. *Suppose that* X *is a reflexive Banach space. Let* $\mathcal{A}$ *be a transitive subalgebra of* $B(X)$ *and let* $\overline{\mathcal{A}}$ *contain a nonzero minimal right ideal* $\mathcal{I}$. *If* δ *is a bounded reflexive transitive derivation from* $\mathcal{A}$ *into* $B(X)$, *then* δ *is spatial and implemented uniquely, or more precisely, there exists* $T \in B(X)$ *such that* $\delta(A) = TA - AT$ *for each* $A \in \mathcal{A}$, *and the implementation* T *is unique only up to an additive constant.*

Proof. Consider the derivation δ^*. δ^* is a bounded reflexive transitive derivation from $\mathcal{A}^* = \{A^* : A \in \mathcal{A}\}$ into $B(X^*)$ by Lemma 4.1.

Set $\mathcal{I}^* = \{B^* : B \in \mathcal{I}\}$. Since $\mathcal{I}$ is a right ideal of $\overline{\mathcal{A}}$, $BA \in \mathcal{I}$ for any $B \in \mathcal{I}$ and $A \in \overline{\mathcal{A}}$. For any $A^* \in (\overline{\mathcal{A}})^*$, $B^* \in \mathcal{I}^*$, that is, $A \in \overline{\mathcal{A}}$, $B \in \mathcal{I}$, we have that $BA \in \mathcal{I}$. Then $A^* B^* = (BA)^* \in \mathcal{I}^*$. It follows that $\mathcal{I}^*$ is a left ideal of $(\overline{\mathcal{A}})^*$. If there exists a nonzero left ideal $\mathcal{I}_1$ of $(\overline{\mathcal{A}})^*$ included in $\mathcal{I}^*$, then $\mathcal{I}_1^*$ is a nonzero right ideal of $\overline{\mathcal{A}}$ included in $\mathcal{I}$. Thus $\mathcal{I}_1^* = \mathcal{I}$ and $\mathcal{I}_1 = \mathcal{I}^*$. It follows that $\mathcal{I}^*$ is a minimal left ideal of $(\overline{\mathcal{A}})^* = \overline{\mathcal{A}}^*$.

By (3.24), there exists a bounded operator $T_1 \in B(X^*)$ such that

$$\operatorname{Imp}(\delta^*) = \{(T_1 + \mu I) : \mu \text{ is a complex scalar}\}, \tag{4.12}$$

where I is the identity operator on X^*. By (4.11),

$$\operatorname{Imp}(\delta) = \operatorname{Imp}(\delta^{**}) = \{(-T_1^* - \overline{\mu} I) : \mu \text{ is a complex scalar}\}, \tag{4.13}$$

where I is the identity operator on X. Set $T = -T_1^*$. Then T is bounded with $\operatorname{Dom}(T) = X$. By (3.7) and (4.13), $\delta = \delta_{\operatorname{Imp}(\delta)} = \Delta_T$. It follows from (2.4) that $\delta(A) = TA - AT$ for each $A \in \mathcal{A}$, and the implementation T is unique only up to an additive constant. □

Corollary 4.3 (Proposition 2.11, [5]). *Suppose that* $\mathcal{A}$ *is a transitive subalgebra of* $B(H)$. *Let* δ *be a bounded reflexive transitive derivation from* $\mathcal{A}$ *into* $B(H)$. *If* $C(H) \subseteq \overline{\mathcal{A}}$, *then there exists* $T \in B(H)$ *such that* $\delta(A) = TA - AT$ *for each* $A \in \mathcal{A}$, *and the implementation* T *is unique only up to an additive constant.*

Proof. Since $C(H) \subseteq \overline{\mathcal{A}}$, for a fixed vector $f \in H$,

$$\mathcal{I} = \{x \otimes f \mid x \in H\} \tag{4.14}$$

is a minimal left ideal of $\overline{\mathcal{A}}$, where $x \otimes f$ is the rank-one operator on H with $(x \otimes f)h = (h, f)x$ for $h \in H$. By Theorem 3.1, there exists $T \in B(H)$ such that $\delta(A) = TA - AT$ for each $A \in \mathcal{A}$, and the implementation T is unique only up to an additive constant. $\qquad\square$

Remark 4.4. In a Banach space without the approximation property (there exists such space as this, e.g., [17]), not all compact operators can be approximated by finite-rank operators in the norm operator topology. Therefore, Theorems 3.1 and 4.2 of this paper improve [5, Proposition 2.11].

Acknowledgments

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