

Research Article

Some Geometric Properties of Lacunary Sequence Spaces Related to Fixed Point Property

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The main purpose of this paper is considering the lacunary sequence spaces defined by Karakaya (2007), by proving the property (β) and Uniform Opial property.

1. Introduction

Let $(X, \|\cdot\|)$ be a real Banach space and let $B(X)$ (resp., $S(X)$) be a closed unit ball (resp., the unit sphere) of X . For any subset A of X , we denote by $\text{conv}(A)$ the convex hull of A . The Banach space X is uniformly convex (UC), if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for $x, y \in S(X)$ the inequality $\|x - y\| > \varepsilon$ implies $\|(x + y)/2\| < 1 - \delta$ (see [1]). A Banach space X has the property (β) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $1 < \|x\| < 1 + \delta$ implies $\alpha(\text{conv}(B(X) \cup \{x\}) \setminus B(X)) < \varepsilon$, where $\alpha(A)$ denotes the Kuratowski measure noncompactness of a subset A of X defined as the infimum of all $\varepsilon > 0$ such that A can be covered by a finite union of sets of diameter less than ε . The following characterization of the property (β) is very useful (see [2]): A Banach space X has the property (β) if and only if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each element $x \in B(X)$ and each sequence (x_n) in $B(X)$ with $\text{sep}(x_n) \geq \varepsilon$ there is an index k for which $\|(x + x_k)/2\| < 1 - \delta$ where $\text{sep}(x_n) = \inf\{\|x_n - x_m\| : n \neq m\} > \varepsilon$. A Banach space X is nearly uniformly convex (NUC) if for each $\varepsilon > 0$ and every sequence (x_n) in $B(X)$ with $\text{sep}(x_n) \geq \varepsilon$, there exists $\delta \in (0, 1)$ such that $\text{conv}(x_n) \cap (1 - \delta)B(X) \neq \emptyset$. Define for any $x \notin B(X)$ the drop $D(x, B(X))$ determined by x by $D(x, B(X)) = \text{conv}(B(X) \cup \{x\})$. A Banach space X has the drop property (write (D)) if for every closed set C disjoint with $B(X)$ there exists an element $x \in C$ such that $D(x, B(X)) \cap C = \{x\}$. A point $x \in S(X)$ is an H -point of $B(X)$ if for any sequence (x_n) in X such that $\|x_n\| \rightarrow 1$ as $n \rightarrow \infty$, the weak convergence of (x_n) to x implies that $\|x_n - x\| \rightarrow 0$ as

$n \rightarrow \infty$. If every point in $S(X)$ is an H -point of $B(X)$, then X is said to have the property (H) . A Banach space is said to have the uniform Kadec-Klee property (abbreviated as (UKK)) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every sequence (x_n) in $S(X)$ with $\text{sep}(x_n) \geq \varepsilon$ and $x_n \xrightarrow{w} x$ as $n \rightarrow \infty$, we have $\|x\| < 1 - \delta$. Every (UKK) Banach space has H -property (see [3]). The following implications are true in any Banach spaces,

$$\begin{array}{c} (D) \implies (Rfx) \\ \uparrow \\ (\text{UC}) \implies \text{property}(\beta) \implies (\text{NUC}) \implies (\text{UKK}) \implies \text{property} (H), \end{array} \quad (1.1)$$

where (Rfx) denotes the property of reflexivity (see [3–6]). A Banach space X is said to have the Opial property (see [7]) if every sequence (x_n) weakly convergent to x_0 satisfies

$$\liminf_{n \rightarrow \infty} \|x_n - x_0\| \leq \liminf_{n \rightarrow \infty} \|x_n - x\|, \quad (1.2)$$

for every $x \in X$. Opial proved in [7] that the sequence space l_p ($1 < p < \infty$) have this property but $L_p[0, \pi]$ ($p \neq 2, 1 < p < \infty$) do not have it. A Banach space X is said to have the uniform Opial property (see [8]), if for each $\varepsilon > 0$ there exists $\tau > 0$ such that for any weakly null sequence (x_n) in $S(X)$ and $x \in X$ with $\|x\| \geq \varepsilon$ there holds

$$1 + \tau \leq \liminf_{n \rightarrow \infty} \|x_n + x\|. \quad (1.3)$$

For example, the space in [9, 10] has the uniform Opial property. The Opial property is important because Banach spaces with this property have the weak fixed point property (see [11]) and the geometric property involving fixed point theory can be found, for example, in [9, 12–14].

For a bounded subset $A \subset X$, the set measure of noncompactness was defined in [15] by

$$\alpha(A) = \inf\{\varepsilon > 0 : A \text{ can be covered by finitely many sets of diameter } \leq \varepsilon\}. \quad (1.4)$$

The ball measure of noncompactness was defined in [16, 17] by

$$\beta(A) = \inf\{\varepsilon > 0 : A \text{ can be covered by finitely many balls of diameter } \leq \varepsilon\}. \quad (1.5)$$

The functions α and β are called the Kuratowski measure of noncompactness and the Hausdorff measure of noncompactness in X , respectively. We can associate these functions with the notions of the set-contraction and ball contraction (see [18]). These notions are very useful tools to study nonlinear operator problems (see [8, 18]). For each $\varepsilon > 0$ define that $\Delta(\varepsilon) = \inf\{1 - \inf\{\|x\| : x \in A\} : A \text{ is closed convex subset of } B(X) \text{ with } \beta(A) \geq \varepsilon\}$. The function Δ is called the modulus of noncompact convexity (see [16]). A Banach space X is said to have property (L) if $\lim_{\varepsilon \rightarrow 1^-} \Delta(\varepsilon) = 1$. It has been proved in [8] that property (L) is a useful tool in

the fixed point theory and that a Banach space X has property (L) if and only if it is reflexive and has the uniform Opial property.

For a real vector space X , a function $\rho : X \rightarrow [0, \infty]$ is called a *modular* if it satisfies the following conditions:

- (i) $\rho(x) = 0$ if and only if $x = 0$;
 - (ii) $\rho(\alpha x) = \rho(x)$ for all scalar α with $|\alpha| = 1$;
 - (iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$, for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$;
- the modular ρ is called convex if
- (iv) $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$, for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

For modular ρ on X , the space

$$X_\rho = \{x \in X : \rho(\lambda x) \rightarrow 0 \text{ as } \lambda \rightarrow 0^+\} \tag{1.6}$$

is called the *modular space*.

A sequence (x_n) in X_ρ is called *modular convergent* to $x \in X_\rho$ if there exists a $\lambda > 0$ such that $\rho(\lambda(x_n - x)) \rightarrow 0$ as $n \rightarrow \infty$.

A modular ρ is said to satisfy the Δ_2 -condition ($\rho \in \Delta_2$) if for any $\varepsilon > 0$ there exist constants $K \geq 2$ and $a > 0$ such that

$$\rho(2u) \leq K\rho(u) + \varepsilon \tag{1.7}$$

for all $u \in X_\rho$ with $\rho(u) \leq a$.

If ρ satisfies the Δ_2 -condition for any $a > 0$ with $K \geq 2$ dependent on a , we say that ρ satisfies the *strong Δ_2 -condition* ($\rho \in \Delta_2^s$).

By a lacunary sequence $\theta = (k_r)$, where $k_0 = 0$, we will mean an increasing sequence of nonnegative integers with $k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. The intervals determined by θ will be denote by $I_r = (k_{r-1}, k_r]$. We write $h_r = k_r - k_{r-1}$ and the ratio k_r/k_{r-1} , will be denoted by q_r . The space of lacunary strongly convergent sequence N_θ was defined by Freedman et al. [19] as

$$N_\theta = \left\{ x = (x_k) : \lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} |x_k - l| = 0, \text{ for some } l \right\}. \tag{1.8}$$

It is well known that there is very closed connection between the space of lacunary strongly convergent sequence and the space of strongly Cesaro summability sequences. This connection can be found in [18–23], because a lot of these connection, a lot of geometric property of Cesaro sequence spaces can generalize the lacunary sequence spaces.

Let w be the space of all real sequences. Let $p = (p_r)$ be a bounded sequence of the positive real numbers. In 2007, Karakaya [24] introduced the new sequence spaces $l(p, \theta)$ involving lacunary sequence as follows:

$$l(p, \theta) = \left\{ x = (x(i)) : \sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{i \in I_r} |x(i)| \right)^{p_r} < \infty \right\} \tag{1.9}$$

and paranorm on $l(p, \theta)$ is given by

$$\|x\|_{l(p, \theta)} = \left(\sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{i \in I_r} |x(i)| \right)^{p_r} \right)^{1/M}, \quad (1.10)$$

where $M = \sup_r p_r$. If $p_r = p$ for all $r \in \mathbb{N}$, we will use the notation $l_p(\theta)$ in place of $l(p, \theta)$. The norm on $l_p(\theta)$ is given by

$$\|x\|_{l_p(\theta)} = \left(\sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{i \in I_r} |x(i)| \right)^p \right)^{1/p}. \quad (1.11)$$

By using the properties of lacunary sequence in the space $l(p, \theta)$, we get the following sequences. If $\theta = (2^r)$, then $l(p, \theta) = \text{ces}(p)$. If $\theta = (2^r)$ and $p_r = p$ for all $r \in \mathbb{N}$, then $l(p, \theta) = \text{ces}_p$. For $x \in l(p, \theta)$ defined the modular on $l(p, \theta)$ by

$$\varrho(x) = \sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{i \in I_r} |x(i)| \right)^{p_r}. \quad (1.12)$$

It is easy to see that if $\lim_{r \rightarrow \infty} \sup p_r < \infty$ then $\varrho \in \Delta_2^s$. The Luxembour norm on $l(p, \theta)$ is defined by

$$\|x\| = \inf \left\{ \varepsilon > 0 : \varrho \left(\frac{x}{\varepsilon} \right) \leq 1 \right\}. \quad (1.13)$$

The Luxembour norm on $l_p(\theta)$ can be reduced to a usual norm on $l_p(\theta)$ [24], that is,

$$\|x\| = \|x\|_{l_p(\theta)}. \quad (1.14)$$

Throughout this paper, we assume that $\lim_{r \rightarrow \infty} \inf p_r > 1$ and $\lim_{r \rightarrow \infty} \sup p_r < \infty$ and for $x \in w$, $i \in \mathbb{N}$, we denote

$$e_i = \left(\overbrace{0, 0, \dots, 0}^{i-1 \text{ times}}, 1, 0, 0, 0, \dots \right), \quad (1.15)$$

$$x|_i = (x(1), x(2), x(3), \dots, x(i), 0, 0, 0, \dots),$$

$$x|_{\mathbb{N}-i} = (0, 0, 0, \dots, x(i+1), x(i+2), \dots).$$

The following results are very important for our consideration.

Lemma 1.1 (see [25, Lemma 2.1]). *If $\rho \in \Delta_2^s$, then for any $L > 0$ and $\varepsilon > 0$, there exists $\delta = \delta(L, \varepsilon) > 0$ such that*

$$|\rho(u + v) - \rho(u)| < \varepsilon, \tag{1.16}$$

whenever $u, v \in X_\rho$ with $\rho(u) \leq L$, and $\rho(v) \leq \delta$.

Lemma 1.2 (see [25, Lemma 2.3]). *Convergence in norm and in modular are equivalent in X_ρ if $\rho \in \Delta_2$.*

Lemma 1.3 (see [25, Corollary 2.2, Lemma 2.3]). *If $\rho \in \Delta_2$, then for any sequence (x_n) in X_ρ , $\|x_n\| \rightarrow 0$ if and only if $\rho(x_n) \rightarrow 0$ as $n \rightarrow \infty$.*

Lemma 1.4 (see [25, Lemma 2.4]). *If $\rho \in \Delta_2^s$, then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|x\| \geq 1 + \delta$ whenever $\rho(x) \geq 1 + \varepsilon$.*

Lemma 1.5 (see [24, Lemma 2.3]). *The functional ϱ is a convex modular on $l(p, \theta)$.*

Lemma 1.6 (see [24, Lemma 2.5]). (i) *For any $x \in l(p, \theta)$, if $\|x\| < 1$, then $\varrho(x) \leq \|x\|$.
(ii) *For any $x \in l(p, \theta)$, $\|x\| = 1$ if and only if $\varrho(x) = 1$.**

2. The Main Results

In this section, we prove the property (β) and uniform Opial property in lacunary sequence and connect to the fixed point property. First we shall give some results which are very important for our consideration.

Lemma 2.1. *For any $x \in l(p, \theta)$, there exists $k_0 \in \mathbb{N}$ and $\lambda \in (0, 1)$ such that $\varrho(x^k/2) \leq ((1 - \lambda)/2)\varrho(x^k)$ for all $k \in \mathbb{N}$ with $k \geq k_0$, where*

$$x^k = \left(\overbrace{0, 0, \dots, 0}^{k-1}, x(k), x(k+1), x(k+2), \dots \right). \tag{2.1}$$

Proof. Let $k \in \mathbb{N}$ be fixed. So there exist $r_k \in \mathbb{N}$ such that $k \in I_{r_k}$. Let α be a real number such that $1 < \alpha \leq \lim_{r \rightarrow \infty} \inf p_r$, then there exists $k_0 \in \mathbb{N}$ such that $\alpha < p_{r_k}$ for all $k \geq k_0$. Choose $\lambda \in (0, 1)$ as a real such that $(1/2)^\alpha \leq (1 - \lambda)/2$. Then for each $x \in l(p, \theta)$ and $k \geq k_0$, we have

$$\begin{aligned} \varrho\left(\frac{x^k}{2}\right) &= \sum_{r=r_k}^{\infty} \left(\frac{1}{h_r} \sum_{i \in I_r} \left| \frac{x(i)}{2} \right| \right)^{p_r} \\ &= \sum_{r=r_k}^{\infty} \left(\frac{1}{2}\right)^{p_r} \left(\frac{1}{h_r} \sum_{i \in I_r} |x(i)| \right)^{p_r} \end{aligned}$$

$$\begin{aligned}
&\leq \left(\frac{1}{2}\right)^\alpha \sum_{r=r_k}^\infty \left(\frac{1}{h_r} \sum_{i \in I_r} |x(i)|\right)^{p_r} \\
&\leq \frac{1-\lambda}{2} \varrho(x^k).
\end{aligned} \tag{2.2}$$

□

Lemma 2.2. For any $x \in l(p, \theta)$ and $\varepsilon \in (0, 1)$ there exists $\delta \in (0, 1)$ such that $\varrho(x) \leq 1 - \varepsilon$ implies $\|x\| \leq 1 - \delta$.

Proof. Suppose that the lemma does not hold, then there exist $\varepsilon > 0$ and $x_n \in l(p, \theta)$ such that $\varrho(x_n) \leq 1 - \varepsilon$ and $1/2 \leq \|x_n\| \nearrow 1$. Let $a_n = (1/\|x_n\|) - 1$. Then $a_n \rightarrow 0$ as $n \rightarrow \infty$. Let $L = \sup\{\varrho(2x_n) : n \in \mathbb{N}\}$. Since $\varrho \in \Delta_2^s$ there exists $K \geq 2$ such that

$$\varrho(2u) \leq K\varrho(u) + 1, \tag{2.3}$$

for every $u \in l(p, \theta)$ with $\varrho(u) < 1$. By (2.3), we have $\varrho(2x_n) \leq K\varrho(x_n) + 1 \leq K + 1$ for all $n \in \mathbb{N}$. Hence $0 < L < \infty$. By Lemmas 1.5 and 1.6(ii), we have

$$\begin{aligned}
1 &= \varrho\left(\frac{x_n}{\|x_n\|}\right) = \varrho(2a_n x_n + (1 - a_n)x_n) \\
&\leq a_n \varrho(2x_n) + (1 - a_n)\varrho(x_n) \\
&\leq a_n L + (1 - \varepsilon) \rightarrow 1 - \varepsilon,
\end{aligned} \tag{2.4}$$

which is a contradiction. □

Theorem 2.3. The space $l(p, \theta)$ is Banach spaces with respect to the Luxemburg norm.

Proof. Let $(x_n) = (x_n(i))$ be a Cauchy sequence in $l(p, \theta)$ and $\varepsilon \in (0, 1)$. Thus there exists $N \in \mathbb{N}$ such that $\|x_n - x_m\| < \varepsilon^M$ for all $n, m \geq N$. By Lemma 1.6 (i), we have

$$\varrho(x_n - x_m) \leq \|x_n - x_m\| < \varepsilon^M \quad \forall n, m \geq N. \tag{2.5}$$

That is,

$$\sum_{r=1}^\infty \left(\frac{1}{h_r} \sum_{i \in I_r} |x_n(i) - x_m(i)|\right)^{p_r} < \varepsilon^M \quad \forall n, m \geq N. \tag{2.6}$$

For fixed r , we get that

$$|x_n(i) - x_m(i)| < \varepsilon \quad \forall n, m \geq N. \tag{2.7}$$

Thus let $(x_n(i))$ be a Cauchy sequence in \mathbb{R} for all $i \in \mathbb{N}$. Since \mathbb{R} is complete, then there exists $x(i) \in \mathbb{R}$ such that $x_m(i) \rightarrow x(i)$ as $m \rightarrow \infty$ for all $i \in \mathbb{N}$. Thus for fixed r , we have

$$|x_n(i) - x(i)| < \varepsilon \quad \text{as } m \rightarrow \infty, \forall n \geq N. \quad (2.8)$$

This implies that, for all $n \geq N$,

$$\varrho(x_n - x_m) \rightarrow \varrho(x_n - x) \quad \text{as } m \rightarrow \infty. \quad (2.9)$$

This means that, for all $n \geq N$,

$$\sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{i \in I_r} |x_n(i) - x_m(i)| \right)^{p_r} \rightarrow \sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{i \in I_r} |x_n(i) - x(i)| \right)^{p_r} \quad (2.10)$$

as $m \rightarrow \infty$. By (2.6), we have

$$\varrho(x_n - x) \leq \|x_n - x_m\| < \varepsilon^M \leq \varepsilon \quad \forall n \geq N. \quad (2.11)$$

This implies that $x_n \rightarrow x$ as $n \rightarrow \infty$. So we have $x_N - x \in l(p, \theta)$. By the linearity of the sequence space $l(p, \theta)$, we have $x = (x_N - x) + x_N \in l(p, \theta)$. Therefore the sequence space $l(p, \theta)$ is Banach space, with respect to the Luxemburg norm and the proof is complete. \square

Theorem 2.4. *The space $l(p, \theta)$ has property (β) .*

Proof. Let $\varepsilon > 0$ and $(x_n) \subset B(l(p, \theta))$ with $\text{sep}(x_n) \geq \varepsilon$. For each $k \in \mathbb{N}$, there exist $r_k \in \mathbb{N}$ such that k is a minimal element in I_{r_k} . Let

$$x_n^k = \left(\overbrace{0, 0, \dots, 0}^{k-1}, x_n(k), x_n(k+1), x_n(k+2), \dots \right). \quad (2.12)$$

Since for each $i \in \mathbb{N}$, $(x_n(i))_{n=1}^{\infty}$ is bounded. By using the diagonal method, we have that for each $k \in \mathbb{N}$ we can find subsequence (x_{n_j}) of (x_n) such that $(x_{n_j}(i))$ converges for each $i \in \mathbb{N}$. Therefore, for any $k \in \mathbb{N}$ there exists an increasing sequence (t_k) such that $\text{sep}((x_{n_j}^k)_{j > t_k}) \geq \varepsilon$. Hence for each $k \in \mathbb{N}$ there exists sequence of positive integers $(s_k)_{k=1}^{\infty}$ with $s_1 < s_2 < s_3 < \dots$ such that $\|x_{s_k}^k\| \geq \varepsilon/2$, and since $\varrho \in \Delta_2^s$, by Lemma 1.3 we may assume that there exists $\eta > 0$ such that $\varrho(x_{s_k}^k) \geq \eta$ for all $k \in \mathbb{N}$, that is,

$$\sum_{r=r_k}^{\infty} \left(\frac{1}{h_r} \sum_{i \in I_r} |x_{s_k}^k(i)| \right)^{p_r} \geq \eta \quad (2.13)$$

for all $k \in \mathbb{N}$. On the other hand by Lemma 2.1, there exist $k_0 \in \mathbb{N}$ and $\lambda \in (0, 1)$ such that

$$\varrho\left(\frac{u^k}{2}\right) \leq \frac{1-\lambda}{2}\varrho(u^k) \quad (2.14)$$

for all $u \in l(p, \theta)$ and $k \geq k_0$. From Lemma 2.2, there exist $\delta \in (0, 1)$ such that for any $y \in l(p, \theta)$,

$$\varrho(y) \leq 1 - \frac{\lambda\eta}{4} \implies \|y\| \leq 1 - \delta. \quad (2.15)$$

Since again $\varrho \in \Delta_2^s$, by Lemma 1.1, there exists δ_0 such that

$$|\varrho(u+v) - \varrho(u)| < \frac{\lambda\eta}{4}, \quad (2.16)$$

whenever $\varrho(u) \leq 1$ and $\varrho(v) \leq \delta_0$. Since $x \in B(l(p, \theta))$, we have that $\varrho(x) \leq 1$. Then there exists $k \geq k_0$ such that $\varrho(x^k) \leq \delta_0$. We put $u = x_{s_k}^k$ and $v = x^k$,

$$\varrho\left(\frac{u}{2}\right) = \sum_{r=r_k}^{\infty} \left(\frac{1}{h_r} \sum_{i \in I_r} \left| \frac{x_{s_k}(i)}{2} \right| \right)^{p_r} < 1, \quad \varrho\left(\frac{v}{2}\right) = \sum_{r=r_k}^{\infty} \left(\frac{1}{h_r} \sum_{i \in I_r} \left| \frac{x(i)}{2} \right| \right)^{p_r} < \delta_0. \quad (2.17)$$

From (2.14) and (2.16), we have

$$\sum_{r=r_k}^{\infty} \left(\frac{1}{h_r} \sum_{i \in I_r} \left| \frac{x(i) + x_{s_k}(i)}{2} \right| \right)^{p_r} = \varrho\left(\frac{u+v}{2}\right) \leq \varrho\left(\frac{u}{2}\right) + \frac{\lambda\eta}{4} \leq \frac{1-\lambda}{2}\varrho(u) + \frac{\lambda\eta}{4}. \quad (2.18)$$

By (2.13), (2.16), (2.18), and convexity of function $f(t) = |t|^{p_r}$, for all $r \in \mathbb{N}$, we have

$$\begin{aligned} \varrho\left(\frac{x + x_{s_k}}{2}\right) &= \sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{i \in I_r} \left| \frac{x(i) + x_{s_k}(i)}{2} \right| \right)^{p_r} \\ &= \sum_{r=1}^{r_k-1} \left(\frac{1}{h_r} \sum_{i \in I_r} \left| \frac{x(i) + x_{s_k}(i)}{2} \right| \right)^{p_r} + \sum_{r=r_k}^{\infty} \left(\frac{1}{h_r} \sum_{i \in I_r} \left| \frac{x(i) + x_{s_k}(i)}{2} \right| \right)^{p_r} \\ &\leq \frac{1}{2} \left(\sum_{r=1}^{r_k-1} \left(\frac{1}{h_r} \sum_{i \in I_r} |x(i)| \right)^{p_r} + \sum_{r=1}^{r_k-1} \left(\frac{1}{h_r} \sum_{i \in I_r} |x_{s_k}(i)| \right)^{p_r} \right) \end{aligned}$$

$$\begin{aligned}
 & + \sum_{r=r_k}^{\infty} \left(\frac{1}{h_r} \sum_{i \in I_r} \left| \frac{x_{s_k}(i)}{2} \right| \right)^{p_r} + \frac{\lambda \eta}{4} \\
 & \leq \frac{1}{2} \left(\sum_{r=1}^{r_k-1} \left(\frac{1}{h_r} \sum_{i \in I_r} |x(i)| \right)^{p_r} + \sum_{r=1}^{r_k-1} \left(\frac{1}{h_r} \sum_{i \in I_r} |x_{s_k}(i)| \right)^{p_r} \right) \\
 & \quad + \frac{1-\lambda}{2} \sum_{r=r_k}^{\infty} \left(\frac{1}{h_r} \sum_{i \in I_r} |x_{s_k}(i)| \right)^{p_r} + \frac{\lambda \eta}{4} \\
 & = \frac{1}{2} \sum_{r=1}^{r_k-1} \left(\frac{1}{h_r} \sum_{i \in I_r} |x(i)| \right)^{p_r} + \frac{1}{2} \sum_{r=1}^{r_k-1} \left(\frac{1}{h_r} \sum_{i \in I_r} |x_{s_k}(i)| \right)^{p_r} \\
 & \quad + \frac{1-\lambda}{2} \sum_{r=r_k}^{\infty} \left(\frac{1}{h_r} \sum_{i \in I_r} |x_{s_k}(i)| \right)^{p_r} + \frac{\lambda \eta}{4} \\
 & = \frac{1}{2} \sum_{r=1}^{r_k-1} \left(\frac{1}{h_r} \sum_{i \in I_r} |x(i)| \right)^{p_r} + \frac{1}{2} \sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{i \in I_r} |x_{s_k}(i)| \right)^{p_r} \\
 & \quad - \frac{\lambda}{2} \sum_{r=r_k}^{\infty} \left(\frac{1}{h_r} \sum_{i \in I_r} |x_{s_k}(i)| \right)^{p_r} + \frac{\lambda \eta}{4} \\
 & \leq \frac{1}{2} + \frac{1}{2} - \frac{\lambda \eta}{2} + \frac{\lambda \eta}{4} \\
 & = 1 - \frac{\lambda \eta}{4}.
 \end{aligned} \tag{2.19}$$

So it follows from (2.15) that

$$\left\| \frac{x + x_{s_k}}{2} \right\| \leq 1 - \delta. \tag{2.20}$$

Therefore, the space $l(p, \theta)$ has property (β) . □

By the facts presented in the introduction, following results are obtained directly from Theorem 2.4.

Corollary 2.5. *The space $l_p(\theta)$ has property (β) .*

Corollary 2.6. *The space $l(p, \theta)$ is nearly uniform convexity, has drop property, and is reflexive.*

Corollary 2.7. *The space $l(p, \theta)$ has property (UKK).*

Corollary 2.8 (see [24, Theorem 2.9]). *The space $l(p, \theta)$ has property (H).*

Corollary 2.9. *The space $l_p(\theta)$ is nearly uniform convexity, has drop property, and is reflexive.*

Corollary 2.10. *The space $l_p(\theta)$ has property (UKK) and (H).*

Theorem 2.11. *The space $l(p, \theta)$ has uniform Opial property.*

Proof. Take any $\varepsilon > 0$ and $x \in l(p, \theta)$ with $\|x\| \geq \varepsilon$. Let (x_n) be weakly null sequence in $S(l(p, \theta))$. By $\lim_{r \rightarrow \infty} \sup p_r < \infty$, that is, $q \in \Delta_2^s$, hence by Lemma 1.2 there exists $\delta \in (0, 1)$ independent of x such that $q(x) > \delta$. Also, by $q \in \Delta_2^s$, Lemma 1.1 asserts that there exists $\delta_1 \in (0, \delta)$ such that

$$|q(y+z) - q(y)| < \frac{\delta}{4}, \quad (2.21)$$

whenever $q(y) \leq 1$ and $q(z) \leq \delta_1$. Choose $r_0 \in \mathbb{N}$ such that

$$\sum_{r=r_0+1}^{\infty} \left(\frac{1}{h_r} \sum_{i \in I_r} |x(i)| \right)^{p_r} < \frac{\delta_1}{4}. \quad (2.22)$$

So, we have

$$\begin{aligned} \delta &< \sum_{r=1}^{r_0} \left(\frac{1}{h_r} \sum_{i \in I_r} |x(i)| \right)^{p_r} + \sum_{r=r_0+1}^{\infty} \left(\frac{1}{h_r} \sum_{i \in I_r} |x(i)| \right)^{p_r} \\ &\leq \sum_{r=1}^{r_0} \left(\frac{1}{h_r} \sum_{i \in I_r} |x(i)| \right)^{p_r} + \frac{\delta_1}{4}, \end{aligned} \quad (2.23)$$

which implies that

$$\sum_{r=1}^{r_0} \left(\frac{1}{h_r} \sum_{i \in I_r} |x(i)| \right)^{p_r} > \delta - \frac{\delta_1}{4} > \delta - \frac{\delta}{4} = \frac{3\delta}{4}. \quad (2.24)$$

Since $x_n \xrightarrow{w} 0$, then there exists $n_0 \in \mathbb{N}$ such that

$$\frac{3\delta}{4} \leq \sum_{r=1}^{r_0} \left(\frac{1}{h_r} \sum_{i \in I_r} |x_n(i) + x(i)| \right)^{p_r} \quad (2.25)$$

for all $n > n_0$, since weak convergence implies coordinatewise convergence. Again, by $x_n \xrightarrow{w} 0$, then there exists $n_1 \in \mathbb{N}$ such that

$$\|x_{n|_{k_0}}\| < 1 - \left(1 - \frac{\delta}{4}\right)^{1/M} \quad (2.26)$$

for all $n > n_1$ where k_0 is a minimal element in I_{r_0+1} and $M \in \mathbb{N}$ with $p_r \leq M$ for all $r \in \mathbb{N}$. Hence, by the triangle inequality of the norm, we get

$$\|x_{n|_{\mathbb{N}-k_0}}\| > \left(1 - \frac{\delta}{4}\right)^{1/M}. \quad (2.27)$$

It follows by the definition of $\|\cdot\|$ that we have

$$\begin{aligned} 1 &< \varrho \left(\frac{x_{n|_{\mathbb{N}-k_0}}}{(1 - (\delta/4))^{1/M}} \right) \\ &= \sum_{r=r_0+1}^{\infty} \left(\frac{(1/h_r) \sum_{i \in I_r} |x_n(i)|}{(1 - (\delta/4))^{1/M}} \right)^{p_r} \\ &\leq \left(\frac{1}{(1 - (\delta/4))^{1/M}} \right)^M \sum_{r=r_0+1}^{\infty} \left(\frac{1}{h_r} \sum_{i \in I_r} |x_n(i)| \right)^{p_r} \end{aligned} \quad (2.28)$$

Which implies that

$$\sum_{r=r_0+1}^{\infty} \left(\frac{1}{h_r} \sum_{i \in I_r} |x_n(i)| \right)^{p_r} > 1 - \frac{\delta}{4} \quad (2.29)$$

for all $n > n_1$. By inequality (2.21), (2.25), and (2.29), it yields for any $n > n_1$ that

$$\begin{aligned} \varrho(x_n + x) &= \sum_{r=1}^{r_0} \left(\frac{1}{h_r} \sum_{i \in I_r} |x_n(i) + x(i)| \right)^{p_r} + \sum_{r=r_0+1}^{\infty} \left(\frac{1}{h_r} \sum_{i \in I_r} |x_n(i) + x(i)| \right)^{p_r} \\ &\geq \frac{3\delta}{4} + \sum_{r=r_0+1}^{\infty} \left(\frac{1}{h_r} \sum_{i \in I_r} |x_n(i)| \right)^{p_r} - \frac{\delta}{4} \\ &\geq \frac{3\delta}{4} + \left(1 - \frac{\delta}{4}\right) - \frac{\delta}{4} \\ &\geq 1 + \frac{\delta}{4}. \end{aligned} \quad (2.30)$$

Since $\varrho \in \Delta_2^s$ and by Lemma 1.4 there exists τ depending on δ only such that $\|x_n + x\| \geq 1 + \tau$, which implies that $\lim_{n \rightarrow \infty} \inf \|x_n + x\| \geq 1 + \tau$, hence the prove is complete. \square

By the facts presented in the introduction and the reflexivity of $l(p, \theta)$, we get the following results.

Corollary 2.12. *The space $l_p(\theta)$ has uniform Opial property.*

Corollary 2.13. *The space $l(p, \theta)$ has property (L) and the fixed point property.*

Corollary 2.14. *The space $l_p(\theta)$ has property (L) and the fixed point property.*

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