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Research Article

Statistical Convergence of Sequences of Functions in Intuitionistic Fuzzy Normed Spaces

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The purpose of this work is to investigate types of convergence of sequences of functions in intuitionistic fuzzy normed spaces and some properties related with these concepts.

1. Introduction and Preliminaries

The fuzzy theory has emerged as one of the most active area of research in many branches of mathematics and engineering. This new theory was introduced by Zadeh [1] in 1965 and since then a large number of research papers have appeared by using the concept of fuzzy set/numbers and fuzzification of many classical theories have also been made. It has also very useful application in various fields, for example, population dynamics [2], chaos control [3], computer programming [4], nonlinear dynamical systems [5], fuzzy physics [6], fuzzy topology [7, 8], and so forth. The notion of intuitionistic fuzzy sets, a generalization of fuzzy sets, was introduced by Atanassov [9] in 1986 and later there has been much progress in the study of intuitionistic fuzzy sets by many authors including Mursaleen et al. [10], Mursaleen et al. [11], and Yılmaz [12]. Using the idea of intuitionistic fuzzy sets, Park [13] defined the notion of intuitionistic fuzzy metric spaces with the help of the continuous t-norms and the continuous t-conorms as a generalization of fuzzy metric spaces due to George and Veeramani [14]. Samanta and Jebril [15] introduced the definitions of intuitionistic fuzzy continuity and sequential intuitionistic fuzzy continuity and proved that they are equivalent.

A few of the algebraic and topological properties of intuitionistic fuzzy continuity and uniformly intuitionistic fuzzy continuity were investigated by Dinda and Samanta [16]. On the other hand, the Fast [17] introduced the concept of statistical convergence for real number sequences. Different types of statistical convergence of sequences of real functions and related notions were first studied in [18], and some important results and references on statistical convergence and function sequences can be found in [19–23].

In this paper, primarily following the line of [18], we define statistical convergence of sequences of functions in intuitionistic fuzzy normed space (IFNS for short), and we investigate some properties related with these concepts. To explain main problems of this work, we have to give some definitions and literature acknowledgment. In [24], Schweizer and Sklar introduced a continuous t-norm and a continuous t-conorm. Afterward Saadati and Park [8] introduced the following definitions by using concepts mentioned above.

We now first recall some basic notions of intuitionistic fuzzy normed spaces.

Definition 1.1 (see [8]). Let $*$ be a continuous t-norm, $\diamond$ a continuous t-conorm, and X a linear space over the intuitionistic fuzzy field ($\mathbb{R}$ or $\mathbb{C}$). If μ and ν are fuzzy sets on $X \times (0,\infty)$ satisfying the following conditions, the five-tuple $(X,\mu,\nu,*,\diamond)$ is said to be an IFNS and (μ,ν) is called an intuitionistic fuzzy norm (IFN for short). For every $x,y \in X$ and $s,t > 0$,

(i) $\mu(x,t) + \nu(x,t) \le 1$,

(ii) $\mu(x,t) > 0$,

(iii) $\mu(x,t) = 1 \Leftrightarrow x = 0$,

(iv) $\mu(ax,t) = \mu(x,t/|a|)$ for each $a \neq 0$,

(v) $\mu(x,t) * \mu(y,s) \le \mu(x+y,t+s)$,

(vi) $\mu(x,\cdot) : (0,\infty) \to [0,1]$ is continuous,

(vii) $\lim_{t\to\infty}\mu(x,t) = 1$ and $\lim_{t\to\infty}\mu(x,t) = 0$,

(viii) $\nu(x,t) < 1$,

(ix) $\nu(x,t) = 0 \Leftrightarrow x = 0$,

(x) $\nu(ax,t) = \nu(x,t/|a|)$ for each $a \neq 0$,

(xi) $\nu(x,t) \diamond \nu(y,s) \ge \nu(x+y,t+s)$,

(xii) $\nu(x,\cdot) : (0,\infty) \to [0,1]$ is continuous,

(xiii) $\lim_{t\to\infty}\nu(x,t) = 0$ and $\lim_{t\to\infty}\nu(x,t) = 1$.
For an IFNS, we further assume that $(X,\mu,\nu,*,\diamond)$ satisfy the following axiom: (see [16])

(xiv)

$$\left.\begin{aligned} a \diamond a &= a \\ a * a &= a \end{aligned}\right\} \quad \forall a \in [0,1]. \tag{1.1}$$

Definition 1.2 (see [8]). Let $(X,\mu,\nu,*,\diamond)$ be a intuitionistic fuzzy normed space. A subset A of X is said to be IF-bounded if there exist $t > 0$ and $0 < r < 1$ such that $\mu(x,t) > 1 - r$ and $\nu(x,t) < r$ for each $x \in A$.

Definition 1.3 (see [25]). Let $(X, \mu, \nu, *, \diamond)$ be a intuitionistic fuzzy metric space. Let A be any subset of X. Define

$$\phi(t) = \inf\{\mu(x, y, t) : x, y \in A\}, \qquad \psi(t) = \sup\{\nu(x, y, t) : x, y \in A\}, \tag{1.2}$$

(i) A is said to be q-bounded if $\lim_{t\to\infty}\phi(t) = 1$ and $\lim_{t\to\infty}\psi(t) = 0$, (ii) A is said to be semibounded if $\lim_{t\to\infty}\phi(t) = k$ and $\lim_{t\to\infty}\psi(t) = 1 - k$, $0 < k < 1$ (iii) A is said to be unbounded if $\lim_{t\to\infty}\phi(t) = 0$ and $\lim_{t\to\infty}\psi(t) = 1$.

Theorem 1.4 (see [25]). *Let $(X, \mu, \nu, *, \diamond)$ be a intuitionistic fuzzy metric space. A subset of X is IF bounded if and only if A is q-bounded or semibounded.*

Definition 1.5 (see [8]). Let $(X, \mu, \nu, *, \diamond)$ be an IFNS and (x_k) be a sequence in X. The sequence (x_k) is said to be convergent to $L \in X$ with respect to IFN (μ, ν) if for every $\varepsilon > 0$ and $t > 0$, there exists a positive integer $k_0(\varepsilon)$ such that $\mu(x_k - L, t) > 1 - \varepsilon$ and $\nu(x_k - L, t) < \varepsilon$ whenever $k > k_0$. In this case, we write $(\mu, \nu) - \lim x_k = L$ as $k \to \infty$.

Definition 1.6 (see [16]). Let $(X, \mu, \nu, *, \diamond)$ and $(Y, \mu', \nu', *, \diamond)$ be two IFNS. A mapping f from $(X, \mu, \nu, *, \diamond)$ to $(Y, \mu', \nu', *, \diamond)$ is said to be intuitionistic fuzzy continuous at $x_0 \in X$ if for any given $\varepsilon > 0$, there exist $\delta = \delta(a, \varepsilon)$, $\beta = \beta(a, \varepsilon) \in (0, 1)$ such that for all $x \in X$ and for all $a \in (0, 1)$,

$$\begin{gathered} \mu(x - x_0, \delta) > 1 - \beta \Longrightarrow \mu'(f(x) - f(x_0), \varepsilon) > 1 - a, \\ \nu(x - x_0, \delta) < \beta \Longrightarrow \nu'(f(x) - f(x_0), \varepsilon) < a. \end{gathered} \tag{1.3}$$

Definition 1.7 (see [16]). Let $f_k : (X, \mu, \nu, *, \diamond) \to (Y, \mu', \nu', *, \diamond)$ be a sequence of functions. The sequence (f_k) is said to be pointwise intuitionistic fuzzy convergent on X to a function f with respect to (μ', ν') if for each $x \in X$, the sequence $(f_k(x))$ is convergent to $f(x)$ with respect to (μ', ν').

Definition 1.8 (see [16]). Let $f_k : (X, \mu, \nu, *, \diamond) \to (Y, \mu', \nu', *, \diamond)$ be a sequence of functions. The sequence (f_k) is said to be uniformly intuitionistic fuzzy convergent on X to a function f with respect to (μ, ν), if given $0 < r < 1$, $t > 0$, there exist a positive integer $k_0 = k_0(r, t)$ such that $\forall x \in X$ and $\forall k > k_0$,

$$\mu'(f_k(x) - f(x), t) > 1 - r, \qquad \nu'(f_k(x) - f(x), t) < r. \tag{1.4}$$

Now, we recall the notion of the statistical convergence of sequences in intuitionistic fuzzy normed spaces.

Definition 1.9 (see [26]). Let $K \subset \mathbb{N}$ and $K_n = \{k \in K : k \leq n\}$. Then the asymptotic density is defined by $\delta(K) = \lim_{n\to\infty}(|K_n|/n)$, where $|K_n|$ denotes the cardinality of K_n.

Definition 1.10. Let A be subset of $\mathbb{N}$. If a property $P(k)$ holds for all $k \in A$ with $\delta(A) = 1$, we say that P holds for almost all $k(a \cdot a \cdot k)$.

Definition 1.11 (see [26]). A sequence $x = (x_k)$ is said to be statistically convergent to the number L, or in short $st - \lim x = L$, if for every $\varepsilon > 0$, the set $K(\varepsilon)$ has asymptotic density zero, where

$$K(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}, \tag{1.5}$$

that is,

$$|x_k - L| < \varepsilon \quad a \cdot a \cdot k. \tag{1.6}$$

Definition 1.12 (see [23]). Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then, sequence (x_k) is said to be statistically convergent to $L \in X$ with respect to IFN (μ, ν) provided that for every $\varepsilon > 0$ and $t > 0$,

$$\delta\left(\left\{k \in \mathbb{N} : \mu(x_k - L, t) \leq 1 - \varepsilon \text{ or } \nu(x_k - L, t) \geq \varepsilon\right\}\right) = 0, \tag{1.7}$$

or equivalently

$$\lim_{n \to \infty} \frac{1}{n} \left|\left\{k \leq n : \mu(x_k - L, t) \leq 1 - \varepsilon \text{ or } \nu(x_k - L, t) \geq \varepsilon\right\}\right| = 0. \tag{1.8}$$

In this case, we write $st_{\mu,\nu} - \lim(x_k) = L$.

Definition 1.13 (see [23]). Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Sequence (x_k) is said to be statistically Cauchy with respect to IFN (μ, ν) provided that for every $\varepsilon > 0$ and $t > 0$, there exists a number $m \in \mathbb{N}$ satisfying

$$\delta\left(\left\{k \in \mathbb{N} : \mu(x_k - x_m, t) \leq 1 - \varepsilon \text{ or } \nu(x_k - x_m, t) \geq \varepsilon\right\}\right) = 0. \tag{1.9}$$

2. Statistical Convergence of Sequences of Functions in Intuitionistic Fuzzy Normed Spaces

In this section, we define pointwise statistically and uniformly statistically convergent sequences of functions in intuitionistic fuzzy normed spaces. Also, we give the statistical analog of the Cauchy convergence criterion for pointwise and uniformly statistical convergent in intuitionistic fuzzy normed space. Finally, we prove that uniformly statistical convergence preserves continuity.

Definition 2.1. Let $(X, \mu, \nu, *, \diamond)$ and $(Y, \mu', \nu', *, \diamond)$ be two IFNS and $f_k : (X, \mu, \nu, *, \diamond) \to (Y, \mu', \nu', *, \diamond)$ a sequence of functions. We say that a sequence (f_k) pointwise statistically converges to f with respect to intuitionistic fuzzy norm (μ', ν') if $f_k(x)$ statistically converges to $f(x)$ for each $x \in X$ with respect to intuitionistic fuzzy norm (μ', ν') and we write $st_{\mu,\nu} - f_k \to f$.

Theorem 2.2. *Let $f_k : (X, \mu, \nu, *, \diamond) \to (Y, \mu', \nu', *, \diamond)$ be a sequence of functions. If (f_k) is pointwise intuitionistic fuzzy convergent on X with respect to (μ, ν), then $st_{\mu,\nu} - f_k \to f$. But the converse of this is not true.*

Proof. Let (f_k) be pointwise intuitionistic fuzzy convergent on X. In this case the sequence $(f_k(x))$ is convergent with respect to (μ', ν') for each $x \in X$. Then for each $\varepsilon > 0$ and $t > 0$, there is number $k_0(\varepsilon) \in \mathbb{N}$ such that

$$\mu'(f_k(x) - f(x), t) > 1 - \varepsilon, \qquad \nu'(f_k(x) - f(x), t) < \varepsilon \tag{2.1}$$

for all $k \geq k_0$ and for each $x \in X$. Hence for each $x \in X$ the set

$$\{k \in \mathbb{N} : \mu'(f_k(x) - f(x), t) \leq 1 - \varepsilon \text{ or } \nu'(f_k(x) - f(x), t) \geq \varepsilon\} \tag{2.2}$$

has finite numbers of terms. Since density of finite subset of $\mathbb{N}$ is 0, hence

$$\delta(\{k \in \mathbb{N} : \mu'(f_k(x) - f(x), t) \leq 1 - \varepsilon \text{ or } \nu'(f_k(x) - f(x), t) \geq \varepsilon\}) = 0. \tag{2.3}$$

That is, $st_{\mu,\nu} - f_k \rightarrow f$. □

Example 2.3. Let $(\mathbb{R}, |\cdot|)$ denote the space of real numbers with the usual norm, and let $a * b = a \cdot b$ and $a \diamond b = \min\{a + b, 1\}$ for $a, b \in [0, 1]$. For all $x \in \mathbb{R}$ and every $t > 0$, we consider

$$\mu(x, t) = \frac{t}{t + |x|}, \qquad \nu(x, t) = \frac{|x|}{t + |x|}. \tag{2.4}$$

In this case, $(\mathbb{R}, \mu, \nu, *, \diamond)$ is an IFNS (also, $([0, 1], \mu, \nu, *, \diamond)$ is intuitionistic fuzzy normed space.). Let $f_k : [0, 1] \rightarrow \mathbb{R}$ be a sequence of functions whose terms are given by

$$f_k(x) = \begin{cases} x^{k^2} + 1 & \text{if } k = m^2 \quad (m \in \mathbb{N}),\ x \in \left[0, \frac{1}{2}\right), \\ 0 & \text{if } k \neq m^2 \quad (m \in \mathbb{N}),\ x \in \left[0, \frac{1}{2}\right), \\ 0 & \text{if } k = m^2 \quad (m \in \mathbb{N}),\ x \in \left[\frac{1}{2}, 1\right), \\ x^k + \frac{1}{2} & \text{if } k \neq m^2 \quad (m \in \mathbb{N}),\ x \in \left[\frac{1}{2}, 1\right), \\ 2 & \text{if } x = 1. \end{cases} \tag{2.5}$$

Then sequence (f_k) is pointwise statistically intuitionistic convergent on $[0,1]$ with respect to (μ,ν). Indeed, for $x\in[0,1/2)$, since

$$\begin{aligned}
K_n(\varepsilon,t) &= \{k\le n:\mu(f_k(x)-f(x),t)\le 1-\varepsilon \text{ or } \nu(f_k(x)-f(x),t)\ge\varepsilon\},\\
K_n(\varepsilon,t) &= \left\{k\le n:\frac{t}{t+|f_k(x)-0|}\le 1-\varepsilon \text{ or } \frac{|f_k(x)-0|}{t+|f_k(x)-0|}\ge\varepsilon\right\}\\
&= \left\{k\le n:|f_k(x)|\ge\frac{\varepsilon t}{1-\varepsilon}\right\}\\
&= \left\{k\le n:f_k(x)=x^{k^2}+1\right\}\\
&= \left\{k\le n:k=m^2,\ m\in\mathbb{N}\right\},
\end{aligned}\tag{2.6}$$

we have

$$\delta(K_n(\varepsilon,t))=\frac{1}{n}\left|\left\{k\le n:k=m^2,\ m\in\mathbb{N}\right\}\right|\le\frac{\sqrt{n}}{n},\tag{2.7}$$

which yields $\lim_{n\to\infty}\delta(K_n(\varepsilon,t))=0$. Thus, for each $x\in[0,1/2)$, sequence (f_k) is statistically convergent to 0 with respect to intuitionistic fuzzy norm (μ,ν).

If we take $x\in[1/2,1)$, then we have

$$\begin{aligned}
K'_n(\varepsilon,t) &= \left\{k\le n:\frac{t}{t+|f_k(x)-(1/2)|}\le 1-\varepsilon \text{ or } \frac{|f_k(x)-(1/2)|}{t+|f_k(x)-(1/2)|}\ge\varepsilon\right\}\\
&= \left\{k\le n:\left|f_k(x)-\frac{1}{2}\right|\ge\frac{\varepsilon t}{1-\varepsilon}\right\}\\
&= \left\{k\le n:k=m^2,\left|f_k(x)-\frac{1}{2}\right|\ge\frac{\varepsilon t}{1-\varepsilon}\right\}\cup\left\{k\le n:k\ne m^2,\left|f_k(x)-\frac{1}{2}\right|\ge\frac{\varepsilon t}{1-\varepsilon}\right\}\\
&= \left\{k\le n:k=m^2\right\}\cup\left\{k\le n:k\ne m^2:\left|x^k+\frac{1}{2}-\frac{1}{2}\right|\ge\frac{\varepsilon t}{1-\varepsilon}\right\}\\
&= \left\{k\le n:k=m^2\right\}\cup\left\{k\le n:k\ne m^2:\left|x^k\right|\ge\frac{\varepsilon t}{1-\varepsilon}\right\}.
\end{aligned}\tag{2.8}$$

Therefore, density of $K'_n(\varepsilon,t)$ is 0 and for each $x\in[1/2,1)$, sequence (f_k) is statistically convergent to $1/2$ with respect to IFN (μ,ν). If we take $x=1$, it can be seen easily that (f_k) is intuitionistic fuzzy convergent to 2. Hence (f_k) is intuitionistic fuzzy statistically convergent to 2. That is

$$st_{\mu,\nu} - f_k(x) = \begin{cases} 0, & x \in \left[0, \dfrac{1}{2}\right) \\ \dfrac{1}{2}, & x \in \left[\dfrac{1}{2}, 1\right) \\ 2, & x = 1. \end{cases} \tag{2.9}$$

Since $f_k(x)$ is statistically convergent to different points with respect to intuitionistic fuzzy norm (μ, ν) for each $x \in X$, it can be seen that (f_k) is pointwise statistically intuitionistic fuzzy convergent on $[0, 1]$.

Lemma 2.4. *Let $f_k : (X, \mu, \nu, *, \diamond) \to (Y, \mu', \nu', *, \diamond)$ be sequence of functions. Then the following statements are equivalent:*

(i) $st_{\mu,\nu} - f_k \to f$,

(ii) $\delta\{k \in \mathbb{N} : \mu'(f_k(x) - f(x), t) \leq 1 - \varepsilon\} = \delta\{k \in \mathbb{N} : \nu'(f_k(x) - f(x), t) \geq \varepsilon\} = 0$ *for each* $x \in X$, *for each* $\varepsilon > 0$ *and* $t > 0$,

(iii) $\delta\{k \in \mathbb{N} : \mu'(f_k(x) - f(x), t) > 1 - \varepsilon$ *and* $\nu'(f_k(x) - f(x), t) < \varepsilon\} = 1$ *for each* $x \in X$, *for each* $\varepsilon > 0$ *and* $t > 0$,

(iv) $st - \lim \mu'(f_k(x) - f(x), t) = 1$ *and* $st - \lim \nu'(f_k(x) - f(x), t) = 0$ *for each* $x \in X$ *and* $t > 0$.

Theorem 2.5. *Let (f_k) and (g_k) be two sequences of functions from $(X, \mu, \nu, *, \diamond)$ to $(Y, \mu', \nu', *, \diamond)$. If $st_{\mu,\nu} - f_k \to f$ and $st_{\mu,\nu} - g_k \to g$, then $st_{\mu,\nu} - (\alpha f_k + \beta g_k) \to \alpha f + \beta g$ where $\alpha, \beta \in$ ($\mathbb{R}$ or $\mathbb{C}$).*

Proof. The proof is clear for $\alpha = 0$ and $\beta = 0$. Now let $\alpha \neq 0$ and $\beta \neq 0$. Since $st_{\mu,\nu} - f_k \to f$ and $st_{\mu,\nu} - g_k \to g$, for each $x \in X$, if we define

$$\begin{aligned} A_1 &= \left\{k \in \mathbb{N} : \mu'\left(f_k(x) - f(x), \frac{t}{2|\alpha|}\right) \leq 1 - \varepsilon \text{ or } \nu'\left(f_k(x) - f(x), \frac{t}{2|\alpha|}\right) \geq \varepsilon\right\}, \\ A_2 &= \left\{k \in \mathbb{N} : \mu'\left(g_k(x) - g(x), \frac{t}{2|\beta|}\right) \leq 1 - \varepsilon \text{ or } \nu'\left(g_k(x) - g(x), \frac{t}{2|\beta|}\right) \geq \varepsilon\right\}, \end{aligned} \tag{2.10}$$

then

$$\delta(A_1) = 0\,, \qquad \delta(A_2) = 0. \tag{2.11}$$

Since $\delta(A_1) = 0$ and $\delta(A_2) = 0$, if we state A by $(A_1 \cup A_2)$ then

$$\delta(A) = 0. \tag{2.12}$$

Hence, $A_1 \cup A_2 \neq \mathbb{N}$ and there exists $\exists m \in \mathbb{N}$ such that

$$\mu'\left(f_m(x) - f(x), \frac{t}{2|\alpha|}\right) > 1 - \varepsilon, \qquad \nu'\left(f_m(x) - f(x), \frac{t}{2|\alpha|}\right) < \varepsilon,$$
$$\mu'\left(g_m(x) - g(x), \frac{t}{2|\beta|}\right) > 1 - \varepsilon, \qquad \nu'\left(g_m(x) - g(x), \frac{t}{2|\beta|}\right) < \varepsilon. \tag{2.13}$$

Let

$$B = \{k \in \mathbb{N} : \mu'((\alpha f_k + \beta g_k)(x) - (\alpha f(x) + \beta g(x)), t) > 1 - \varepsilon,$$
$$\nu'((\alpha f_k + \beta g_k)(x) - (\alpha f(x) + \beta g(x)), t) < \varepsilon\}. \tag{2.14}$$

We will show that for each $x \in X$

$$A^c \subset B. \tag{2.15}$$

Let $m \in A^c$. In this case,

$$\mu'\left(f_m(x) - f(x), \frac{t}{2|\alpha|}\right) > 1 - \varepsilon, \qquad \nu'\left(f_m(x) - f(x), \frac{t}{2|\alpha|}\right) < \varepsilon,$$
$$\mu'\left(g_m(x) - g(x), \frac{t}{2|\beta|}\right) > 1 - \varepsilon, \qquad \nu'\left(g_m(x) - g(x), \frac{t}{2|\beta|}\right) < \varepsilon. \tag{2.16}$$

Using those above, we have

$$\begin{aligned}
\mu'((\alpha f_m + \beta g_m)(x) - (\alpha f + \beta g)(x), t) &\geq \mu'\left(\alpha f_m(x) - \alpha f(x), \frac{t}{2}\right) * \mu'\left(\beta g_m(x) - \beta g(x), \frac{t}{2}\right) \\
&= \mu'\left(f_m(x) - f(x), \frac{t}{2|\alpha|}\right) * \mu'\left(g_m(x) - g(x), \frac{t}{2|\beta|}\right) \\
&> (1 - \varepsilon) * (1 - \varepsilon) \\
&= (1 - \varepsilon), \\
\nu'((\alpha f_m + \beta g_m)(x) - (\alpha f + \beta g)(x), t) &\leq \nu'\left(\alpha f_m(x) - \alpha f(x), \frac{t}{2}\right) * \nu'\left(\beta g_m(x) - \beta g(x), \frac{t}{2}\right) \\
&= \nu'\left(f_m(x) - f(x), \frac{t}{2|\alpha|}\right) * \nu'\left(g_m(x) - g(x), \frac{t}{2|\beta|}\right) \\
&< \varepsilon \diamond \varepsilon \\
&= \varepsilon.
\end{aligned} \tag{2.17}$$

This implies that

$$A^c \subset B. \tag{2.18}$$

Since $B^c \subset A$ and $\delta(A) = 0$, hence

$$\delta(B^c) = 0, \tag{2.19}$$

that is

$$\begin{gathered}\delta(\{k \in \mathbb{N} : \mu'((\alpha f_k + \beta g_k)(x) - (\alpha f + \beta g)(x), t) \leq 1 - \varepsilon, \\ \nu'((\alpha f_k + \beta g_k)(x) - (\alpha f + \beta g)(x), t) \geq \varepsilon\}) = 0\end{gathered} \tag{2.20}$$

which means

$$st_{\mu,\nu} - (\alpha f_k + \beta g_k) \longrightarrow \alpha f + \beta g. \tag{2.21}$$

□

Definition 2.6. Let $f_k : (X, \mu, \nu, *, \diamond) \to (Y, \mu', \nu', *, \diamond)$ be a sequence of functions. The sequence (f_k) is a pointwise statistically Cauchy sequence in IFNS provided that for each $\varepsilon > 0$ and $t > 0$ there exists $N = N(\varepsilon, t, x)$ such that

$$\delta(\{k \in \mathbb{N} : \mu'(f_k(x) - f_N(x), t) \leq 1 - \varepsilon \text{ or } \nu'(f_k(x) - f_N(x), t) \geq \varepsilon \text{ for each } x \in X\}) = 0. \tag{2.22}$$

That is, there exists a number $N = N(\varepsilon, t, x)$ for each $x \in X$ such that

$$\mu'(f_k(x) - f_N(x), t) > 1 - \varepsilon, \; \nu'(f_k(x) - f_N(x), t) < \varepsilon \quad \text{for } a \cdot a \cdot k. \tag{2.23}$$

Theorem 2.7. *Let $f_k : (X, \mu, \nu, *, \diamond) \to (Y, \mu', \nu', *, \diamond)$ be a sequence of functions. If (f_k) is a pointwise statistically convergent sequence with respect to intuitionistic fuzzy norm (μ, ν), then (f_k) is a pointwise statistically Cauchy sequence with respect to intuitionistic fuzzy norm (μ, ν).*

Proof. Suppose that $st_{\mu,\nu} - f_k \to f$ and let $\varepsilon > 0$, $t > 0$. For given each $\varepsilon > 0$, choose $s > 0$ such that $(1 - \varepsilon) * (1 - \varepsilon) > 1 - s$ and $\varepsilon \diamond \varepsilon < s$. If we state, respectively, $A_x(\varepsilon, t)$ and $A_x^c(\varepsilon, t)$ by

$$\begin{gathered}\left\{k \in \mathbb{N} : \mu'\left(f_k(x) - f(x), \frac{t}{2}\right) \leq 1 - \varepsilon \text{ or } \nu'\left(f_k(x) - f(x), \frac{t}{2}\right) \geq \varepsilon\right\}, \\ \left\{k \in \mathbb{N} : \mu'\left(f_k(x) - f(x), \frac{t}{2}\right) > 1 - \varepsilon, \; \nu'\left(f_k(x) - f(x), \frac{t}{2}\right) < \varepsilon\right\},\end{gathered} \tag{2.24}$$

for each $x \in X$. Then, we have

$$\delta(A_x(\varepsilon, t)) = 0, \tag{2.25}$$

which implies that

$$\delta(A_x^c(\varepsilon,t)) = 1. \tag{2.26}$$

Let $N \in A_x^c(\varepsilon,t)$. Then

$$\mu'\left(f_N(x) - f(x), \frac{t}{2}\right) > 1-\varepsilon, \qquad \nu'\left(f_N(x) - f(x), \frac{t}{2}\right) < \varepsilon. \tag{2.27}$$

We want to show that there exists a number $N = N(x,\varepsilon,t)$ such that

$$\delta(\{k \in \mathbb{N} : \mu'(f_k(x) - f_N(x), t) \leq 1-s \text{ or } \nu'(f_k(x) - f_N(x), t) \geq s \text{ for each } x \in X\}) = 0. \tag{2.28}$$

Therefore, define for each $x \in X$,

$$B_x(\varepsilon,t) = \{k \in \mathbb{N} : \mu'(f_k(x) - f_N(x), t) \leq 1-s \text{ or } \nu'(f_k(x) - f_N(x), t) \geq s\}. \tag{2.29}$$

We have to show that

$$B_x(\varepsilon,t) \subset A_x(\varepsilon,t). \tag{2.30}$$

Suppose that

$$B_x(\varepsilon,t) \not\subseteq A_x(\varepsilon,t). \tag{2.31}$$

In this case, $B_x(\varepsilon,t)$ has at least one different element which $A_x(\varepsilon,t)$ does not has. Let $k \in B_x(\varepsilon,t) \setminus A_x(\varepsilon,t)$. Then we have

$$\mu'(f_k(x) - f_N(x), t) \leq 1-s, \qquad \mu'\left(f_k(x) - f(x), \frac{t}{2}\right) > 1-\varepsilon, \tag{2.32}$$

in particularly $\mu'(f_N(x) - f(x), t/2) > 1-\varepsilon$. In this case,

$$\begin{aligned} 1-s &\geq \mu'(f_k(x) - f_N(x), t) \geq \mu'\left(f_k(x) - f(x), \frac{t}{2}\right) * \mu'\left(f_N(x) - f(x), \frac{t}{2}\right) \\ &\geq (1-\varepsilon) * (1-\varepsilon) > 1-s, \end{aligned} \tag{2.33}$$

which is not possible. On the other hand

$$\nu'(f_k(x) - f_N(x), t) \geq s, \qquad \nu'(f_k(x) - f(x), t) < \varepsilon, \tag{2.34}$$

in particularly $\nu'(f_N(x) - f(x), t) < \varepsilon$. In this case,

$$s \leq \nu'(f_k(x) - f_N(x), t) \leq \nu'\left(f_k(x) - f(x), \frac{t}{2}\right) \diamond \nu'\left(f_N(x) - f(x), \frac{t}{2}\right) < \varepsilon \diamond \varepsilon < s \tag{2.35}$$

which is not possible. Hence $B_x(\varepsilon, t) \subset A_x(\varepsilon, t)$. Therefore, by $\delta(A_x(\varepsilon, t)) = 0$, $\delta(B_x(\varepsilon, t)) = 0$. That is, (f_k) is a pointwise statistical Cauchy sequence with respect to intuitionistic fuzzy norm (μ, ν). □

Afterward this step, we introduce a uniformly statistical convergence of sequences of function in an IFNS. To do this, we need the following definition

Definition 2.8. Let $(X, \mu, \nu, *, \diamond)$ and $(Y, \mu', \nu', *, \diamond)$ be two intuitionistic fuzzy normed linear space over the same field IF and $f_k : (X, \mu, \nu, *, \diamond) \to (Y, \mu', \nu', *, \diamond)$ be a sequence of functions. (f_k) converges uniform statistically to f with respect to $(\mu, \nu) \Leftrightarrow \forall \varepsilon > 0$, $\exists M \subset \mathbb{N}$, $\delta(M) = 1$ and $\exists k_0 = k_0(\varepsilon, t) \in M \ni \forall k > k_0$ and $k \in M$ and $\forall x \in X$,

$$\mu'(f_k(x) - f(x), t) > 1 - \varepsilon, \qquad \nu'(f_k(x) - f(x), t) < \varepsilon. \tag{2.36}$$

Lemma 2.9. *Let $f_k : (X, \mu, \nu, *, \diamond) \to (Y, \mu', \nu', *, \diamond)$ be a sequence of functions. Then the following statements are equivalent:*

(i) $st_{\mu,\nu} - f_k \rightrightarrows f$.

(ii) $\delta\{k \in \mathbb{N} : \mu'(f_k(x) - f(x), t) \leq 1 - \varepsilon\} = \delta\{k \in \mathbb{N} : \nu'(f_k(x) - f(x), t) \geq \varepsilon\} = 0$ *for all* $x \in X$, *for every* $\varepsilon > 0$ *and* $t > 0$.

(iii) $\delta\{k \in \mathbb{N} : \mu'(f_k(x) - f(x), t) > 1 - \varepsilon$ *and* $\nu'(f_k(x) - f(x), t) < \varepsilon\} = 1$ *for all* $x \in X$, *for every* $\varepsilon > 0$ *and* $t > 0$.

(iv) $st - \lim \mu'(f_k(x) - f(x), t) = 1$ *and* $st - \lim \nu'(f_k(x) - f(x), t) = 0$ *for all* $x \in X$ *and* $t > 0$.

Proposition 2.10. *Let the sequence (f_k) and f be bounded functions from $(X, \mu, \nu, *, \diamond)$ to $(Y, \mu', \nu', *, \diamond)$. f_k is intuitionistic fuzzy uniformly statistically convergent to f if and only if*

$$st - \lim \inf \mu'(f_k(x) - f(x), t) = 1, \qquad st - \lim \sup \nu'(f_k(x) - f(x), t) = 0, \tag{2.37}$$

where the supremum and infimum are taken over all $x \in X$.

Proof. Suppose that $st_{\mu,\nu} - f_k \rightrightarrows f$ on X. Since (f_k) and f are bounded in $(X, \mu, \nu, *, \diamond)$ for each $k \in \mathbb{N}$, by using Definition 1.3 and Theorem 1.4, we have $\inf \mu'(f_k(x) - f(x), t) = r_k$ and $\sup \nu'(f_k(x) - f(x), t) = 1 - r_k$ for each $k \in \mathbb{N}$ and for each $t > 0$, $(0 < r_k < 1)$. By using (iv) of Lemma 2.9, we get

$$st - \lim \inf \mu'(f_k(x) - f(x), t) = 1, \qquad st - \lim \sup \nu'(f_k(x) - f(x), t) = 0, \tag{2.38}$$

where the supremum and infimum are taken over all $x \in X$.

Conversely, suppose that

$$st - \lim \inf \mu'(f_k(x) - f(x), t) = 1, \qquad st - \lim \sup \nu'(f_k(x) - f(x), t) = 0, \tag{2.39}$$

where the supremum and infimum are taken over all $x \in X$. Since

$$\inf \mu'(f_k(x) - f(x), t) \in (0, 1], \qquad \sup \nu'(f_k(x) - f(x), t) \in [0, 1), \tag{2.40}$$

from definition statistical convergence, for $a \cdot a \cdot k$, for every $\varepsilon > 0$ and $t > 0$

$$\left|\inf \mu'(f_k(x) - f(x), t) - 1\right| < \varepsilon, \qquad \left|\sup \nu'(f_k(x) - f(x), t) - 0\right| < \varepsilon. \tag{2.41}$$

For all $x \in X$

$$\begin{aligned}
\mu'(f_k(x) - f(x), t) \geq{} & \inf \mu'(f_k(x) - f(x), t) \\
& \Longrightarrow -\mu'(f_k(x) - f(x), t) < -\inf \mu'(f_k(x) - f(x), t) \\
& \Longrightarrow 1 - \mu'(f_k(x) - f(x), t) < 1 - \inf \mu'(f_k(x) - f(x), t) \\
\text{since } \mu'(f_k(x) - f(x), t) \in{} & (0, 1] \ \forall \ x \in X \\
& \Longrightarrow \left|1 - \mu'(f_k(x) - f(x), t)\right| < \left|1 - \inf \mu'(f_k(x) - f(x), t)\right| \\
& \Longrightarrow \left|1 - \mu'(f_k(x) - f(x), t)\right| < \left|1 - \inf \mu'(f_k(x) - f(x), t)\right| < \varepsilon \\
& \Longrightarrow \left|1 - \mu'(f_k(x) - f(x), t)\right| < \varepsilon \quad a \cdot a \cdot k, \\
\nu'(f_k(x) - f(x), t) \leq{} & \sup \nu'(f_k(x) - f(x), t) \\
& \Longrightarrow \left|\nu'(f_k(x) - f(x), t) - 0\right| \leq \left|\sup \nu'(f_k(x) - f(x), t) - 0\right| < \varepsilon \\
& \Longrightarrow \left|\nu'(f_k(x) - f(x), t) - 0\right| < \varepsilon \quad a \cdot a \cdot k.
\end{aligned} \tag{2.42}$$

Therefore, $st - \lim \mu'(f_k(x) - f(x), t) = 1$ and $st - \lim \nu'(f_k(x) - f(x), t) = 0$ for all $x \in X$ and $t > 0$. From Lemma 2.9, we get $st_{\mu,\nu} - f_k \rightrightarrows f$. □

Example 2.11. Let $(\mathbb{R}, \mu, \nu, *, \diamond)$ be as Example 2.3. Consider $f_k : [0, 1) \to \mathbb{R}$ be sequence of functions whose terms are given by

$$f_k = \begin{cases} x^k + 1, & \text{if } k \neq m^2 \quad (m \in \mathbb{N}) \\ 2, & \text{otherwise.} \end{cases} \tag{2.43}$$

Then, for every $0 < \varepsilon < 1$ and for every $t > 0$, we define

$$K_n = \{k \leq n : \mu(f_k(x) - f(x), t) \leq 1 - \varepsilon \text{ or } \nu(f_k(x) - f(x), t) \geq \varepsilon\}. \tag{2.44}$$

For all $x \in X$, we have

$$\delta(K_n) = \delta\left(\left\{k \leq n : \mu\left(f_k(x) - 1, t\right) \leq 1 - \varepsilon \text{ or } \nu\left(f_k(x) - 1, t\right) \geq \varepsilon\right\}\right) = 0. \tag{2.45}$$

Since

$$st_{\mu,\nu} - f_k(x) = 1, \tag{2.46}$$

for all $x \in X$, the sequence (f_k) is uniformly statistically intuitionistic fuzzy convergent to 1 on $[0,1)$.

Example 2.12. Let $(\mathbb{R}, \mu, \nu, *, \diamond)$ be as Example 2.3. Consider $f_k : [-1,1] \to \mathbb{R}$ be sequence of functions whose terms are given by

$$f_k = \begin{cases} \left(x - \dfrac{1}{k}\right)^2, & \text{if } k \neq m^2 \quad (m \in \mathbb{N}) \\ 2, & \text{otherwise.} \end{cases} \tag{2.47}$$

Since $st_{\mu,\nu} - f_k(x) = x^2$ for all $x \in [-1,1], f_k$ is uniformly statistically intuitionistic fuzzy convergent to x^2 on $[-1,1]$. We can show this using Proposition 2.10 as the following, since f_k is bounded functions sequence on $[-1,1]$.

We want to find

$$\begin{aligned} \lim_{k\to\infty} \inf_{x\in[-1,1]} \mu\left(f_k(x) - f(x), t\right) &= \frac{t}{t + \left|-(2x/k) + (1/k^2)\right|}, \\ \lim_{k\to\infty} \sup_{x\in[-1,1]} \nu\left(f_k(x) - f(x), t\right) &= \frac{\left|-(2x/k) + 1/k^2\right|}{t + \left|-(2x/k) + (1/k^2)\right|}. \end{aligned} \tag{2.48}$$

Firstly, we need to find $\lim_{k\to\infty} \inf \mu(f_k(x) - f(x), t)$ and $\lim_{k\to\infty} \sup \nu(f_k(x) - f(x), t)$ for over all $x \in [-1, 1/2k]$ and for over all $x \in (1/2k, 1]$, respectively. In case of $x \in [-1, 1/2k]$, we have $-(2x/k) + (1/k^2) \geq 0$. Since

$$\begin{aligned} 0 \leq -\frac{2x}{k} + \frac{1}{k^2} \leq \frac{2}{k} + \frac{1}{k^2} &\Longrightarrow t \leq t - \frac{2x}{k} + \frac{1}{k^2} \leq t + \frac{2}{k} + \frac{1}{k^2} \\ &\Longrightarrow \frac{1}{t + (2/k) + (1/k^2)} \leq \frac{1}{t - (2x/k) + (1/k^2)} \leq \frac{1}{t} \\ &\Longrightarrow \frac{t}{t + (2/k) + (1/k^2)} \leq \frac{t}{t - (2x/k) + (1/k^2)} \leq 1, \end{aligned} \tag{2.49}$$

we get

$$\begin{aligned} \lim_{k\to\infty} \inf_{x\in[-1,1/2k]} \mu\left(f_k(x) - x^2, t\right) &= \lim_{k\to\infty} \inf_{x\in[-1,1/2k]} \frac{t}{t - (2x/k) + (1/k^2)} \\ &= \lim_{k\to\infty} \frac{t}{t + (2/k) + (1/k^2)} = 1. \end{aligned} \tag{2.50}$$

On the other hand, we have

$$\frac{-(2x/k)+\left(1/k^2\right)}{t-(2x/k)+\left(1/k^2\right)} \leq \frac{(2/k)+\left(1/k^2\right)}{t}, \tag{2.51}$$

and so

$$\begin{aligned}\lim_{k\to\infty} \sup_{x\in[-1,1/2k]} \nu\left(f_k(x)-x^2,t\right) &= \lim_{k\to\infty} \sup_{x\in[-1,1/2k]} \frac{-(2x/k)+\left(1/k^2\right)}{t-(2x/k)+\left(1/k^2\right)} \\ &= \lim_{k\to\infty} \frac{(2/k)+\left(1/k^2\right)}{t} = 0.\end{aligned} \tag{2.52}$$

In case of $x \in (1/2k, 1]$, we have $-(2x/k)+(1/k^2) \ < 0$. Since

$$\begin{aligned}-\frac{2}{k}+\frac{1}{k^2} < -\frac{2x}{k}+\frac{1}{k^2} \leq 0 &\Longrightarrow 0 \leq \frac{2x}{k}-\frac{1}{k^2} < \frac{2}{k}-\frac{1}{k^2} \\ &\Longrightarrow t < t+\frac{2x}{k}-\frac{1}{k^2} \leq t+\frac{2}{k}-\frac{1}{k^2} \\ &\Longrightarrow \frac{1}{t+(2/k)-\left(1/k^2\right)} < \frac{1}{t+(2x/k)-\left(1/k^2\right)} \leq \frac{1}{t} \\ &\Longrightarrow \frac{t}{t+(2/k)-\left(1/k^2\right)} \leq \frac{t}{t+(2x/k)-\left(1/k^2\right)} < 1,\end{aligned} \tag{2.53}$$

we get

$$\begin{aligned}\lim_{k\to\infty} \inf_{x\in(1/2k,1]} \mu\left(f_k(x)-x^2,t\right) &= \lim_{k\to\infty} \inf_{x\in(1/2k,1]} \frac{t}{t+(2x/k)-\left(1/k^2\right)} \\ &= \lim_{k\to\infty} \frac{t}{t+(2/k)-\left(1/k^2\right)} = 1.\end{aligned} \tag{2.54}$$

On the other hand, we have

$$\frac{(2x/k)-\left(1/k^2\right)}{t+(2x/k)-\left(1/k^2\right)} \leq \frac{(2/k)-\left(1/k^2\right)}{t}, \tag{2.55}$$

and so

$$\begin{aligned}\lim_{k\to\infty} \sup_{x\in(1/2k,1]} \nu\left(f_k(x)-x^2,t\right) &= \lim_{k\to\infty} \sup_{x\in(1/2k,1]} \frac{(2x/k)-\left(1/k^2\right)}{t+(2x/k)-\left(1/k^2\right)} \\ &= \lim_{k\to\infty} \frac{(2/k)-\left(1/k^2\right)}{t} = 0.\end{aligned} \tag{2.56}$$

That is, the sequence (f_k) is uniformly statistically intuitionistic fuzzy convergent to x^2 on $[-1,1]$.

Remark 2.13. If $st_{\mu,\nu} - f_k \rightrightarrows f$, then $st_{\mu,\nu} - f_k \rightarrow f$. But the converse of this is not true.

We prove this with the following example.

Example 2.14. Let us define the sequence of functions

$$f_k(x) = \begin{cases} 0 & k = n^2 \\ \dfrac{k^2 x}{1+k^3x^2} & \text{otherwise,} \end{cases} \tag{2.57}$$

on $[0,1]$. This sequence of functions is pointwise statistically intuitionistic fuzzy convergent to 0 (indeed, for $x = 1/k$, $st_{\mu,\nu} - f_k(1/k) = 1$ and for $x = 0$, $st_{\mu,\nu} - f_k(0) = 0$). But, it is not uniformly statistical intuitionistic fuzzy convergent. Since the sequence of functions is bounded on $[0,1]$, we can use Proposition 2.10 to prove our claim. Let us take infumum for $\mu(f_k(x) - 0, t) = \mu(k^2x/(1+k^3x^2\ ), t) = t/(t + (k^2x)/(1+k^3x^2))$ and supremum for $\nu(f_k(x) - 0, t) = (\nu(k^2x/(1+k^3x^2), t)) = (k^2x/(1+k^3x^2)/(t+(k^2x)/(1+k^3x^2\ ))$, over all $x \in [0,1]$. Firstly, we try to find $\sup_{x\in[0,1]}(k^2x/(1+k^3x^2\ ))$ and $\inf_{x\in[0,1]}(k^2x/(1+k^3x^2))$. For this, we have

$$\left(\frac{k^2x}{1+k^3x^2}\right)' = \frac{k^2(1-k^3x^2)}{(1+n^3x^2)^2} = 0 \Longrightarrow x = k^{-3/2}. \tag{2.58}$$

Since $f_k(0) = 0$, $f_k(1) = k^2/(1+k^3)$ and $f_k(k^{-3/2}) = \sqrt{k}/2$, we get $\sup_{x\in[0,1]}(k^2x/(1+k^3x^2)) = \sqrt{k}/2$ and $\inf_{x\in[0,1]}(k^2x/(1+k^3x^2)) = 0$. Then,

$$\begin{aligned}
\frac{k^2x}{1+k^3x^2} \leq \frac{\sqrt{k}}{2} &\Longrightarrow t + \frac{k^2x}{1+k^3x^2} \leq t + \frac{\sqrt{k}}{2} \\
&\Longrightarrow \frac{1}{t+\left(\sqrt{k}/2\right)} \leq \frac{1}{t+(k^2x/(1+k^3x^2))} \\
&\Longrightarrow \frac{t}{t+\left(\sqrt{k}/2\right)} \leq \frac{t}{t+(k^2x/(1+k^3x^2))}, \\
0 \leq \frac{k^2x}{1+k^3x^2} &\Longrightarrow t+0 \leq t + \frac{k^2x}{1+k^3x^2} \\
&\Longrightarrow \frac{1}{t+(k^2x/(1+k^3x^2))} \leq \frac{1}{t} \\
&\Longrightarrow \frac{k^2x/(1+k^3x^2)}{t+(k^2x/(1+k^3x^2))} \leq \frac{k^2x/(1+k^3x^2)}{t} \leq \frac{\sqrt{k}}{2t},
\end{aligned} \tag{2.59}$$

so, we get

$$\lim_{k\to\infty}\inf \mu\left(\frac{k^2x}{1+k^3x^2},t\right) = \lim_{k\to\infty}\inf \frac{t}{(t+k^2x/(1+k^3x^2))} = \lim_{k\to\infty}\frac{t}{t+\sqrt{k}/2}$$
$$= \lim_{k\to\infty}\frac{2t}{2t+\sqrt{k}} = 0\neq 1, \tag{2.60}$$
$$\lim_{k\to\infty}\sup v\left(\frac{k^2x}{1+k^3x^2},t\right) = \lim_{k\to\infty}\sup \frac{k^2x/(1+k^3x^2)}{t+(k^2x/(1+k^3x^2))} = \lim_{k\to\infty}\frac{\sqrt{k}}{2t} = \infty\neq 0.$$

Therefore, we conclude that (f_k) does not intuitionistic fuzzy uniformly statistical convergent to 0.

Theorem 2.15. *Let $f_k : (X,\mu,\nu,*,\diamond) \to (Y,\mu',\nu',*,\diamond)$ be a sequence of functions. If (f_k) is uniformly intuitionistic fuzzy convergent on X to a function f with respect to (μ,ν), then $st_{\mu,\nu}-f_k \rightrightarrows f$. But the converse of this is not true.*

Proof. Let f_k be uniformly intuitionistic fuzzy convergent on X to a function f. In this case, given $0<\varepsilon<1, t>0$, there exist a positive integer $k_0=k_0(r,t)$ such that $\forall x\in X$ and $\forall k>k_0$,

$$\mu'(f_k(x)-f(x),t) > 1-\varepsilon, \qquad \nu'(f_k(x)-f(x),t) < \varepsilon. \tag{2.61}$$

That is, for $k \le k_0$

$$\mu'(f_k(x)-f(x),t) \le 1-\varepsilon, \qquad \nu'(f_k(x)-f(x),t) \ge \varepsilon, \tag{2.62}$$

is satisfied and these k's are finite. Since finite set has 0-density, density of complement of finite set is 1. If complement of this finite set is stated by M, for every $\varepsilon>0$, there exist $M\subset\mathbb{N}$, $\delta(M)=1$ and $\exists k_0=k_0(\varepsilon,t)\in M$ such that $\forall k>k_0$ and $k\in M$ and $\forall x\in X$,

$$\mu'(f_k(x)-f(x),t) > 1-\varepsilon, \qquad \nu'(f_k(x)-f(x),t) < \varepsilon. \tag{2.63}$$

This shows that $st_{\mu,\nu}-f_k \rightrightarrows f$. □

Definition 2.16. Let $f_k : (X,\mu,\nu,*,\diamond) \to (Y,\mu',\nu',*,\diamond)$ be a sequence of functions.The sequence (f_k) is a uniformly statistically Cauchy sequence in intuitionistic fuzzy normed space provided that for every $\varepsilon>0$ and $t>0$, there exists a number $N=N(\varepsilon,t)$ such that

$$\delta\left(\left\{k\in\mathbb{N} : \mu'(f_k(x)-f_N(x),t) \le 1-\varepsilon \text{ or } \nu'(f_k(x)-f_N(x),t)\ge\varepsilon \ \ \forall x\in X\right\}\right)=0. \tag{2.64}$$

Theorem 2.17. *Let $f_k : (X,\mu,\nu,*,\diamond) \to (Y,\mu',\nu',*,\diamond)$ be sequence of functions. If (f_k) is a uniformly statistically convergent sequence with respect to intuitionistic fuzzy norm (μ,ν), then (f_k) is uniformly statistically Cauchy sequence with respect to intuitionistic fuzzy norm (μ,ν).*

Proof. Suppose that $st_{\mu,\nu} - f_k \rightrightarrows f$. In this case, $\forall\varepsilon > 0$, there exists $M \subset \mathbb{N}$, $\delta(M) = 1$ and $k_0 = k_0(\varepsilon, t) \in M$ such that $\forall k > k_0$, $k \in M$ and $\forall x \in X$,

$$\mu'\left(f_k(x) - f(x), \frac{t}{2}\right) > 1 - \varepsilon, \qquad \nu'\left(f_k(x) - f(x), \frac{t}{2}\right) < \varepsilon. \tag{2.65}$$

Choose $N = N(\varepsilon, t) \in M$, $N > k_0$. So, $\mu'(f_N(x) - f(x), t/2) > 1-\varepsilon$ and $\nu'(f_N(x) - f(x), t/2) < \varepsilon$. We investigate $N = N(\varepsilon, t)$ such that

$$\delta(\{k \in \mathbb{N} : \mu'(f_k(x) - f_N(x), t) \leq 1 - \varepsilon \text{ or } \nu'(f_k(x) - f_N(x), t) \geq \varepsilon \ \forall x \in X\}) = 0, \tag{2.66}$$

or

$$\delta\left(K^{'}\right) = \delta(\{k \in \mathbb{N} : \mu'(f_k(x) - f_N(x), t) > 1 - \varepsilon \text{ or } \nu'(f_k(x) - f_N(x), t) < \varepsilon \ \forall\, x \in X\}) = 1. \tag{2.67}$$

For every $k \in M$, we have

$$\begin{aligned}
\mu'(f_k(x) - f_N(x), t) &= \mu'(f_k(x) - f(x) + f(x) - f_N(x), t) \\
&\geq \mu'\left(f_k(x) - f(x), \frac{t}{2}\right) * \mu'\left(f(x) - f_N(x), \frac{t}{2}\right) \\
&> (1 - \varepsilon) * (1 - \varepsilon) \\
&= (1 - \varepsilon), \\
\nu'(f_k(x) - f_N(x), t) &= \nu'(f_k(x) - f(x) + f(x) - f_N(x), t) \\
&\leq \nu''\left(f_k(x) - f(x), \frac{t}{2}\right) * \nu'\left(f(x) - f_N(x), \frac{t}{2}\right) \\
&> \varepsilon \diamond \varepsilon \\
&= \varepsilon.
\end{aligned} \tag{2.68}$$

Since $\delta(M) = 1, (f_k)$ is a uniformly statistically Cauchy sequence in intuitionistic fuzzy normed space. □

Theorem 2.18. *Let $(X, \mu, \nu, *, \diamond)$ and $(Y, \mu', \nu', *, \diamond)$ be two IFNS and the mapping f_k : $(X, \mu, \nu, *, \diamond) \rightarrow (Y, \mu', \nu', *, \diamond)$ be the intuitionistic fuzzy continuous on X of sequence of functions. If $st_{\mu,\nu} - f_k \rightrightarrows f$, the mapping $f : X \rightarrow Y$ is the intuitionistic fuzzy continuous on X.*

Proof. Let $x_0 \in X$ be an arbitrary point. By the intuitionistic fuzzy continuity of f_k's, for every $\varepsilon > 0$ and $t > 0$ there exists $\delta = \delta(x_0, \varepsilon, t/3) > 0$ such that

$$\mu'\left(f_k(x_0) - f_k(x), \frac{t}{3}\right) > 1 - \varepsilon, \qquad \nu'\left(f_k(x_0) - f_k(x), \frac{t}{3}\right) < \varepsilon, \tag{2.69}$$

for every $k \in \mathbb{N}$ and all x such that $\mu(x_0 - x, t) > 1 - \delta$ and $\nu(x_0 - x, t) < \delta$. Let $x \in B(x_0, \delta, t)$ be fixed ($B(x_0, \delta, t)$ stands for an open ball in $(X, \mu, \nu, *, \diamond)$ with center x_0 and radius δ). Since $st_{\mu,\nu} - f_k \rightrightarrows f$ on X, for all $x \in X$, if we state, respectively, A and B by these sets

$$A = \left\{k \in \mathbb{N} : \mu'\left(f_k(x) - f(x), \frac{t}{3}\right) \leq 1 - \varepsilon \text{ or } \nu'\left(f_k(x) - f(x), \frac{t}{3}\right) \geq \varepsilon \ \forall x \in X\right\},$$
$$B = \left\{k \in \mathbb{N} : \mu'\left(f_k(x_0) - f(x_0), \frac{t}{3}\right) \leq 1 - \varepsilon \text{ or } \nu'\left(f_k(x_0) - f(x_0), \frac{t}{3}\right) \geq \varepsilon \ \forall x \in X\right\} \tag{2.70}$$

then, $\delta(A) = 0$ and $\delta(B) = 0$, hence $\delta(A \cup B) = 0$ and $A \cup B$ is different from $\mathbb{N}$. Thus, there exists $m \in \mathbb{N}$ such that

$$\mu'\left(f_m(x) - f(x), \frac{t}{3}\right) > 1 - \varepsilon, \qquad \nu'\left(f_m(x) - f(x), \frac{t}{3}\right) < \varepsilon,$$
$$\mu'\left(f_m(x_0) - f(x_0), \frac{t}{3}\right) > 1 - \varepsilon, \qquad \nu'\left(f_m(x_0) - f(x_0), \frac{t}{3}\right) < \varepsilon. \tag{2.71}$$

Now, we will show that f is intuitionistic fuzzy contunious at x_0. Using Definition 1.1, we have

$$\begin{aligned}
\mu'(f(x) - f(x_0), t) &= \mu'(f(x) - f_m(x) + f_m(x) - f_m(x_0) + f_m(x_0) - f(x_0), t) \\
&\geq \mu'\left(f(x) - f_m(x), \frac{t}{3}\right) * \mu'\left(f_m(x) - f_m(x_0), \frac{t}{3}\right) * \mu'\left(f_m(x_0) - f(x_0), \frac{t}{3}\right) \\
&> (1 - \varepsilon) * (1 - \varepsilon) * (1 - \varepsilon) \\
&= 1 - \varepsilon, \\
\nu'(f(x) - f(x_0), t) &= \nu'(f(x) - f_m(x) + f_m(x) - f_m(x_0) + f_m(x_0) - f(x_0), t) \\
&\leq \nu'\left(f(x) - f_m(x), \frac{t}{3}\right) * \nu'\left(f_m(x) - f_m(x_0), \frac{t}{3}\right) * \nu'\left(f_m(x_0) - f(x_0), \frac{t}{3}\right) \\
&< \varepsilon \diamond \varepsilon \diamond \varepsilon \\
&= \varepsilon.
\end{aligned} \tag{2.72}$$

Thus, the proof is completed. □

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