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Research Article

On the Second Order of Accuracy Stable Implicit Difference Scheme for Elliptic-Parabolic Equations

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We are interested in studying a second order of accuracy implicit difference scheme for the solution of the elliptic-parabolic equation with the nonlocal boundary condition. Well-posedness of this difference scheme is established. In an application, coercivity estimates in Hölder norms for approximate solutions of multipoint nonlocal boundary value problems for elliptic-parabolic differential equations are obtained.

1. Introduction

Methods of solutions of nonlocal boundary value problems for mixed-type differential equations have been studied extensively by various researchers (see, e.g., [1–19] and the references therein).

In [20], we considered the well-posedness of the following multipoint nonlocal boundary value problem:

$$\begin{gathered}
-\frac{d^2u(t)}{dt^2} + Au(t) = g(t), \quad (0 \le t \le 1), \\
\frac{du(t)}{dt} - Au(t) = f(t), \quad (-1 \le t \le 0), \\
u(1) = \sum_{i=1}^{J} \alpha_i u(\lambda_i) + \varphi, \\
-1 \le \lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots < \lambda_J \le 0,
\end{gathered} \tag{1.1}$$

in a Hilbert space H with the self-adjoint positive definite operator A under assumption

$$\sum_{i=1}^{J}|\alpha_i| \le 1. \tag{1.2}$$

The well-posedness of multipoint nonlocal boundary value problem (1.1) in Hölder spaces with a weight was established. Moreover, coercivity estimates in Hölder norms for the solutions of nonlocal boundary value problems for elliptic-parabolic equations were obtained.

In [21], we studied the well-posedness of the first order of accuracy difference scheme for the approximate solution of boundary value problem (1.1) under assumption (1.2).

Throughout this work, we consider the following second order of accuracy difference scheme:

$$\begin{gathered}
-\tau^{-2}(u_{k+1}-2u_k+u_{k-1})+Au_k=g_k,\\
g_k=g(t_k),\quad t_k=k\tau,\ 1\le k\le N-1,\ N\tau=1,\\
\tau^{-1}(u_k-u_{k-1})-\left(I+\frac{\tau}{2}A\right)Au_{k-1}=\left(I+\frac{\tau}{2}A\right)f_k,\qquad f_k=f(t_{k-1/2}),\\
t_{k-1/2}=\left(k-\frac{1}{2}\right)\tau,\quad -(N-1)\le k\le 0,\\
u_2-4u_1+3u_0=-3u_0+4u_{-1}-u_{-2},\\
u_N=\sum_{k=1}^{J}\alpha_i\left(u_{[\lambda_i/\tau]}+\left(\lambda_i-\left[\frac{\lambda_i}{\tau}\right]\tau\right)\left(f_{[\lambda_i/\tau]}+Au_{[\lambda_i/\tau]}\right)\right)+\varphi,
\end{gathered} \tag{1.3}$$

for the approximate solution of boundary value problem (1.1) under assumption (1.2).

The well-posedness of difference scheme (1.3) in Hölder spaces with a weight is established. As an application, the stability, almost coercivity stability, and coercivity stability estimates for solutions of second order of accuracy difference scheme for the approximate solution of the nonlocal boundary elliptic-parabolic problem are obtained.

2. Main Theorems

Throughout the paper, H is a Hilbert space and we denote $B=(1/2)(\tau A+\sqrt{A(4+\tau^2 A)})$, where A is a self-adjoint positive definite operator. Then, it is clear that B is the self-adjoint positive definite operator and $B\ge\delta^{1/2}I$ where $\delta>\delta_0>0$, and $R=(I+\tau B)^{-1}$, which is defined on the whole space H, is a bounded operator. Here, I is the identity operator. The following operators

$$D = \left(I + \tau A + \frac{(\tau A)^2}{2}\right), \qquad G = \left(I - \frac{\tau^2 A}{2}\right), \qquad P = \left(I + \frac{\tau}{2} A\right), \qquad R = (I + \tau B)^{-1},$$

$$T_\tau = \left(I + B^{-1} A\left(I + \tau A + \frac{\tau}{2} P^{-2}\right) K\left(I - R^{2N-1}\right) + GKP^{-2} R^{2N-1} \right.$$
$$\left. - GKP^{-2}(2I + \tau B) R^N \left[\sum_{i=1}^{n} \alpha_i \left(I + \left(\lambda_i - \left[\frac{\lambda_i}{\tau}\right]\tau\right) A\right) D^{-[\lambda_i/\tau]} u_0\right]\right)^{-1} \tag{2.1}$$

exist and are bounded for a self-adjoint positive operator A. Here,

$$B = \frac{1}{2}\left(\tau A + \sqrt{A(4 + \tau^2 A)}\right), \qquad K = \left(I + 2\tau A + \frac{5}{4}(\tau A)^2\right)^{-1}. \tag{2.2}$$

Furthermore, positive constants will be indicated by M which can differ in time. On the other hand $M_i(\alpha, \beta, \ldots)$ is used to focus on the fact that the constant depends only on $\alpha, \beta, \ldots$ and the subindex i is used to indicate a different constant.

First of all, let us start with some auxiliary lemmas from [16, 22–24] that are essential below.

Lemma 2.1. *For a self-adjoint positive operator A, the following estimates are satisfied:*

$$\begin{gathered}
\left\|R^k\right\|_{H \to H} \le M_1(\delta)(1 + \delta\tau)^{-k}, \qquad \left\|D^k\right\|_{H \to H} \le M_1(\delta), \\
\left\|BR^k\right\|_{H \to H} \le \frac{M_1(\delta)}{k\tau}, \qquad \left\|P^{-1}\right\|_{H \to H} \le M_1(\delta), \qquad \left\|AD^k\right\|_{H \to H} \le \frac{M_1(\delta)}{k\tau}, \\
\left\|D^k - e^{-k\tau A}\right\|_{H \to H} \le \frac{M_1(\delta)}{k^2}, \qquad \left\|\left(I - R^{2N}\right)^{-1}\right\|_{H \to H} \le M_1(\delta), \\
\left\|R^k - e^{-k\tau A^{1/2}}\right\|_{H \to H} \le \frac{M_1(\delta)}{k}, \quad k \ge 1,\ \delta > 0.
\end{gathered} \tag{2.3}$$

From these estimates, it follows that

$$\left\|\left(I + B^{-1} A\left(I + \tau A + \frac{\tau}{2} P^{-2}\right) K\left(I - R^{2N-1}\right) + GKP^{-2} R^{2N-1} - GKP^{-2}(2I + \tau B)\right.\right.$$
$$\left.\left. \times R^N \left[\sum_{i=1}^{n} \alpha_i \left(I + \left(\lambda_i - \left[\frac{\lambda_i}{\tau}\right]\tau\right) A\right) D^{-[\lambda_i/\tau]}\right]\right)^{-1}\right\|_{H \to H} \le M_2(\delta). \tag{2.4}$$

Lemma 2.2. *For any* g_k, $1 \le k \le N-1$ *and* $f_k, -N+1 \le k \le 0$, *the solution of problem* (1.3) *exists, and the following formulas hold:*

$$
\begin{aligned}
u_k &= \left(I - R^{2N}\right)^{-1} \\
&\quad \times \Bigg\{ \left[R^k - R^{2N-k}\right] u_0 + \left[R^{N-k} - R^{N+k}\right] \\
&\qquad \times \Bigg[\sum_{i=1}^{n} \alpha_i \Bigg[\left(I + \left(\lambda_i - \left[\frac{\lambda_i}{\tau}\right]\tau\right) A \right) \left(D^{-[\lambda_i/\tau]} u_0 - \tau \sum_{s=[\lambda_i/\tau]+1}^{0} P D^{s-[\lambda_i/\tau]} f_s \right) \\
&\qquad\qquad + \left(\lambda_i - \left[\frac{\lambda_i}{\tau}\right]\tau\right) f_{[\lambda_i/\tau]} \Bigg] + \varphi \Bigg] \\
&\qquad - \left[R^{N-k} - R^{N+k}\right] (I + \tau B)(2I + \tau B)^{-1} B^{-1} \sum_{s=1}^{N-1} \left[R^{N-s} - R^{N+s}\right] g_s \tau \Bigg\} \\
&\quad + (I + \tau B)(2I + \tau B)^{-1} B^{-1} \sum_{s=1}^{N-1} \left[R^{|k-s|} - R^{k+s}\right] g_s \tau, \quad 1 \le k \le N, \\
u_k &= D^{-k} u_0 - \tau \sum_{s=k+1}^{0} P D^{s-k} f_s, \quad -N \le k \le -1, \\
u_0 &= \frac{1}{2} T_\tau K P^{-2} \\
&\quad \times \Bigg\{ \left(2I - \tau^2 A\right) \times \Bigg\{ (2 + \tau B) R^N \\
&\qquad \times \Bigg[\sum_{i=1}^{n} \alpha_i \Bigg[\left(I + \left(\lambda_i - \left[\frac{\lambda_i}{\tau}\right]\tau\right) A \right) \\
&\qquad\qquad \times \left(D^{-[\lambda_i/\tau]} u_0 - \tau \sum_{s=[\lambda_i/\tau]+1}^{0} P D^{s-[\lambda_i/\tau]} f_s \right) \\
&\qquad\qquad + \left(\lambda_i - \left[\frac{\lambda_i}{\tau}\right]\tau\right) f_{[\lambda_i/\tau]} \Bigg] + \varphi \Bigg] \\
&\qquad - R^{N-1} B^{-1} \sum_{s=1}^{N-1} \left[R^{N-s} - R^{N+s}\right] g_s \tau + \left(I - R^{2N}\right) B^{-1} \sum_{s=1}^{N-1} R^{s-1} g_s \tau \Bigg\} \\
&\quad + \left(I - R^{2N}\right)(I + \tau B)\left(\tau B^{-1} g_1 - 4 P B^{-1} f_0 + P D B^{-1} f_0 + P B^{-1} f_{-1}\right) \Bigg\}, \\
T_\tau &= \Bigg(I + B^{-1} A \left(I + \tau A + \frac{\tau}{2} P^{-2} \right) K \left(I - R^{2N-1} \right) + G K P^{-2} R^{2N-1} \\
&\quad - G K P^{-2} (2I + \tau B) R^N \left[\sum_{i=1}^{n} \alpha_i \left(I + \left(\lambda_i - \left[\frac{\lambda_i}{\tau}\right]\tau\right) A \right) D^{-[\lambda_i/\tau]} u_0 \right] \Bigg)^{-1}.
\end{aligned}
\tag{2.5}
$$

Now, we study well-posedness of problem (1.3). Let $F_\tau(H) = F([a,b]_\tau, H)$ be the linear space of mesh functions $\varphi^\tau = \{\varphi_k\}_{\widetilde{N}}^{\widetilde{\widetilde{N}}}$ defined on $[a,b]_\tau = \{t_k = kh, \widetilde{N} \leq k \leq \widetilde{\widetilde{N}}, \widetilde{N}\tau = a, \widetilde{\widetilde{N}}\tau = b\}$ with values in the Hilbert space H. Next, on $F_\tau(H)$ we denote $C([a,b]_\tau, H)$, $C_{0,1}^\alpha([-1,1]_\tau, H)$, $C_{0,1}^\alpha([-1,0]_\tau, H)$, $C_0^\alpha([0,1]_\tau, H)$, $\widetilde{C}_{0,1}^\alpha([-1,1]_\tau, H)$, and $\widetilde{C}_0^\alpha([-1,0]_\tau, H)$, $0 < \alpha < 1$ Banach spaces with the following norms:

$$\begin{aligned}
\|\varphi^\tau\|_{C([a,b]_\tau,H)} &= \max_{N_a \leq k \leq N_b} \|\varphi_k\|_H, \\
\|\varphi^\tau\|_{C_{0,1}^\alpha([-1,1]_\tau,H)} &= \|\varphi^\tau\|_{C([-1,1]_\tau,H)} + \sup_{-N \leq k < k+r \leq 0} \|\varphi_{k+r} - \varphi_k\|_E (-k)^\alpha r^{-\alpha} \\
&\quad + \sup_{1 \leq k < k+r \leq N-1} \|\varphi_{k+r} - \varphi_k\|_E ((k+r)\tau)^\alpha (N-k)^\alpha r^{-\alpha}, \\
\|\varphi^\tau\|_{C_0^\alpha([-1,0]_\tau,H)} &= \|\varphi^\tau\|_{C([-1,0]_\tau,H)} + \sup_{-N \leq k < k+r \leq 0} \|\varphi_{k+r} - \varphi_k\|_E (-k)^\alpha r^{-\alpha}, \\
\|\varphi^\tau\|_{C_{0,1}^\alpha([0,1]_\tau,H)} &= \|\varphi^\tau\|_{C([0,1]_\tau,H)} \\
&\quad + \sup_{1 \leq k < k+r \leq N-1} \|\varphi_{k+r} - \varphi_k\|_E ((k+r)\tau)^\alpha (N-k)^\alpha r^{-\alpha}, \\
\|\varphi^\tau\|_{\widetilde{C}_{0,1}^\alpha([-1,1]_\tau,H)} &= \|\varphi^\tau\|_{C([-1,1]_\tau,H)} + \sup_{-N \leq k < k+2r \leq 0} \|\varphi_{k+2r} - \varphi_k\|_E (-k)^\alpha (2r)^{-\alpha} \\
&\quad + \sup_{1 \leq k < k+r \leq N-1} \|\varphi_{k+r} - \varphi_k\|_E ((k+r)\tau)^\alpha (N-k)^\alpha r^{-\alpha}, \\
\|\varphi^\tau\|_{\widetilde{C}_0^\alpha([-1,0]_\tau,H)} &= \|\varphi^\tau\|_{C([-1,0]_\tau,H)} \\
&\quad + \sup_{-N \leq k < k+2r \leq 0} \|\varphi_{k+2r} - \varphi_k\|_E (-k)^\alpha (2r)^{-\alpha}, \text{ respectively.}
\end{aligned} \tag{2.6}$$

Theorem 2.3. *Nonlocal boundary value problem* (1.3) *is stable in* $C([-1,1]_\tau, H)$ *space.*

Proof. By [22], we have

$$\left\| \{u_k\}_1^{N-1} \right\|_{C([0,1]_\tau,H)} \leq M_3(\delta) \left[\|g^\tau\|_{C([0,1]_\tau,H)} + \|\xi\|_H + \|\varphi\|_H \right], \tag{2.7}$$

for the solution of the following boundary value problem:

$$\begin{gathered}
-\tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + Au_k = g_k, \\
g_k = g(t_k), \quad t_k = k\tau, \ 1 \leq k \leq N-1, \\
u_0 = \xi, \qquad u_N = \varphi.
\end{gathered} \tag{2.8}$$

By [24], we have

$$\left\| \{u_k\}_{-N}^0 \right\|_{C([-1,0]_\tau,H)} \leq M_4(\delta) \left[\|f^\tau\|_{C([-1,0]_\tau,H)} + \|\xi\|_H \right] \tag{2.9}$$

for the solution of an inverse Cauchy difference problem:

$$\tau^{-1}(u_k - u_{k-1}) - \left(I + \frac{\tau}{2}A\right)Au_{k-1} = \left(I + \frac{\tau}{2}A\right)f_k, \quad -(N-1) \le k \le 0, \quad u_0 = \xi. \tag{2.10}$$

Then, the proof of Theorem 2.3 is based on stability inequalities (2.7) and (2.9) and on the following estimates:

$$\begin{aligned} \|\xi\|_H &\le M_5(\delta)\left[\|f^\tau\|_{C([-1,0]_\tau,H)} + \|g^\tau\|_{C([0,1]_\tau,H)} + \|\varphi\|_H\right], \\ \|\varphi\|_H &\le M_6(\delta)\left[\|f^\tau\|_{C([-1,0]_\tau,H)} + \|g^\tau\|_{C([0,1]_\tau,H)} + \|\varphi\|_H\right], \end{aligned} \tag{2.11}$$

for the solution of boundary value problem (1.3). Estimates (2.11) follow from estimates (2.3) and (2.4) and formula (2.5). This finishes the proof of Theorem 2.3. □

Theorem 2.4. *Assume that $\varphi \in D(A)$ and $f_0, f_{-1}, g_1 \in D(I+\tau B)$. Then, for the solution of difference problem* (1.3)*, the following almost coercivity inequality holds:*

$$\begin{aligned} &\left\|\left\{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\right\}_1^{N-1}\right\|_{C([0,1]_\tau,H)} + \left\|\left\{\tau^{-1}(u_k - u_{k-1})\right\}_{-N+1}^{0}\right\|_{C([-1,0]_\tau,H)} \\ &\quad + \left\|\{Au_k\}_1^{N-1}\right\|_{C([0,1]_\tau,H)} + \left\|\left\{\left(I + \frac{\tau}{2}A\right)Au_{k-1}\right\}_{-N+1}^{0}\right\|_{C([-1,0]_\tau,H)} \\ &\le M_7(\delta)\left[\min\left\{\ln\frac{1}{\tau}, 1 + |\ln\|A\|_{H\to H}|\right\}\left[\|f^\tau\|_{C([-1,0]_\tau,H)} + \|g^\tau\|_{C([0,1]_\tau,H)}\right]\right. \\ &\quad \left. + \|A\varphi\|_H + \|(I+\tau B)f_0\|_H + \|(I+\tau B)g_1\|_H + \|(I+\tau B)f_{-1}\|_H\right]. \end{aligned} \tag{2.12}$$

Proof. We have

$$\begin{aligned} &\left\|\left\{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\right\}_1^{N-1}\right\|_{C([0,1]_\tau,H)} + \left\|\{Au_k\}_1^{N-1}\right\|_{C([0,1]_\tau,H)} \\ &\le M_8(\delta)\left[\min\left\{\ln\frac{1}{\tau}, 1 + |\ln\|A\|_{H\to H}|\right\}\|g^\tau\|_{C([0,1]_\tau,H)} + \|A\xi\|_H + \|A\varphi\|_H\right], \end{aligned} \tag{2.13}$$

for the solution of boundary value problem (2.8) (see [22]), and we get

$$\begin{aligned} &\left\|\left\{\tau^{-1}(u_k - u_{k-1})\right\}_{-N+1}^{0}\right\|_{C([-1,0]_\tau,H)} + \left\|\left\{\left(I + \frac{\tau}{2}A\right)Au_{k-1}\right\}_{-N+1}^{0}\right\|_{C([-1,0]_\tau,H)} \\ &\le M_9(\delta)\left[\min\left\{\ln\frac{1}{\tau}, 1 + |\ln\|A\|_{H\to H}|\right\}\|f^\tau\|_{C([-1,0]_\tau,H)} + \|A\xi\|_H\right], \end{aligned} \tag{2.14}$$

for the solution of inverse Cauchy difference problem (2.10) (see [24]). Then, the proof of Theorem 2.4 is based on almost coercivity inequalities (2.13) and (2.14) and on the following estimates:

$$\begin{aligned}
\|A\xi\|_H &\leq M_{10}(\delta)\left[\|A\varphi\|_H + \|(I+\tau B)f_0\|_H + \min\left\{\ln\frac{1}{\tau}, 1 + |\ln\|A\|_{H\to H}|\right\}\right. \\
&\qquad \left.\times\left[\|f^\tau\|_{C([-1,0]_\tau,H)} + \|g^\tau\|_{C([0,1]_\tau,H)}\right]\right], \\
\|A\psi\|_H &\leq M_{11}(\delta)\left[\|A\varphi\|_H + \|(I+\tau B)f_0\|_H + \min\left\{\ln\frac{1}{\tau}, 1 + |\ln\|A\|_{H\to H}|\right\}\right. \\
&\qquad \left.\times\left[\|f^\tau\|_{C([-1,0]_\tau,H)} + \|g^\tau\|_{C([0,1]_\tau,H)}\right]\right]
\end{aligned} \tag{2.15}$$

for the solution of boundary value problem (1.3). Proofs of these estimates follow the scheme of the papers [23, 24] and rely on both formula (2.5) and estimates (2.3) and (2.4). Theorem 2.4 is proved. □

Theorem 2.5. *Let assumptions of Theorem 2.5 be satisfied. Then, boundary value problem* (1.3) *is well-posed in Hölder spaces* $C^\alpha_{0,1}([-1,1]_\tau, H)$, *and* $\widetilde{C}^\alpha_{0,1}([-1,1]_\tau, H)$, *and the following coercivity inequalities hold:*

$$\begin{aligned}
&\left\|\left\{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\right\}_1^{N-1}\right\|_{C^\alpha_{0,1}([0,1]_\tau,H)} + \left\|\left\{\tau^{-1}(u_k-u_{k-1})\right\}_{-N+1}^{0}\right\|_{\widetilde{C}^\alpha_0([-1,0]_\tau,H)} \\
&\quad + \left\|\{Au_k\}_1^{N-1}\right\|_{C^\alpha_{0,1}([0,1]_\tau,H)} + \left\|\left\{\left(I+\frac{\tau}{2}A\right)Au_{k-1}\right\}_{-N+1}^{0}\right\|_{\widetilde{C}^\alpha_0([-1,0]_\tau,H)} \\
&\quad \leq M_{12}(\delta)\left[\frac{1}{\alpha(1-\alpha)}\left[\|f^\tau\|_{C^\alpha_0([-1,0]_\tau,H)} + \|g^\tau\|_{C^\alpha_{0,1}([0,1]_\tau,H)}\right] + \|A\varphi\|_H + \|(I+\tau B)f_0\|_H\right. \\
&\qquad\qquad \left.+\|(I+\tau B)g_1\|_H + \|(I+\tau B)f_{-1}\|_H\right], \\
&\left\|\left\{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\right\}_1^{N-1}\right\|_{C^\alpha_{0,1}([0,1]_\tau,H)} + \left\|\left\{\tau^{-1}(u_k-u_{k-1})\right\}_{-N+1}^{0}\right\|_{\widetilde{C}^\alpha_0([-1,0]_\tau,H)} \\
&\quad + \left\|\{Au_k\}_1^{N-1}\right\|_{C^\alpha_{0,1}([0,1]_\tau,H)} + \left\|\left\{\left(I+\frac{\tau}{2}A\right)Au_{k-1}\right\}_{-N+1}^{0}\right\|_{\widetilde{C}^\alpha_0([-1,0]_\tau,H)} \\
&\quad \leq M_{13}(\delta)\left[\frac{1}{\alpha(1-\alpha)}\left[\|f^\tau\|_{\widetilde{C}^\alpha_0([-1,0]_\tau,H)} + \|g^\tau\|_{C^\alpha_{0,1}([0,1]_\tau,H)}\right] + \|A\varphi\|_H\right. \\
&\qquad\qquad \left.+\|(I+\tau B)f_0\|_H + \|(I+\tau B)g_1\|_H + \|(I+\tau B)f_{-1}\|_H\right].
\end{aligned} \tag{2.16}$$

Proof. By [22, 24], we have

$$\begin{aligned} &\left\| \left\{ \tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) \right\}_1^{N-1} \right\|_{C_{0,1}^{\alpha}([0,1]_\tau, H)} + \left\| \{Au_k\}_1^{N-1} \right\|_{C_{0,1}^{\alpha}([0,1]_\tau, H)} \\ &\quad \leq M_{14}(\delta) \left[\frac{1}{\alpha(1-\alpha)} \|g^\tau\|_{C_{0,1}^{\alpha}([0,1]_\tau, H)} + \|A\xi\|_H + \|A\psi\|_H \right], \end{aligned} \tag{2.17}$$

for the solution of boundary value problem (2.8), and

$$\begin{aligned} &\left\| \left\{ \tau^{-1}(u_k - u_{k-1}) \right\}_{-N+1}^{0} \right\|_{\widetilde{C}_0^{\alpha}([-1,0]_\tau, H)} + \left\| \left\{ \left(I + \frac{\tau}{2} A \right) A u_{k-1} \right\}_{-N+1}^{0} \right\|_{\widetilde{C}_0^{\alpha}([-1,0]_\tau, H)} \\ &\quad \leq M_{15}(\delta) \left[\frac{1}{\alpha(1-\alpha)} \|f^\tau\|_{C_0^{\alpha}([-1,0]_\tau, H)} + \|A\xi\|_H \right], \end{aligned} \tag{2.18}$$

$$\begin{aligned} &\left\| \left\{ \tau^{-1}(u_k - u_{k-1}) \right\}_{-N+1}^{0} \right\|_{\widetilde{C}_0^{\alpha}([-1,0]_\tau, H)} + \left\| \left\{ \left(I + \frac{\tau}{2} A \right) A u_{k-1} \right\}_{-N+1}^{0} \right\|_{\widetilde{C}_0^{\alpha}([-1,0]_\tau, H)} \\ &\quad \leq M_{16}(\delta) \left[\frac{1}{\alpha(1-\alpha)} \|f^\tau\|_{\widetilde{C}_0^{\alpha}([-1,0]_\tau, H)} + \|A\xi\|_H \right] \end{aligned} \tag{2.19}$$

for the solution of inverse Cauchy difference problem (2.10), respectively. Then, the proof of Theorem 2.5 is based on coercivity inequalities (2.17)–(2.19) and the following estimates:

$$\begin{aligned} \|A\xi\|_H &\leq M_{17}(\delta) \left[\frac{1}{\alpha(1-\alpha)} \left[\|f^\tau\|_{\widetilde{C}_0^{\alpha}([-1,0]_\tau, H)} + \|g^\tau\|_{C_{0,1}^{\alpha}([0,1]_\tau, H)} \right] \right. \\ &\qquad \left. + \|A\varphi\|_H + \|(I + \tau B) f_0\|_H + \|(I + \tau B) g_1\|_H + \|(I + \tau B) f_{-1}\|_H \right], \\ \|A\psi\|_H &\leq M_{18}(\delta) \left[\frac{1}{\alpha(1-\alpha)} \left[\|f^\tau\|_{\widetilde{C}_0^{\alpha}([-1,0]_\tau, H)} + \|g^\tau\|_{C_{0,1}^{\alpha}([0,1]_\tau, H)} \right] \right. \\ &\qquad \left. + \|A\varphi\|_H + \|(I + \tau B) f_0\|_H + \|(I + \tau B) g_1\|_H + \|(I + \tau B) f_{-1}\|_H \right] \end{aligned} \tag{2.20}$$

for the solution of difference scheme (1.3). Proofs of these estimates follow the scheme of the papers [22, 24] and rely on both estimates (2.3) and (2.4) and formula (2.5). This concludes the proof of Theorem 2.5. □

3. An Application

In this section, an application of these abstract Theorems 2.3, 2.4, and 2.5 is considered. In $[-1,1] \times \Omega$, let us consider the following boundary value problem for multidimensional elliptic-parabolic equation:

$$\begin{gathered}
-u_{tt} - \sum_{r=1}^{n} (a_r(x) u_{x_r})_{x_r} = g(t,x), \quad 0 < t < 1,\ x \in \Omega, \\
u_t + \sum_{r=1}^{n} (a_r(x) u_{x_r})_{x_r} = f(t,x), \quad -1 < t < 0,\ x \in \Omega, \\
u(t,x) = 0, \quad x \in S,\ -1 \le t \le 1, \\
u(1,x) = \sum_{i=1}^{J} \alpha_i u(\lambda_i, x) + \varphi(x), \quad \sum_{i=1}^{J} |\alpha_i| \le 1, \\
-1 \le \lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots < \lambda_J \le 0, \\
u(0+,x) = u(0-,x), \quad u_t(0+,x) = u_t(0-,x), \quad x \in \overline{\Omega},
\end{gathered} \tag{3.1}$$

where $a_r(x)$ $(x \in \Omega), \varphi(x)$ $(\varphi(x) = 0, x \in S)$, $g(t,x)$ $(t \in (0,1), x \in \overline{\Omega})$, and $f(t,x)$ $(t \in (-1,0), x \in \overline{\Omega})$ are given smooth functions. Here, Ω is the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < 1, 1 \le k \le n)$ with boundary S, $\overline{\Omega} = \Omega \cup S$, and $a_r(x) \geqslant a > 0$.

The discretization of problem (3.1) is carried out in two steps. In the first step, let us define the following grid sets:

$$\begin{gathered}
\widetilde{\Omega}_h = \{x = x_m = (h_1 m_1, \ldots, h_n m_n), m = (m_1, \ldots, m_n), \\
0 \le m_r \le N_r,\ h_r N_r = 1,\ r = 1, \ldots, n\}, \\
\Omega_h = \widetilde{\Omega}_h \cap \Omega, \qquad S_h = \widetilde{\Omega}_h \cap S.
\end{gathered} \tag{3.2}$$

We introduce the Hilbert spaces $L_{2h} = L_2(\overline{\Omega}_h), W_{2h}^1 = W_2^1(\overline{\Omega}_h)$, and $W_{2h}^2 = W_2^2(\overline{\Omega}_h)$ of the grid functions $\varphi^h(x) = \{\varphi(h_1 m_1, \ldots, h_n m_n)\}$ defined on $\overline{\Omega}_h$, equipped with the following norms:

$$\begin{aligned}
\left\|\varphi^h\right\|_{L_{2h}} &= \left(\sum_{x\in\overline{\Omega}_h}\left|\varphi^h(x)\right|^2 h_1\cdots h_n\right)^{1/2}, \\
\left\|\varphi^h\right\|_{W_{2h}^1} &= \left\|\varphi^h\right\|_{L_{2h}} + \left(\sum_{x\in\overline{\Omega}_h}\sum_{r=1}^{n}\left|(\varphi^h)_{x_r}\right|^2 h_1\cdots h_n\right)^{1/2}, \\
\left\|\varphi^h\right\|_{W_{2h}^2} &= \left\|\varphi^h\right\|_{L_{2h}} + \left(\sum_{x\in\overline{\Omega}_h}\sum_{r=1}^{n}\left|(\varphi^h)_{x_r}\right|^2 h_1\cdots h_n\right)^{1/2} \\
&\quad + \left(\sum_{x\in\overline{\Omega}_h}\sum_{r=1}^{n}\left|(\varphi^h)_{x_r\overline{x_r},m_r}\right|^2 h_1\cdots h_n\right)^{1/2}.
\end{aligned} \tag{3.3}$$

To the differential operator A generated by problem (3.1), we assign the difference operator A_h^x by formula

$$A_h^x u^h = -\sum_{r=1}^{n}\left(a_r(x)u_{\overline{x_r}}^h\right)_{x_r,m_r} \tag{3.4}$$

acting in the space of grid functions $u^h(x)$, satisfying the conditions $u^h(x) = 0$ for all $x \in S_h$. With the help of A_h^x, we arrive at the following nonlocal boundary value problem:

$$\begin{aligned}
&-\frac{d^2u^h(t,x)}{dt^2} + A_h^x u^h(t,x) = g^h(t,x), \quad 0<t<1,\ x\in\Omega_h, \\
&\frac{du^h(t,x)}{dt} - A_h^x u^h(t,x) = f^h(t,x), \quad -1<t<0,\ x\in\Omega_h, \\
&u^h(1,x) = \sum_{k=1}^{n}\alpha_k u^h(\lambda_k,x) + \varphi^h(x), \quad \sum_{k=1}^{n}|\alpha_k| \le 1, \quad x\in\overline{\Omega}_h, \\
&u^h(0+,x) = u^h(0-,x), \quad \frac{du^h(0+,x)}{dt} = \frac{du^h(0-,x)}{dt}, \quad x\in\overline{\Omega}_h,
\end{aligned} \tag{3.5}$$

for an infinite system of ordinary differential equations.

In the second step, we replace problem (3.5) by difference scheme (1.3) accurate to the following second order (see [22, 24]):

$$
\begin{gathered}
-\frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+A_h^x u_k^h(x)=g_k^h(x),\\
g_k^h(x)=g^h(t_k,x),\quad t_k=k\tau,\quad 1\le k\le N-1,\quad N\tau=1,\quad x\in\Omega_h,\\
\frac{u_k^h(x)-u_{k-1}^h(x)}{\tau}-\left(A_h^x+\frac{\tau}{2}\left(A_h^x\right)^2\right)u_{k-1}^h(x)=\left(I+\frac{\tau}{2}A_h^x\right)f_k^h(x),\\
f_k^h(x)=f^h(t_{k-1/2},x),\qquad t_{k-1/2}=\left(k-\frac{1}{2}\right)\tau,\quad -N+1\le k\le 0,\ x\in\Omega_h,\\
-u_2^h(x)+4u_1^h(x)-3u_0^h(x)=3u_0^h(x)-4u_{-1}^h(x)+u_{-2}^h(x),\quad x\in\widetilde{\Omega}_h,\\
u_N^h(x)=\sum_{k=1}^{J}\alpha_i\left(u^h_{[\lambda_i/\tau]}(x)+\left(\lambda_k-\left[\frac{\lambda_i}{\tau}\right]\tau\right)\left(f^h_{[\lambda_i/\tau]}+A_h^x u^h_{[\lambda_i/\tau]}(x)\right)\right)+\varphi^h(x),\quad x\in\widetilde{\Omega}_h.
\end{gathered}
\tag{3.6}
$$

Theorem 3.1. *Let τ and $|h|=\sqrt{h_1^2+\cdots+h_n^2}$ be sufficiently small positive numbers. Then, solutions of difference scheme* (3.6) *satisfy the following stability and almost coercivity estimates:*

$$
\begin{aligned}
&\left\|\left\{u_k^h\right\}_{-N}^{N-1}\right\|_{C([-1,1]_\tau,L_{2h})}\le M_{19}(\delta)\left[\left\|\left\{f_k^h\right\}_{-N+1}^{-1}\right\|_{C([-1,0]_\tau,L_{2h})}+\left\|\left\{g_k^h\right\}_{1}^{N-1}\right\|_{C([0,1]_\tau,L_{2h})}+\left\|\varphi^h\right\|_{L_{2h}}\right],\\
&\left\|\left\{\tau^{-2}\left(u_{k+1}^h-2u_k^h+u_{k-1}^h\right)\right\}_1^{N-1}\right\|_{C([0,1]_\tau,L_{2h})}+\left\|\left\{u_k^h\right\}_1^{N-1}\right\|_{C([0,1]_\tau,W_{2h}^2)}\\
&\quad+\left\|\left\{\tau^{-1}\left(u_k^h-u_{k-1}^h\right)\right\}_{-N+1}^{0}\right\|_{C([-1,0]_\tau,L_{2h})}+\left\|\left\{u_{k-1}^h\right\}_{-N+1}^{0}\right\|_{C([-1,0]_\tau,W_{2h}^2)}\\
&\quad\le M_{20}(\delta)\Bigg[\left\|f_0^h\right\|_{L_{2h}}+\left\|f_{-1}^h\right\|_{L_{2h}}+\left\|g_1^h\right\|_{L_{2h}}+\left\|\varphi^h\right\|_{W_{2h}^2}+\tau\left\|f_0^h\right\|_{W_{2h}^1}+\tau\left\|f_{-1}^h\right\|_{W_{2h}^1}+\tau\left\|g_1^h\right\|_{W_{2h}^1}\\
&\qquad+\ln\frac{1}{\tau+|h|}\left[\left\|\left\{f_k^h\right\}_{-N+1}^{-1}\right\|_{C([-1,0]_\tau,L_{2h})}+\left\|\left\{g_k^h\right\}_{1}^{N-1}\right\|_{C([0,1]_\tau,L_{2h})}\right]\Bigg].
\end{aligned}
\tag{3.7}
$$

The proof of Theorem 3.1 is based on Theorem 2.3, Theorem 2.4, the symmetry property of the difference operator A_h^x defined by formula (3.4), the estimate

$$
\min\left\{\ln\frac{1}{\tau},1+\left|\ln\left\|A_h^x\right\|_{L_{2h}\to L_{2h}}\right|\right\}\le M_{21}(\delta)\ln\frac{1}{\tau+|h|},
\tag{3.8}
$$

and the following theorem on the coercivity inequality for the solution of elliptic difference equation in L_{2h}.

Theorem 3.2. *For the solution of the following elliptic difference problem:*

$$A_h^x u^h(x) = \omega^h(x), \quad x \in \Omega_h, \qquad u^h(x) = 0, \quad x \in S_h, \tag{3.9}$$

the following coercivity inequality holds [25]:

$$\sum_{r=1}^{n} \left\| \left(u^h\right)_{\overline{x}_r x_r, m_r} \right\|_{L_{2h}} \leq M_{22}(\delta) \left\| \omega^h \right\|_{L_{2h}}. \tag{3.10}$$

Theorem 3.3. *Let τ and $|h|$ be sufficiently small positive numbers. Then, solutions of difference scheme* (3.6) *satisfy the following coercivity stability estimates:*

$$\begin{aligned}
&\left\| \left\{ \tau^{-2}\left(u_{k+1}^h - 2u_k^h + u_{k-1}^h\right) \right\}_1^{N-1} \right\|_{C_{0,1}^{\alpha}([0,1]_\tau, L_{2h})} \\
&\quad + \left\| \left\{ \tau^{-1}\left(u_k^h - u_{k-1}^h\right) \right\}_{-N+1}^{0} \right\|_{\widetilde{C}_0^{\alpha}([-1,0]_\tau, L_{2h})} + \left\| \left\{ u_k^h \right\}_1^{N-1} \right\|_{C_{0,1}^{\alpha}([0,1]_\tau, W_{2h}^2)} \\
&\quad + \left\| \left\{ u_{k-1}^h \right\}_{-N+1}^{0} \right\|_{\widetilde{C}_0^{\alpha}([-1,0]_\tau, W_{2h}^2)} \\
&\quad \leq M_{23}(\delta) \Bigg[\left\| \varphi^h \right\|_{W_{2h}^2} + \tau \left\| f_0^h \right\|_{W_{2h}^1} + \tau \left\| f_{-1}^h \right\|_{W_{2h}^1} + \tau \left\| g_1^h \right\|_{W_{2h}^1} \\
&\qquad + \frac{1}{\alpha(1-\alpha)} \left[\left\| \left\{ f_k^h \right\}_{-N+1}^{-1} \right\|_{\widetilde{C}_0^{\alpha}([-1,0]_\tau, L_{2h})} + \left\| \left\{ g_k^h \right\}_1^{N-1} \right\|_{C_{0,1}^{\alpha}([0,1]_\tau, L_{2h})} \right] \Bigg],
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
&\left\| \left\{ \tau^{-2}\left(u_{k+1}^h - 2u_k^h + u_{k-1}^h\right) \right\}_1^{N-1} \right\|_{C_{0,1}^{\alpha}([0,1]_\tau, L_{2h})} + \left\| \left\{ u_{k-1}^h \right\}_{-N+1}^{0} \right\|_{\widetilde{C}_0^{\alpha}([-1,0]_\tau, W_{2h}^2)} \\
&\quad + \left\| \left\{ \tau^{-1}\left(u_k^h - u_{k-1}^h\right) \right\}_{-N+1}^{0} \right\|_{\widetilde{C}_0^{\alpha}([-1,0]_\tau, L_{2h})} + \left\| \left\{ u_k^h \right\}_1^{N-1} \right\|_{C_{0,1}^{\alpha}([0,1]_\tau, W_{2h}^2)} \\
&\quad \leq M_{24}(\delta) \Bigg[\left\| \varphi^h \right\|_{W_{2h}^2} + \tau \left\| f_0^h \right\|_{W_{2h}^1} + \tau \left\| f_{-1}^h \right\|_{W_{2h}^1} + \tau \left\| g_1^h \right\|_{W_{2h}^1} \\
&\qquad + \frac{1}{\alpha(1-\alpha)} \left[\left\| \left\{ f_k^h \right\}_{-N+1}^{-1} \right\|_{\widetilde{C}_0^{\alpha}([-1,0]_\tau, L_{2h})} + \left\| \left\{ g_k^h \right\}_1^{N-1} \right\|_{C_{0,1}^{\alpha}([0,1]_\tau, L_{2h})} \right] \Bigg].
\end{aligned}$$

The proof of Theorem 3.3 is based on the abstract Theorem 2.5, Theorem 3.2, and the symmetry property of the difference operator A_h^x defined by formula (3.4).

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