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Research Article

Existence Theorem for Integral and Functional Integral Equations with Discontinuous Kernels

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Existence of extremal solutions of nonlinear discontinuous integral equations of Volterra type is proved. This result is extended herein to functional Volterra integral equations (FVIEs) and to a system of discontinuous VIEs as well.

1. Introduction

In this work the existence of extremal solutions of nonlinear discontinuous integral as well as functional integral equations is proved by weakening all forms of Caratheodory's condition. We consider the nonlinear Volterra integral equation (for short VIEs):

$$x(t) = u(t) + \int_0^t f(t,\tau,x(\tau))d\tau, \tag{1.1}$$

and the functional Volterra integral equations (for short FVIEs):

$$x(t) = u(t) + \int_0^t f(t,\tau,x(\tau),x)d\tau, \tag{1.2}$$

where f may be discontinuous with respect to all of their arguments. The special case of (1.1)

$$x(t) = \int_0^t k(t-\tau)g(x(\tau))d\tau, \tag{1.3}$$

has been studied extensively under continuity and/or monotonicity [1–4]. Meehan and O'Regan [5] established, by placing some monotonicity assumption on a nonlinear L^1-Carathéodory kernel of the form $k(t,s,x(s))$, existence of a $C[0,T]$ solution to (1.1). It is proven in [6] that, providing some type of discontinuous nonlinearities, (1.1) has extremal solutions. Dhage [7] proved under mixed Lipschitz, Carathéodory, and monotonicity conditions existence of extremal solutions of nonlinear discontinuous functional integral equations. Other remarkable work was done in [8–11].

The main objective in this paper is to emphasize that the kernel f is not required to be neither continuous nor monotonic in any of its arguments to establish an existence of extremal solutions for (1.1) (in $\mathbb{R}$) which generalizes in some aspects some of the previously mentioned works. A monotonicity type condition with respect to the functional term is needed to establish existence of extremal solutions to (1.2). We base the proof of the main result on, among other tools, the following lemmas which could analogously be proved as Lemma 1.1 and Lemma 1.2, see [12], and hence the proofs are omitted.

Lemma 1.1. *Suppose that $f : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies conditions (C1) and (C3). Let $x_1, x_2 : [0,1] \to \mathbb{R}$ be continuous and satisfy the inequality $x_1(\tau) < x_2(\tau)$ for all $\tau \in [0,1]$. Then the functions*

$$\varphi(t,\tau) = \inf_{y\in(x_1(\tau),x_2(t,\tau))} f(t,\tau,y), \qquad \psi(t,\tau) = \sup_{y\in(x_1(\tau),x_2(\tau))} f(t,\tau,y), \tag{1.4}$$

are Lebesgue measurable for each fixed $t \in [0,1]$. In particular, for each $t \in [0,1]$, $f(t,\cdot,x(\cdot))$ is Lebesgue measurable for each fixed $x \in C([0,1])$.

Lemma 1.2. *Suppose that $f : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies conditions (C1), (C2), and (C3). Let $x_1, x_2 : [0,1] \to \mathbb{R}$ be continuous and satisfy the inequality $x_1(\tau) < x_2(\tau)$ for all $\tau \in [0,1]$. Let, for each $(t,\tau,x) \in [0,1] \times [0,1] \times \mathbb{R}$,*

$$f^*(t,\tau,x) = \liminf_{y\downarrow x} f(t,\tau,y), \qquad f_*(t,\tau,x) = \limsup_{y\uparrow x} f(t,\tau,y). \tag{1.5}$$

The compositions $f^*(t,\cdot,x(\cdot))$ and $f_*(t,\cdot,x(\cdot))$ are Lebesgue measurable for all $t \in [0,1]$ any continuous $x : [0,1] \to \mathbb{R}$, and, for almost all $\tau \in [0,1]$,

$$\begin{aligned} \inf_{y\in(x_1(\tau),x_2(\tau))} f_*(t,\tau,y) &= \inf_{y\in(x_1(\tau),x_2(\tau))} f(t,\tau,y) = \inf_{y\in(x_1(\tau),x_2(\tau))} f^*(t,\tau,y), \\ \sup_{y\in(x_1(\tau),x_2(\tau))} f_*(t,\tau,y) &= \sup_{y\in(x_1(\tau),x_2(\tau))} f(t,\tau,y) = \sup_{y\in(x_1(\tau),x_2(\tau))} f^*(t,\tau,y). \end{aligned} \tag{1.6}$$

The outline of the work is as follows. In Section 2 we present our existence theorem for (1.1) in $\mathbb{R}$. In Sections 3 and 4 generalizations of this established existence theorem for functional Volterra integral equation as well as for system of nonlinear Volterra integral equations are presented. Comparison with the literature is provided throughout the paper.

2. Volterra Integral Equations

Theorem 2.1. *Let* $u : [0,1] \to \mathbb{R}$ *and let* $f : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ *be given. Suppose that* (C1)–(C4) *are fulfilled.*

(C1) u *is continuous.*

(C2) *For each* $(t,x) \in [0,1] \times \mathbb{R}$*, the function* $\tau \mapsto f(t,\tau,x)$ *is Lebesgue measurable. For all* $(t,x) \in [0,1] \times \mathbb{R}$ *and for almost all* $\tau \in [0,1]$*,*

$$\left| f(t,\tau,x) \right| < M(\tau), \tag{2.1}$$

where $M : [0,1] \to [0,\infty]$ *is a Lebesgue integrable function.*

(C3) *For each* $(t,\tau,x) \in [0,1] \times [0,1] \times \mathbb{R}$*,*

$$\limsup_{y \uparrow x} f(t,\tau,y) \le f(t,\tau,x) = \liminf_{y \downarrow x} f(t,\tau,y). \tag{2.2}$$

(C4) *Let* $F = \{y \in \mathbb{R}; |y| \le |u| + |\int_0^1 M(\tau)d\tau|\}$*, where,* $|u| = \max\{|u(t)|; t \in [0,1]\}$*. For every* $y \in F$ *and all* $n \in N$*, the functions*

$$t \mapsto \int_0^t \sup_{|x-y| \le 1/3^n} f(t,\tau,x)d\tau, \tag{2.3}$$

are equicontinuous and tend to zero as $t \downarrow 0$*.*

Under the above assumptions VIE expressed by (1.1) has extremal solutions in the interval $[0,1]$.

Proof. We will prove the existence of a maximal solution the proof of the existence of a minimal solution is analogous and hence is omitted. The pattern of the proof consists of four steps. Similarly as it was done in [13, 14] we define the maximal solution as the limit of an appropriate sequence of approximations x_n, $n \in N$.

Step 1. Since u, being a continuous function on compact set, is uniformly continuous and $E \mapsto \int_E M d\tau$, being absolutely continuous with respect to Lebesgue measure, is uniformly continuous on $[0,1]$; then for all $n \in N$ there exists $\delta_n > 0$ such that

$$|u(s) - u(t)| + \left| \int_s^t M(\tau)d\tau \right| \le \frac{1}{3^{n+2}}, \tag{2.4}$$

for all $s, t \in [0,1]$, with $|s - t| \le \delta_n$. Next, we take for $n \in N$ subdivisions D_n

$$t_0^n < t_1^n < \cdots < t_{k_n}^n \tag{2.5}$$

of $[0,1]$ in such a way that $0 = t_0^n < t_1^n < \cdots < t_{k_n}^n = 1$, D_{n+1} is a refinement of D_n, that is, $D_n \subset D_{n+1}$ and

$$t_{k+1}^n - t_k^n \leq \delta_n, \quad k = 0, 1, \ldots, k_n - 1. \tag{2.6}$$

For any $n \in N$, $g_n : [0,1] \times [0,1] \to \mathbb{R}$ and $x_n : [0,1] \to \mathbb{R}$ are recursively defined by, for $s, t \in [0, t_1^n]$,

$$\begin{aligned} g_n(t,\tau) &= \sup_{|x-u(0)| \leq 2/3^n} f(t,\tau,x), \\ x_n(t) &= u(t) + \int_0^t g_n(t,\tau)d\tau. \end{aligned} \tag{2.7}$$

Once g_n, x_n have already been defined on $[0, t_k^n]$, with $k < k_n$, they are defined in $[t_k^n, t_{k+1}^n]$ by putting

$$g_n(t,\tau) = \sup_{|x-x_n(t_k^n)| \leq 2/3^n} f(t,\tau,x), \tag{2.8}$$

$$x_n(t) = x_n(t_k^n) + u(t) - u(t_k^n) + \int_{t_k^n}^t g_n(t,\tau)d\tau. \tag{2.9}$$

It follows, by Lemma 1.1, that functions g_n are Lebesgue measurable; taking into account this together with $(C2)$, x_n is well defined. Moreover it is easy to see that for all $t \in [0,1]$ and all $n \in N$

$$x_n(t) = u(t) + \int_0^t g_n(t,\tau)d\tau. \tag{2.10}$$

Step 2. We claim that, for all $n \in N$,

(i) $x_{n+1} \leq x_n$,

(ii) if $x : [0,1] \to \mathbb{R}$ is a continuous function, which serves as a dummy function, satisfying, $x(0) \leq u(0)$ and $x(t) \leq x(s) + u(t) - u(s) + \int_s^t f(t,\tau,x(\tau))d\tau$, for $s, t \in [0,1]$, then $x \leq x_n$ in $[0,1]$.

To prove these assertions we shall proceed inductively. Clearly, $x_{n+1}(0) = x_n(0) = p(0)$. Let us suppose that $x_{n+1}(t) \leq x_n(t)$, for $t \in [0, t_k^n]$, with some $k < k_n$. Since $D_n \subset D_{n+1}$, there exist $i, j \in \{0, 1 \ldots, k_{n+1}\}$, $i < j$, such that $t_i^{n+1} = t_k^n$ and $t_j^{n+1} = t_{k+1}^n$. Let us suppose that $x_{n+1}(t) \leq x_n(t)$, for $[0, t_m^{n+1}]$, with an $m \in \{i, i+1, \ldots, j-1\}$. At this point we have just two possibilities:

(P_{11}) $x_{n+1}(t_m^{n+1}) < x_n(t_m^{n+1}) - 2/3^{n+1}$,

(P_{12}) $x_n(t_m^{n+1}) - 2/3^{n+1} \leq x_{n+1}(t_m^{n+1}) \leq x_n(t_m^{n+1})$.

If (P_{11}) holds, since $t_{m+1}^{n+1}-t_m^{n+1} \le \delta_{n+1}$, it follows, by (2.4) and (2.9), that for $t \in [t_m^{n+1}, t_{m+1}^{n+1}]$

$$\begin{aligned} x_{n+1}(t) &= x_{n+1}\left(t_m^{n+1}\right) + u(t) - u\left(t_m^{n+1}\right) + \int_{t_m^{n+1}}^{t} g_n(t,\tau)d\tau \\ &< x_{n+1}\left(t_m^{n+1}\right) + \frac{1}{3^{n+2}} < x_n\left(t_m^{n+1}\right) - \frac{2}{3^{n+1}} + \frac{1}{3^{n+2}} \\ &= x_n(t) + \left[x_n\left(t_m^{n+1}\right) - x_n(t)\right] - \frac{5}{3^{n+2}} \\ &< x_n(t) + \frac{1}{3^{n+2}} - \frac{5}{3^{n+2}} = x_n(t) + \frac{1}{3^{n+1}} - \frac{5}{3^{n+2}} < x_n(t). \end{aligned} \tag{2.11}$$

Assume the validity of (P_{12}); it follows, by (2.4) and (2.9), that

$$\begin{aligned} x_{n+1}\left(t_m^{n+1}\right) - \frac{2}{3^{n+1}} &\ge x_n\left(t_m^{n+1}\right) - \frac{2}{3^{n+1}} - \frac{2}{3^{n+1}} \\ &= x_n\left(t_k^n\right) + \left[x_n\left(t_m^{n+1}\right) - x_n\left(t_k^n\right)\right] - \frac{4}{3^{n+1}} \\ &> x_n\left(t_k^n\right) - \frac{1}{3^{n+2}} - \frac{4}{3^{n+1}} = x_n\left(t_k^n\right) - \frac{1}{3^{n+1}} - \frac{4}{3^{n+1}} > x_n\left(t_k^n\right) - \frac{2}{3^n}. \end{aligned} \tag{2.12}$$

On the other hand we have

$$\begin{aligned} x_{n+1}\left(t_m^{n+1}\right) + \frac{2}{3^{n+1}} &\le x_n\left(t_m^{n+1}\right) + \frac{2}{3^{n+1}} \\ &= x_n\left(t_k^n\right) + \left[x_n\left(t_m^{n+1}\right) - x_n\left(t_k^n\right)\right] + \frac{2}{3^{n+1}} \\ &< x_n\left(t_k^n\right) + \frac{1}{3^{n+2}} + \frac{2}{3^{n+1}} = x_n\left(t_k^n\right) + \frac{1}{3^{n+1}} + \frac{2}{3^{n+1}} < x_n\left(t_k^n\right) + \frac{2}{3^n}. \end{aligned} \tag{2.13}$$

We thus have

$$\left(x_{n+1}\left(t_m^{n+1}\right) - \frac{2}{3^{n+1}}, x_{n+1}\left(t_m^{n+1}\right) + \frac{2}{3^{n+1}}\right) \subset \left(x_n\left(t_k^n\right) - \frac{2}{3^n}, x_n\left(t_k^n\right) + \frac{2}{3^n}\right), \tag{2.14}$$

and hence, for each $\tau \in [t_m^{n+1}, t_{m+1}^{n+1}]$,

$$g_{n+1}(t,\tau) = \sup_{\left|x - x_{n+1}\left(t_m^{n+1}\right)\right| \le 2/3^n} f(t,\tau,x) \le \sup_{\left|x - x_n\left(t_k^n\right)\right| \le 2/3^n} f(t,\tau,x) = g_n(t,\tau). \tag{2.15}$$

By (2.9),

$$x_n\left(t_m^{n+1}\right) = x_n\left(t_k^n\right) + u\left(t_m^{n+1}\right) - u\left(t_k^n\right) + \int_{t_k^n}^{t_m^{n+1}} g_n(t,\tau)d\tau. \tag{2.16}$$

This in turn implies that, for all $t \in [t_m^{n+1}, t_{m+1}^{n+1}]$,

$$\begin{aligned} x_{n+1}(t) &= x_{n+1}\left(t_m^{n+1}\right) + u(t) - u\left(t_m^{n+1}\right) + \int_{t_m^{n+1}}^{t} g_{n+1}(t,\tau)d\tau \\ &\leq x_n\left(t_m^{n+1}\right) + u(t) - u\left(t_m^{n+1}\right) + \int_{t_m^{n+1}}^{t} g_n(t,\tau)d\tau \\ &= x_n\left(t_i^n\right) + u\left(t_m^{n+1}\right) - u\left(t_k^n\right) + \int_{t_k^n}^{t_m^{n+1}} g_n(t,\tau)d\tau + u(t) - u\left(t_m^{n+1}\right) + \int_{t_m^{n+1}}^{t} g_n(t,\tau)d\tau \\ &= x_n\left(t_k^n\right) + u(t) - u\left(t_k^n\right) + \int_{t_k^n}^{t} g_n(t,\tau)d\tau = x_n(t), \end{aligned} \tag{2.17}$$

which completes the proof of (i). Now we shall handle (ii) inductively too. Let us fix an arbitrary $n \in N$, by assumption, $x(0) \leq u(0) = x_n(0)$. Let us suppose that $x(t) \leq x_n(t)$, for $t \in [0, t_k^n]$, with $k < k_n$. At this point we have just two possibilities:

(P_{21}) $x(t_k^n) < x_n(t_k^n) - 2/3^{n+1}$,

(P_{22}) $x_n(t_k^n) - 2/3^{n+1} \leq x(t_k^n) \leq x_n(t_k^n)$.

Suppose that we are in (P_{21}); since $t_{k+1}^n - t_k^n \leq \delta_n$, $k = 0, 1, 2, \ldots, k_n - 1$, it follows, by (2.4) and (2.9); and (ii), that

$$\begin{aligned} x(t) &\leq x\left(t_k^n\right) + u(t) - u\left(t_k^n\right) + \int_{t_k^n}^{t} f(t,\tau,x(\tau))d\tau \\ &< x\left(t_k^n\right) + \frac{1}{3^{n+2}} < x_n\left(t_k^n\right) - \frac{2}{3^{n+1}} + \frac{1}{3^{n+2}} \\ &= x_n(t) + \left[x_n\left(t_k^n\right) - x_n(t)\right] - \frac{5}{3^{n+2}} < x_n(t) + \frac{1}{3^{n+2}} - \frac{5}{3^{n+2}} < x_n(t), \end{aligned} \tag{2.18}$$

for $t \in [t_k^n, t_{k+1}^n]$.

If we are in (P_{22}), it follows, by (2.4) and (2.9), that, for $t \in [t_k^n, t_{k+1}^n]$,

$$\begin{aligned} x(t) &= x\left(t_k^n\right) + \left[x(t) - x\left(t_k^n\right)\right] \geq x_n\left(t_k^n\right) - \frac{2}{3^{n+1}} + \left[x(t) - x\left(t_k^n\right)\right] \\ &> x_n\left(t_k^n\right) - \frac{2}{3^{n+1}} - \frac{1}{3^{n+2}} > x_n\left(t_k^n\right) - \frac{2}{3^n}. \end{aligned} \tag{2.19}$$

By (ii) and (2.9),

$$\begin{aligned} x(t) &\leq x\left(t_k^n\right) + u(t) - u\left(t_k^n\right) + \int_{t_k^n}^{t} f(t,\tau,x(\tau))d\tau \\ &\leq x_n\left(t_k^n\right) + u(t) - u\left(t_k^n\right) + \int_{t_k^n}^{t} f(t,\tau,x(\tau))d\tau < x_n\left(t_k^n\right) + \frac{1}{3^{n+2}} < x_n\left(t_k^n\right) + \frac{2}{3^n}. \end{aligned} \tag{2.20}$$

We thus have for all $\tau \in [t_k^n, t_{k+1}^n]$, $x(\tau) \in (x_n(t_k^n) - 2/3^n, x_n(t_k^n) + 2/3^n)$, and hence

$$f(t,\tau,x(\tau)) \le \sup_{|x - x_n(t_k^n)| \le 2/3^n} f(t,\tau,x) = g_n(t,\tau), \tag{2.21}$$

for all $\tau \in [t_k^n, t_{k+1}^n]$, so it follows, by (2.4) and (2.9), that

$$\begin{aligned} x(t) &\le x_n(t_k^n) + u(t) - u(t_k^n) + \int_{t_k^n}^t f(t,\tau,x(\tau))d\tau \\ &\le x_n(t_k^n) + u(t) - u(t_k^n) + \int_{t_k^n}^t g_n(t,\tau)d\tau = x_n(t), \end{aligned} \tag{2.22}$$

for $t \in [t_k^n, t_{k+1}^n]$, which completes the proof of the assertion (ii).

Step 3. It follows, by $(C4)$, that the constructed bounded nonincreasing sequence x_n, $n \in N$ is uniformly convergence, and hence let us set

$$x^+(t) = \lim_{n\to\infty} x_n(t), t \in [0,1]. \tag{2.23}$$

By (2.4) and (2.9), we have $|x_n(\tau) - x_n(t_k^n)| \le 1/3^{n+2}$, for $\tau \in [t_k^n, t_{k+1}^n]$. Thus,

$$g_n(t,\tau) = \sup_{|x - x_n(t_k^n)| \le 2/3^n} f(t,\tau,x) \ge f(t,\tau,x_n(\tau)), \tag{2.24}$$

for $\tau \in [t_k^n, t_{k+1}^n]$. Therefore, by formula just before Step 2, for $t \in [0,1]$, and $n \in N$

$$x_n(t) = u(t) + \int_0^t g_n(t,\tau)d\tau \ge u(t) + \int_0^t f(t,\tau,x_n(\tau))d\tau. \tag{2.25}$$

Applying Fatou Lemma and taking into account the condition $(C3)$ and in view of Lemma 1.2, we obtain

$$\begin{aligned} x^+(t) = \lim_{n\to\infty} x_n(t) = \liminf_{n\to\infty} x_n(t) &\ge u(t) + \int_0^t \liminf_{n\to\infty} f(t,\tau,x_n(\tau))d\tau \\ &= u(t) + \int_0^t f(t,\tau,x^+(\tau))d\tau, \end{aligned} \tag{2.26}$$

for each $t \in [0,1]$. Let us observe that

$$\limsup_{n\to\infty} g_n(t,\tau) \le f(t,\tau,x^+(\tau)), \tag{2.27}$$

almost everywhere in $[0,1]$. Indeed, let us fix an $n \in N$. For any $\tau \in [0,1]$ there exists a $k = 0,\ldots,k_n$ such that $\tau \in [t_k^n, t_{k+1}^n]$. Since

$$g_n(t,\tau) = \sup_{|x - x_n(t_k^n)| \le 2/3^n} f(t,\tau,x), \tag{2.28}$$

there exists an $\widehat{x_n}$, with $|\widehat{x_n} - x_n(t_k^n)| \le 2/3^n$ such that

$$g_n(t,\tau) - \frac{1}{3^n} \le f(t,\tau,\widehat{x_n}) \le g_n(t,\tau). \tag{2.29}$$

Whence,

$$\limsup_{n\to\infty} g_n(t,\tau) = \limsup_{n\to\infty} f(t,\tau,\widehat{x_n}). \tag{2.30}$$

$$|\widehat{x_n} - x_n(\tau)| \le \left|\widehat{x_n} - x_n(t_k^n)\right| + \left|x_n(t_k^n) - x_n(\tau)\right| \le \frac{4}{3^n}. \tag{2.31}$$

We thus have

$$\lim_{n\to\infty} \widehat{x_n} = \lim_{n\to\infty} x_n(\tau) = x^+(\tau), \tag{2.32}$$

which together with $(C3)$ and (2.30) implies that

$$\limsup_{n\to\infty} g_n(t,\tau) \le f(t,\tau,x^+(\tau)), \tag{2.33}$$

almost everywhere in $[0,1]$. Applying Fatou lemma once again we obtain

$$x^+(t) = \lim_{n\to\infty} x_n(t) = \limsup_{n\to\infty} x_n(t) \le u(t) + \int_0^t \limsup_{n\to\infty} g_n(t,\tau)d\tau \le u(t) + \int_0^t f(t,\tau,x^+(\tau))d\tau, \tag{2.34}$$

which together with (2.26) means that x^+ is a solution (1.1).

Step 4. Let $x : [0,1] \to \mathbb{R}$ be a solution of (1.1). Clearly, x is continuous and satisfies the conditions of the assertion (ii), so, $x \le x_n$, for any $n \in N$. Since $x_n \to x^+$ as $n \to \infty$, then $x \le x^+$. This shows that x^+ is a maximal solution of (1.1).

We proceed similarly to prove the existence of minimal solution; we first define recursively the functions $h_n(t,\tau)$ and $x_n(t)$, by setting

$$h_n(t,\tau) = \inf_{|x - x_n(t_k^n)| \le 2/3^n} f(t,\tau,x), \quad \tau \in [0,1],$$
$$x_n(t) = x_n(t_k^n) + u(t) - u(t_k^n) + \int_{t_k^n}^t h_n(t,\tau)d\tau, \quad t \in [t_k^n, t_{k+1}^n]. \tag{2.35}$$

Taking $x : [0,1] \to \mathbb{R}$ to be a continuous function, which serves as a dummy function, satisfying $x(0) \geq u(0)$ and $x(t) \geq x(s) + u(t) - u(s) + \int_s^t f(t,\tau,x(\tau))d\tau$, for $s,t \in [0,1]$, and just following the previous steps one can show that (1.1) has a minimal solution x^- in $[0,1]$. This completes the proof. □

It is interesting to point out that "=" in condition $(C3)$ could not be replaced with "≤" as it was done in [12]. Probably the reason consists in the fact that the composition $f(t,\cdot,x(\cdot))$ is no longer measurable for any continuous function $x : [0,1] \to \mathbb{R}$; the following example illustrates this fact.

Example 2.2. Let $S \subset [0,1]$ be any non-Lebesgue measurable subset. Define $f : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ by

$$f(t,\tau,x) = \begin{cases} 1 & \text{if } x > \tau, \\ 1 & \text{if } x = \tau, \tau \in S, \\ 0 & \text{otherwise.} \end{cases} \tag{2.36}$$

It is easy to see that conditions $(C1)$ and $(C2)$ are satisfied. Since f is nondecreasing in x, we have, for all $(t,\tau,x) \in [0,1] \times [0,1] \times \mathbb{R}$,

$$\limsup_{y \uparrow x} f(t,\tau,y) \leq f(t,\tau,x) \leq \liminf_{y \downarrow x} f(t,\tau,y). \tag{2.37}$$

However the composition $f(t,\cdot,x(\cdot))$ is not Lebesgue measurable if $x(\tau) = \tau$, for $\tau \in [0,1]$.

3. Functional Volterra Integral Equations

Our main concern in this section is to extend result established herein (Theorem 2.1) to a functional Volterra integral equation in deriving existence of extremal solutions for a class of FVIEs (1.2).

Notations. $M : [0,1] \to [0,\infty]$ is a Lebesgue integrable function, $C_M([0,1];\mathbb{R})$ is the set of all continuous functions $x : [0,1] \to \mathbb{R}$ satisfying $|x(t) - x(s)| \geq |u(t) - u(s)| + |\int_s^t M(\tau)d\tau|$ for all $s,t \in [0,1]$. For a fixed $\varphi \in C_M([0,1])$ let $f_\varphi : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ be the function defined by $f_\varphi(t,\tau,x) = f(t,\tau,x,\varphi)$, and let $\Gamma_\varphi = \{x \in C_M([0,1];\mathbb{R}) \mid x(0) \geq u(0),\ x(t) \geq x(s) + u(t) - u(s) + \int_s^t f_\varphi(t,\tau,x(\tau))d\tau, \text{for } s,t \in [0,1]\}$.

Theorem 3.1. *Let Υ denote the set of all $f : [0,1] \times [0,1] \times \mathbb{R} \times C_M([0,1];\mathbb{R}) \to \mathbb{R}$ satisfying the following conditions $(F1)$–$(F5)$.*

- $(F1)$ *$u : [0,1] \to \mathbb{R}$ is continuous,*
- $(F2)$ *For each $(t,x) \in [0,1] \times \mathbb{R}$ and $\varphi \in C_M([0,1];\mathbb{R}), \tau \mapsto f(t,\tau,x,\varphi)$ is Lebesgue measurable, and for almost all $\tau \in [0,1]$,*

$$|f(t,\tau,x,\varphi)| < M(\tau). \tag{3.1}$$

($F3$) *For each* $(t,\tau,x,\varphi) \in [0,1] \times [0,1] \times \mathbb{R} \times C_M([0,1];\mathbb{R})$,

$$\limsup_{y\uparrow x} f(t,\tau,y,\varphi) \le f(t,\tau,x,\varphi) = \liminf_{y\downarrow x} f(t,\tau,y,\varphi). \tag{3.2}$$

($F4$) *For each* $(t,\tau,x) \in [0,1] \times [0,1] \times \mathbb{R}$, $f(t,\tau,x,\varphi) \le f(t,\tau,x,\psi)$ *whenever* $\varphi,\psi \in C_M([0,1];\mathbb{R})$ *with* $\varphi \le \psi$.

($F5$) *Let* $\digamma = \{y \in \mathbb{R}; |y| \le |u| + |\int_0^1 M(\tau)d\tau|\}$, *where,* $|u| = \max\{|u(t)|; t \in [0,1]\}$. *Let* $\varphi \in C_M([0,1];\mathbb{R})$ *be fixed, for every* $y \in \digamma$ *and all* $n \in N$; *the functions*

$$t \longmapsto \int_0^t \sup_{|x-y|\le 1/3^n} f_\varphi(t,\tau,x)d\tau, \tag{3.3}$$

are equicontinuous and tend to zero as $t \downarrow 0$.

Under the previous assumptions FVIE expressed by (1.2) has extremal solutions in the interval $[0,1]$.

Proof. Since the proofs of existence of maximal and minimal solutions are similar, we concentrate our attention on showing the existence of the minimal solution.

For a fixed $\varphi \in C_M([0,1];\mathbb{R})$ let us consider the nonfunctional Volterra integral equation N-FVIE:

$$x(t) = u(t) + \int_0^t f_\varphi(t,\tau,x(\tau))d\tau. \tag{3.4}$$

Obviously, the function $f_\varphi(t,\tau,x(\tau))$ satisfies the hypotheses of Theorem 2.1; we thus conclude that N-FVIE (3.4) has a maximal solution which is given by

$$x_\varphi(t) = \inf_{x\in\Gamma_\varphi} x(t). \tag{3.5}$$

Let $\beth = \{\varphi \in C_M([0,1];\mathbb{R}) \mid x_\varphi \le \varphi\}$. Since $-u(t) - \int_s^t M(\tau)d\tau$ belongs to $\beth$, then the set $\beth$ is not empty. Define

$$\overline{x}(t) = \inf_{\varphi\in\beth} \varphi(t). \tag{3.6}$$

Given $\varphi \in \beth$ and $x \in \Gamma_\varphi$, it follows, by ($F4$), that

$$x(t) \ge x(s) + u(t) - u(s) + \int_s^t f(t,\tau,x(\tau),\varphi)d\tau \ge x(s) + u(t) - u(s) + \int_s^t f(t,\tau,x(\tau),\overline{x}). \tag{3.7}$$

We thus have $x \in \Gamma_{\overline{x}}$, $\Gamma_\varphi \subseteq \Gamma_{\overline{x}}$ and $x_\varphi \geq x_{\overline{x}}$. Since $\varphi \in \beth$ is arbitrary, then $\varphi \geq x_\varphi \geq x_{\overline{x}}$, and hence

$$\overline{x} \geq x_{\overline{x}}. \tag{3.8}$$

On the other hand, for $x \in \Gamma_\varphi$ we have

$$\begin{aligned} x(t) &\geq x(s) + u(t) - u(s) + \int_s^t f(t,\tau,x(\tau),\varphi)d\tau \geq x(s) + u(t) - u(s) + \int_s^t f(t,\tau,x(\tau),\overline{x}) \\ &\geq x(s) + u(t) - u(s) + \int_s^t f(t,\tau,x(\tau),x_{\overline{x}}). \end{aligned} \tag{3.9}$$

Thus $x \in \Gamma_\varphi \subseteq \Gamma_{\overline{x}} \subseteq \Gamma_{x_{\overline{x}}}$. Consequently, $x_\varphi \geq x_{\overline{x}} \geq x_{x_{\overline{x}}}$ which implies that $x_{\overline{x}} \in \beth$, and thus $\overline{x} \leq x_{\overline{x}}$ which together with (3.8) implies that $\overline{x} = x_{\overline{x}}$. Since every solution of (1.2) belongs to $\beth$, then $\overline{x}$ is a minimal solution. This completes the proof. □

4. System of Volterra Integral Equations

The main obstacle to extending the results of the previous section for vector-valued functions is that the usual order in $\mathbb{R}^n$ makes the condition,

$$\limsup_{y\uparrow x} f(t,\tau,y) \leq f(t,\tau,x) = \liminf_{y\downarrow x} f(t,\tau,y), \tag{4.1}$$

used for scalar functions, does not have a good equivalence for vector-valued functions. We now show how Theorem 2.1 may be exploited to derive existence of extremal solutions for a class of systems of discontinuous VIEs. The proof is based on a technique similar to that used for systems of differential and functional differential equations [12, 15].

Theorem 4.1. *Given $u = (u_1,\dots,u_n) : [0,1] \to \mathbb{R}^n$ and $f = (f_1,\dots,f_n) : [0,1]\times[0,1]\times\mathbb{R}^n \to \mathbb{R}^n$ satisfying the conditions (B1)–(B4) below*

- (B1) *$f(t,\cdot,x(\cdot))$ is Lebesgue measurable for any continuous $x : [0,1] \to \mathbb{R}^n$,*
- (B2) *for each $i = 1,\dots,n$ and Lebesgue almost all $t \in [0,1]$, f_i is nondecreasing in x_k, $k = 1,\dots,i-1,i+1,\dots,n$ and for all $x = (x_1,\dots,x_n) \in \mathbb{R}^n$,*

$$\begin{aligned} \limsup_{y\uparrow x_i} f_i(t,\tau,x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_n) &\leq f_i(t,\tau,x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_n) \\ &= \lim_{y\downarrow x_i} f_i(t,\tau,x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_n), \end{aligned} \tag{4.2}$$

- (B3) *for each $x \in \mathbb{R}^n$ and Lebesgue almost all $t \in [0,1]$, $\|f(t,\tau,x)\| \leq M(\tau)$, where $\|x\| = \max\{|x_i|; i = 1,\dots,n\}$ and $M : [0,1] \to [0,\infty]$ is a Lebesgue integrable function,* (*B*4) *let* $F = \{y \in \mathbb{R}^n; \|y\| \le \|u\| + |\int_0^1 M(\tau)d\tau|\}$, *for every* $y \in F$ *and all* $k \in N$, *the functions*

$$t \mapsto \int_0^t \sup_{|x-y|\le 1/3^k} f(t,\tau,x)d\tau, \tag{4.3}$$

are equicontinuous and tend to zero as $t \downarrow 0$, *where* $\sup f$ *and* $\int f$ *are interpreted componentwise.*

Under the above assumptions VIE expressed by (1.1) (in $\mathbb{R}^n$) has extremal solutions in the interval $[0,1]$.

Proof. We shall only prove the existence of a maximal solution, since the same pattern could be followed to prove existence of a minimal solution. Note that, for $x, y \in \mathbb{R}^n$, we write $x \le y$ if $x_i \le y_i$, for each $i = 1,\dots,n$. Let us denote by X the set of all $x = (x_1,\dots,x_n) : [0,1] \to \mathbb{R}^n$ satisfying the following conditions

$$\begin{aligned} x_i(t) &\le u_i(t) + \int_0^t f_i(t,\tau,x_1(\tau),\dots,x_n(\tau))d\tau, i = 1,2,\dots,n, \\ \|x(t)-x(s)\| &\le \|u(t)-u(s)\| + \left|\int_s^t M(\tau)d\tau\right|, s,t \in [0,1]. \end{aligned} \tag{4.4}$$

For every $i = 1,2,\dots,n$, we let

$$x_i^+(t) = \sup_{(x_1,\dots,x_n)\in X} x_i(t), t \in [0,1]. \tag{4.5}$$

It follows, by monotonicity, that for every $x \in X$, for each $i = 1,\dots,n$, and for all $t \in [0,1]$,

$$x_i(t) \le u_i(t) + \int_0^t f_i\left(t,\tau,x_1^+(\tau),\dots,x_{i-1}^+(\tau),x_i(\tau),x_{i+1}^+(\tau),\dots,x_n^+(\tau)\right)d\tau. \tag{4.6}$$

Let us define a nonincreasing sequence, whose existence is guaranteed by hypotheses and for its recursively construction we follow arguments developed in *Step 1* of the proof of Theorem 2.1; (y_m), $y_m = (y_m^1,\dots,y_m^n) : [0,1] \to \mathbb{R}^n$ such that, for each $i = 1,\dots,n$

$$\begin{aligned} y_m^i(0) = u_i(0), \qquad y_m^i(t) &= u_i(t) + \int_0^t g_m^i(\tau)d\tau, \\ \|y(t)-y(s)\| &\le \|u(t)-u(s)\| + \left|\int_s^t M(\tau)d\tau\right|, s,t \in [0,1], \end{aligned} \tag{4.7}$$

where

$$g_m^i(\tau) = \sup_{|x-y_m^i(t_k^n)|\le 2/3^m} f_i\left(t,\tau,x_1^+(\tau),\dots,x_{i-1}^+(\tau),x,x_{i+1}^+(\tau),\dots,x_n^+(\tau)\right). \tag{4.8}$$

Let, for each $i = 1, \ldots, n$,

$$\lim_{m \to \infty} y_m^i(\tau) = y_i^+(\tau), \tau \in [0,1]. \tag{4.9}$$

Clearly $x_i^+(t) \leq y_i^+(t)$. Regarding f_i as a function only of y_i^+ while the remaining variables are considered to be constant. It follows, by similar arguments used in the proof of Theorem 2.1, that for every $i = 1, \ldots, n$, and for all $t \in [0,1]$,

$$y_i^+(t) = u_i(t) + \int_0^t f_i\left(t, \tau, x_1^+(\tau), \ldots, x_{i-1}^+(\tau), y_i^+(\tau), x_{i+1}^+(\tau), \ldots, x_n^+(\tau)\right) d\tau. \tag{4.10}$$

By monotonicity

$$y_i^+(t) \leq u_i(t) + \int_0^t f_i\left(t, \tau, y_1^+(\tau), \ldots, y_{i-1}^+(\tau), y_i^+(\tau), y_{i+1}^+(\tau), \ldots, y_n^+(\tau)\right) d\tau, i = 1, \ldots, n. \tag{4.11}$$

We thus have $y^+ = (y_1^+, \ldots, y_n^+) \in X$, so, $y^+ = x^+$, which together with (4.10) implies that y^+ is a solution of (1.1) on $[0,1]$. Proceeding analogously as *Step 4* of the proof of Theorem 2.1, one can show that y^+ is a maximal solution of (1) on $[0,1]$. □

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