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Research Article

Characteristic Functions and Borel Exceptional Values of E-Valued Meromorphic Functions

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The main purpose of this paper is to investigate the characteristic functions and Borel exceptional values of E-valued meromorphic functions from the $\mathbb{C}_R = \{z : |z| < R\}$, $0 < R \le +\infty$ to an infinite-dimensional complex Banach space E with a Schauder basis. Results obtained extend the relative results by Xuan, Wu and Yang, Bhoosnurmath, and Pujari.

1. Introduction and Preliminaries

In 1980s, Ziegler [1] succeeded in extending the Nevanlinna theory of meromorphic functions to the vector-valued meromorphic functions in finite dimensional spaces. Later, Hu and Yang [2] established the Nevanlinna theory of meromorphic mappings with the range in an infinite-dimensional Hilbert spaces. In 2006, C.-G. Hu and Q. Hu [3] established the Nevanlinna's first and second main theorems of meromorphic mappings with the range in an infinite-dimensional Banach spaces E with a Schauder basis. Recently, Xuan and Wu [4] established the Nevanlinna's first and second main theorems for an E-valued meromorphic mapping from a generic domain $D \subsetneq \mathbb{C}$ to an infinite-dimensional Banach spaces E with a Schauder basis.

In [4], Xuan and Wu also proved Chuang's inequality (see, e.g., [5]) of E-valued meromorphic mapping $f(z)$ in the whole complex plane, which compares the relationship between $T(r, f)$ and $T(r, f')$, and also obtained that the order and the lower order of E-valued meromorphic mapping $f(z)$ and those of its derivative $f'(z)$ are the same. In Section 2, we shall prove that Chuang's inequality is valid for E-valued meromorphic mapping $f(z)$ in the unit disc and prove that for any infinite-order E-valued meromorphic function $f(z)$ defined in the unit disc has the same Xiong's proximate order as its derivative $f'(z)$.

In [5], Yang obtained much stronger results than those of Gopalakrishna and Bhoosnurmath [6] for the Borel exceptional values of meromorphic functions dealing with multiple values. In Section 3, we shall extend Le Yang's result to E-valued meromorphic functions of finite and infinite orders in

$$\mathbb{C}_R := \{z : |z| < R\}, \quad 0 < R \leq +\infty. \tag{1.1}$$

In the following, we introduce the definitions, notations, and results of [3, 4] which will be used in this paper.

Let $(E, \|\bullet\|)$ be an infinite dimension complex Banach space with Schauder basis $\{e_j\}$ and the norm $\|\bullet\|$. Thus, an E-valued meromorphic function $f(z)$ defined in $\mathbb{C}_R$, $0 < R \leq +\infty$ can be written as

$$f(z) = \left(f_1(z), f_2(z), \ldots, f_k(z), \ldots\right). \tag{1.2}$$

Let E_n be an n-dimensional projective space of E with a basis $\{e_j\}_1^n$. The projective operator $P_n : E \to E_n$ is a realization of E_n associated with basis.

The elements of E are called vectors and are usually denoted by letters from the alphabet: $a, b, c, \ldots$. The symbol 0 denotes the zero vector of E. We denote vector infinity, complex number infinity, and the norm infinity by $\widehat{\infty}, \infty$, and $+\infty$, respectively. A vector-valued mappings is called holomorphic (meromorphic) if all $f_j(z)$ are holomorphic (some of $f_j(z)$ are meromorphic). The jth derivative $j = 1, 2, \ldots$ of $f(z)$ is defined by

$$f^{(j)}(z) = \left(f_1^{(j)}(z), f_2^{(j)}(z), \ldots, f_k^{(j)}(z), \ldots\right). \tag{1.3}$$

A point $z_0 \in \mathbb{C}_r$ is called a "pole" (or $\widehat{\infty}$ point) of

$$f(z) = \left(f_1(z), f_2(z), \ldots, f_k(z), \ldots\right) \tag{1.4}$$

if z_0 is a pole (or ∞ point) of at least one of the component functions $f_k(z)$ $(k = 1, 2, \ldots)$. A point $z_0 \in \mathbb{C}_r$ is called a "zero" of $f(z) = (f_1(z), f_2(z), \ldots, f_k(z), \ldots)$ if z_0 is a zero of all the component functions $f_k(z)$ $(k = 1, 2, \ldots)$. A point $z_0 \in \mathbb{C}_r$ is called a pole or an $\widehat{\infty}$-point of $f(z)$ of multiplicity $q \in \mathbb{N}^+$, meaning that in such a point z_0 at least one of the meromorphic component functions $f_j(z)$ has a pole of this multiplicity in the ordinary sense of function theory. A point $z_0 \in \mathbb{C}_r$ is called a zero of $f(z)$ of multiplicity $q \in \mathbb{N}^+$, meaning that in such a point z_0 all component functions $f_j(z)$ vanish, each with at least this multiplicity.

Let $n(r, f)$ or $n(r, \widehat{\infty})$ denote the number of poles of $f(z)$ in $|z| \leq r$ and let $n(r, a, f)$ denote the number of a-points of $f(z)$ in $|z| \leq r$, counting with multiplicities. Define the volume function associated with E-valued meromorphic function $f(z)$ by

$$\begin{aligned} V(r, \widehat{\infty}, f) = V(r, f) &= \frac{1}{2\pi} \int_{C_r} \log\left|\frac{r}{\xi}\right| \Delta \log\|f(\xi)\| dx \wedge dy, \quad \xi = x + iy, \\ V(r, a, f) &= \frac{1}{2\pi} \int_{C_r} \log\left|\frac{r}{\xi}\right| \Delta \log\|f(\xi) - a\| dx \wedge dy, \quad \xi = x + iy, \end{aligned} \tag{1.5}$$

and the counting function of finite or infinite a-points by

$$N(r, f) = n(0, f) \log r + \int_0^r \frac{n(t, f) - n(0, f)}{t} dt, \tag{1.6}$$

$$N(r, \widehat{\infty}) = n(0, \widehat{\infty}) \log r + \int_0^r \frac{n(t, \widehat{\infty}) - n(0, \widehat{\infty})}{t} dt, \tag{1.7}$$

$$N(r, a, f) = n(0, a, f) \log r + \int_0^r \frac{n(t, a, f) - n(0, a, f)}{t} dt, \tag{1.8}$$

respectively. Next, we define

$$\begin{aligned} m(r, f) = m(r, \widehat{\infty}, f) &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left\| f\left(re^{i\theta}\right) \right\| d\theta, \\ m(r, a) = m(r, a, f) &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{\|f(re^{i\theta}) - a\|} d\theta, \\ T(r, f) &= m(r, f) + N(r, f). \end{aligned} \tag{1.9}$$

Let $\overline{n}(r, f)$ or $\overline{n}(r, \widehat{\infty})$ denote the number of poles of $f(z)$ in $|z| \leq r$, and let $\overline{n}(r, a, f)$ denote the number of a-points of $f(z)$ in $|z| \leq r$, ignoring multiplicities. Similarly, we can define the counting functions $\overline{N}(r, f)$, $\overline{N}(r, \widehat{\infty})$, and $\overline{N}(r, a, f)$ of $\overline{n}(r, f)$, $\overline{n}(r, \widehat{\infty})$, and $\overline{n}(r, a, f)$.

If $f(z)$ is an E-valued meromorphic function in the whole complex plane, then the order and the lower order of $f(z)$ are defined by

$$\begin{aligned} \lambda(f) &= \limsup_{r \to +\infty} \frac{\log^+ T(r, f)}{\log r}, \\ \mu(f) &= \liminf_{r \to +\infty} \frac{\log^+ T(r, f)}{\log r}. \end{aligned} \tag{1.10}$$

If $f(z)$ is an E-valued meromorphic function in $\mathbb{C}_R$, $0 < R < +\infty$, then the order and the lower order of $f(z)$ are defined by

$$\lambda(f) = \limsup_{r\to R^-}\frac{\log T(r,f)}{\log^+(1/(R-r))},$$
$$\mu(f) = \liminf_{r\to R^-}\frac{\log T(r,f)}{\log^+(1/(R-r))}. \tag{1.11}$$

Lemma 1.1. *Let $B(x)$ be a positive and continuous function in $[0,+\infty)$ which satisfies $\limsup_{x\to+\infty}(\log B(x)/\log x) = \infty$. Then there exists a continuously differentiable function $\rho(x)$, which satisfies the following conditions.*

(i) *$\rho(x)$ is continuous and nondecreasing for $x \geq x_0$ $(x_0 > 0)$ and tends to $+\infty$ as $x \to +\infty$.*

(ii) *The function $U(x) = x^{\rho(x)}$ $(x \geq x_0)$ satisfies the following:*

$$\lim_{x\to+\infty}\frac{\log U(X)}{\log U(x)} = 1, \quad X = x + \frac{x}{\log U(x)}. \tag{1.12}$$

(iii) $\limsup_{x\to+\infty}(\log B(x)/\log U(x)) = 1$.

Lemma 1.1 is due to K. L. Hiong (also Qinglai Xiong) and $\rho(x)$ is called the proximate order of Hiong. A simple proof of the existence of $\rho(r)$ was given by Chuang [7]. Suppose that $f(z)$ is an E-valued meromorphic function of infinite order in the unit disk $\mathbb{C}_1$. Let $x = 1/(1-r)$ and $X = 1/(1-R)$. From (ii) and (iii) in Lemma 1.1, we have

$$\lim_{r\to1^-}\frac{\log U(1/(1-R))}{\log U(1/(1-r))} = 1, \quad R = \frac{r\log U(1/(1-r))+1}{\log U(1/(1-r))+1},$$
$$\limsup_{x\to1^-}\frac{\log T(r,f)}{\log U(1/(1-r))} = 1. \tag{1.13}$$

Here, the functions $\rho(1/(1-r))$ and $U(1/(1-r))$ are called the proximate order and type function of $f(z)$, respectively.

Definition 1.2. An E-valued meromorphic function $f(z)$ in $\mathbb{C}_R$, $0 < R \leq +\infty$ is of compact projection, if for any given $\varepsilon > 0$, $\|P_n(f(z)) - f(z)\| < \varepsilon$ has sufficiently larg n in any fixed compact subset $D \subset C_R$.

Throughout this paper, we say that $f(z)$ is an E-valued meromorphic function meaning that $f(z)$ is of compact projection. C.-G. Hu and Q. Hu [3] established the following Nevanlinna's first and second main theorems of E-valued meromorphic functions.

Theorem 1.3. *Let $f(z)$ be a nonconstant E-valued meromorphic function in $\mathbb{C}_R$, $0 < R \leq +\infty$. Then for $0 < r < R$, $a \in E$, $f(z) \not\equiv a$,*

$$T(r, f) = V(r, a) + N(r, a) + m(r, a) + \log^+ \|c_q(a)\| + \varepsilon(r, a). \tag{1.14}$$

Here, $\varepsilon(r, a)$ is a function satisfying that

$$|\varepsilon(r, a)| \leq \log^+ \|a\| + \log 2, \quad \varepsilon(r, 0) \equiv 0, \tag{1.15}$$

and $c_q(a) \in E$ is the coefficient of the first term in the Laurent series at the point a.

Theorem 1.4. *Let $f(z)$ be a nonconstant E-valued meromorphic function in $\mathbb{C}_R$, $0 < R \leq +\infty$ and $a^{[k]} \in E \cup \{\widehat{\infty}\}$ $(k = 1, 2, \ldots, q)$ be $q \geq 3$ distinct points. Then for $0 < r < R$,*

$$(q-2)T(r, f) \leq \sum_{k=1}^{q} \left[V\left(r, a^{[k]}\right) + \overline{N}\left(r, a^{[k]}\right)\right] + S(r, f). \tag{1.16}$$

If $R = +\infty$, then

$$S(r, f) = O(\log T(r, f) + \log r) \tag{1.17}$$

holds as $r \to +\infty$ without exception if $f(z)$ has finite order and otherwise as $r \to +\infty$ outside a set J of exceptional intervals of finite measure $\int_J dr < +\infty$. If the order of $f(z)$ is infinite and $\rho(r)$ is the proximate order of $f(z)$, then

$$S(r, f) = O(\log U(r)) \tag{1.18}$$

holds as $r \to +\infty$ without exception.

If $0 < R < +\infty$, then

$$S(r, f) = O\left(\log T(r, f) + \log \frac{1}{R-r}\right) \tag{1.19}$$

holds as $r \to R$ without exception if $f(z)$ has finite order and otherwise as $r \to R$ outside a set J of exceptional intervals of finite measure $\int_J d((r/(R-r)) < +\infty$.

In all cases, the exceptional set J is independent of the choice of $a^{[k]}$.

2. Characteristic Function of E-Valued Meromorphic Functions in the Unit Disc $\mathbb{C}_1$

In [4], Xuan and Wu proved the following.

Theorem A. *Let $f(z)$ $(z \in \mathbb{C})$ be a nonconstant E-valued meromorphic function and $f(0) \neq \widehat{\infty}$. Then for $\tau > 1$ and $0 < r < R$, one has*

$$T(r, f) < C_\tau T(\tau r, f') + \log^+ \tau r + 4 + \log^+ \|f(0)\|, \tag{2.1}$$

where C_τ is a positive constant.

Theorem B. *Let $f(z)$ $(z \in \mathbb{C})$ be a nonconstant E-valued meromorphic function. Then we have*

$$T(r, f') < 2T(r, f) + O(\log r + \log^+ T(r, f)). \tag{2.2}$$

Theorem C. *For a nonconstant E-valued meromorphic function $f(z)$ $(z \in \mathbb{C})$ of order $\lambda(f) < +\infty$, one has $\lambda(f) = \lambda(f')$, $\mu(f) = \mu(f')$.*

In this section, we shall prove that Theorems A, B, and C are valid for E-valued meromorphic function in the unit disc $\mathbb{C}_1$.

Lemma 2.1. *Let $f(z)$ be an E-valued meromorphic function defined in the unit disc, and $f(0) \neq \widehat{\infty}$. If $0 < R < R' < 1$, then there exists a $\theta_0 \in [0, 2\pi)$, such that for any $0 \leq r \leq R$, one has*

$$\log^+ \left\| f\left(re^{i\theta_0}\right) \right\| \leq \frac{R' + R}{R' - R} m(R', f) + n(R', f) \log 4 + N(R', f). \tag{2.3}$$

Lemma 2.2. *Let $f(z)$ be an E-valued meromorphic function defined in the unit disc, and let $0 < R < R' < R'' < 1$. Then there exists a positive number $R \leq \rho \leq R'$, such that for $|z| = \rho$, one has*

$$\log^+ \left\| f\left(re^{i\theta_0}\right) \right\| \leq \frac{R'' + R'}{R'' - R'} m(R'', f) + n(R'', f) \log \frac{8eR''}{R' - R}. \tag{2.4}$$

Lemmas 2.1 and 2.2 are due to Xuan and Wu [4] for the E-valued meromorphic function defined in the whole complex plane. From the proof of Xuan and Wu [4], we know that Lemmas 2.1 and 2.2 are also valid for the E-valued meromorphic function defined in the unit disc $\mathbb{C}_1$.

Lemma 2.3. *Let $f(z)$ $(z \in \mathbb{C}_1)$ be a nonconstant E-valued meromorphic function and $f(0) \neq \widehat{\infty}$. Suppose that $h(r) \geq 1$, $R = (1 + rh(r))/(1 + h(r))$, then when r sufficiently tends to 1, one has*

$$n(r, f) \leq \frac{6h(r)}{1 - r} N(R, f). \tag{2.5}$$

Proof.

$$
\begin{aligned}
N(R,f) &= n(0,f)\log r + \int_0^R \frac{n(t,f)-n(0,f)}{t}dt = \int_0^R \frac{n(t,f)}{t}dt \\
&\geq \int_r^R \frac{n(t,f)}{t}dt \geq n(r,f)\log\frac{R}{r} \\
&= n(r,f)\log\left(1+\frac{1-r}{r(1+h(r))}\right) \geq n(r,f)\left(\frac{1-r}{r(1+h(r))} - \frac{((1-r)/r(+h(r)))^2}{2}\right) \\
&\geq n(r,f)\left(\frac{(1-r)/r(1+h(r))}{2}\right) \geq n(r,f)\frac{1-r}{6h(r)}.
\end{aligned}
\tag{2.6}
$$

□

Lemma 2.4 (see [4]). *Let $f(z)$ $(z \in \mathbb{C}_R,\ 0 < R \leq +\infty)$ be a nonconstant E-valued meromorphic function and $f(0) \neq \widehat{\infty}$, and L a curve from the origin along the segment $\arg z = \theta_0$ to $\rho e^{i\theta_0}$, and along $\{|z| = \rho < r\}$ turn a rotation to $\rho e^{i\theta_0}$. Then for any $\{|z| = r \leq \rho\}$, one has*

$$\log^+\|f(z)\| \leq \log^+ M + O(1), \tag{2.7}$$

where $M = \max\{\|f'(z)\|, z \in L\}$.

Lemma 2.5 (see [3]). *Let $f(z)$ be a nonconstant E-valued meromorphic function in $\mathbb{C}_1$. Then for $0 < r < 1$,*

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+\frac{\|f'(re^{i\theta})\|}{\|f(re^{i\theta})\|}d\theta < K\left(\log T(r,f) + \log\frac{1}{1-r}\right), \tag{2.8}$$

where K is a sufficiently large constant.

We are now in the position to establish the main results of this section.

Theorem 2.6. *Let $f(z)$ $(z \in \mathbb{C}_1)$ be a nonconstant E-valued meromorphic function and $f(0) \neq \widehat{\infty}$. Then for $\varepsilon > 1$ and any real function $h(x) \geq 1$, when r sufficiently tend to 1, one has*

$$T(r,f) < \frac{ch^{1+\varepsilon}(r)}{(1-r)^{1+\varepsilon}}T(R,f'), \quad R = \frac{1+rh(r)}{1+h(r)}. \tag{2.9}$$

Proof. Denote $R_1 = (R + 2r)/3, R_2 = (r + 2R)/3$, we can get

$$r < R_1 < R_2 < R, \quad R_1 - r = R_2 - R_1 = R - R_2 = \frac{R - r}{3},$$
$$R = \frac{1 - 3R_2 h(r)}{1 + 3h(r)}, \quad R_2 + R_1 = r + R < 2, \qquad 1 - R_2 = \frac{(1 - r)(1 + 3h(r))}{3(1 + h(r))} \geq \frac{1 - r}{2}; \tag{2.10}$$
$$R - r = \frac{1 - r}{1 + h(r)} \geq \frac{1 - r}{2h(r)}.$$

Applying Lemma 2.1 to $f'(z)$ and combining Lemma 2.3, we can find a real number $\theta_0 \in [0, 2\pi)$ such that for any $0 \leq t \leq R_1$, one has

$$\begin{aligned}
\log^+ \left\| f'\left(te^{i\theta_0}\right) \right\| &\leq \frac{R_2 + R_1}{R_2 - R_1} m(R_2, f') + n(R_2, f') \log 4 + N(R_2, f') \\
&\leq \left(\frac{6}{R - r} + \frac{6h(r)}{1 - R_2} \log 4 + 1 \right) T(R_2, f') \\
&\leq \left(\frac{6 + 6h(r)}{1 - r} + \frac{12h(r)}{1 - r} \log 4 + \frac{1 - r}{1 - r} \right) T(R, f') \\
&\leq \frac{6 + 6h(r) + 24h(r) + 1 - r}{1 - r} T(R, f') \leq \frac{40h(r)}{1 - r} T(R, f').
\end{aligned} \tag{2.11}$$

In view of Lemma 2.2, there is a $\rho \in [r, R_1]$ such that for any $z \in \{|z| = \rho\}$, one has

$$\begin{aligned}
\log^+ \| f'(z) \| &\leq \frac{R_2 + R_1}{R_2 - R_1} m(R_2, f') + n(R_2, f') \log \frac{8eR_2}{R_1 - R} \\
&\leq \left(\frac{6}{R - r} + \frac{6h(r)}{1 - R_2} \log \frac{48eh(r)}{1 - r} \right) T(R_2, f') \\
&\leq \left(\frac{6 + 6h(r)}{1 - r} + \frac{12h(r)}{1 - r} \log \frac{144h(r)}{1 - r} \right) T(R, f') \\
&\leq \left(\frac{12h(r)}{1 - r} \left(9 + \log \frac{h(r)}{1 - r} \right) \right) T(R, f') \\
&\leq \left(\frac{12h(r)}{1 - r} \left(9 + \left(\frac{h(r)}{1 - r} \right)^{\varepsilon} \right) \right) T(R, f') \\
&\leq 120 \left(\frac{h(r)}{1 - r} \right)^{1+\varepsilon} T(R, f').
\end{aligned} \tag{2.12}$$

From the origin along the segment $\arg z = \theta_0$ to $\rho e^{i\theta_0}$ and along $\{|z| = \rho\}$, turn a rotation to $\rho e^{i\theta_0}$. We denote this curve by L. In virtue of Lemma 2.4, we have

$$\log^+ \| f(z) \| \leq \log^+ M + O(1) \tag{2.13}$$

holds for any $\{|z| = r \leq \rho\}$, where $M = \max\{\|f'(z)\|, z \in L\}$. In virtue of (2.11), (2.12), and (2.13), we have

$$m(r,f) \leq m(\rho,f) \leq m(\rho,f') \leq \frac{1}{2\pi}\int_0^{2\pi} \log^+ M d\theta \leq 121\left(\frac{h(r)}{1-r}\right)^{1+\varepsilon} T(R,f'). \tag{2.14}$$

Hence,

$$T(r,f) = m(r,f) + N(r,f) \leq m(r,f) + 2N(r,f') \leq 123\left(\frac{h(r)}{1-r}\right)^{1+\varepsilon} T(R,f'). \tag{2.15}$$

□

Theorem 2.7. *Let $f(z)$ $(z \in \mathbb{C}_1)$ be a nonconstant E-valued meromorphic function and $f(0) \neq 0, \widehat{\infty}$. Then for any $0 < r < R < 1$, one has*

$$T(r,f') < 2T(r,f) + O\left(\log^+ \frac{1}{1-r} + \log^+ T(r,f)\right). \tag{2.16}$$

Proof. By Lemma 2.5, we have

$$\begin{aligned} T(r,f') &= m(r,f') + N(r,f') \\ &\leq m(r,f) + m\left(r,\frac{f'}{f}\right) + 2N(r,f) \\ &\leq 2T(r,f) + m\left(r,\frac{f'}{f}\right) \\ &\leq 2T(r,f) + O\left(\log^+ \frac{1}{1-r} + \log^+ T(r,f)\right). \end{aligned} \tag{2.17}$$

□

Theorem 2.8. *For a nonconstant E-valued meromorphic function $f(z)$ $(z \in \mathbb{C}_1)$ of order $\lambda(f) < +\infty$, one has $\lambda(f) = \lambda(f')$, $\mu(f) = \mu(f')$.*

Theorem 2.8 only discussed the E-valued meromorphic function of finite order. In fact, for any E-valued meromorphic function of infinite order, we have the following.

Theorem 2.9. *If $f(z)$ $(z \in \mathbb{C}_1)$ is a nonconstant E-valued meromorphic function of order $\lambda(f) = +\infty$, then the proximate orders of $f(z)$ and $f'(z)$ are the same.*

Proof. Let $h(r) = \log U(1/(1-r))$, in view of Theorems 2.6 and 2.7, we can easily derive Theorem 2.9. □

3. E-Valued Borel Exceptional Values of Meromorphic Functions in $\mathbb{C}_{\mathbb{R}}$

Some definitions in this section can be found in [8].

Definition 3.1. Let $f(z)$ $(z \in \mathbb{C}_R, 0 < R \leq +\infty)$ be an E-valued meromorphic function and $a \in E \cup \{\widehat{\infty}\}$, if k is a positive integer, let $\overline{n}_k(r,f)$ or $\overline{n}_k(r,\widehat{\infty})$ denote the number of distinct poles of $f(z)$ of order $\leq k$ in $|z| \leq r$, and let $\overline{n}_k(r,a)$ denote the number of distinct a-points of $f(z)$ of order $\leq k$ in $|z| \leq r$. Similarly, we can define the counting functions $\overline{N}_k(r,f)$, $\overline{N}_k(r,\widehat{\infty})$, and $\overline{N}_k(r,a)$ of $\overline{n}_k(r,f)$, $\overline{n}_k(r,\widehat{\infty})$, and $\overline{n}_k(r,a)$.

Definition 3.2. Let $f(z)$ $(z \in \mathbb{C}_R, 0 < R \leq +\infty)$ be an E-valued meromorphic function and $a \in E \cup \{\widehat{\infty}\}$. If $R = +\infty$, we define

$$\begin{aligned} \overline{\rho}_k(a,f) &= \limsup_{r\to+\infty} \frac{\log^+\left[V(a,f) + \overline{N}_k(r,a)\right]}{\log r}, \\ \overline{\rho}(a,f) &= \limsup_{r\to+\infty} \frac{\log^+\left[V(a,f) + \overline{N}(r,a)\right]}{\log r}, \\ \rho(a,f) &= \limsup_{r\to+\infty} \frac{\log^+\left[V(a,f) + N(r,a)\right]}{\log r}. \end{aligned} \tag{3.1}$$

If $R < +\infty$, we define

$$\begin{aligned} \overline{\rho}_k(a,f) &= \limsup_{r\to R^-} \frac{\log^+\left[V(a,f) + \overline{N}_k(r,a)\right]}{\log(1/(R-r))}, \\ \overline{\rho}(a,f) &= \limsup_{r\to R^-} \frac{\log^+\left[V(a,f) + \overline{N}(r,a)\right]}{\log(1/(R-r))}, \\ \rho(a,f) &= \limsup_{r\to R^-} \frac{\log^+\left[V(a,f) + N(r,a)\right]}{\log(1/(R-r))}. \end{aligned} \tag{3.2}$$

Definition 3.3. Let $f(z)$ $(z \in \mathbb{C}_R, 0 < R \leq +\infty)$ be an E-valued meromorphic function and $a \in E \cup \{\widehat{\infty}\}$ and k is a positive integer, we say that a is an

(i) E-valued evB (exceptional value in the sense of Borel) for f for distinct zeros of order $\leq k$ if $\overline{\rho}_k(a,f) < \lambda(f)$;

(ii) E-valued evB for f for distinct zeros if $\overline{\rho}(a,f) < \lambda(f)$;

(iii) E-valued evB for f (for the whole aggregate of zeros) if $\rho(a,f) < \lambda(f)$.

In [5], Yang proved the following result.

Theorem D. *Let $f(z)$ $(z \in \mathbb{C}_R,\ R = +\infty)$ be a meromorphic function with finite order $\lambda > 0$ and k_j $(j = 1, 2, \ldots, q)$ be q positive integers. a is called a pseudo-Borel exceptional value of $f(z)$ of order k if*

$$\limsup_{r \to +\infty} \frac{\log^+ \overline{n}_k(r, a)}{\log r} < \lambda(f). \tag{3.3}$$

If $f(z)$ has q distinct pseudo-Borel exceptional values a_j of order k_j $(j = 1, 2, \ldots, q)$, then

$$\sum_{j=1}^{q} \left(1 - \frac{1}{k_j + 1}\right) \leq 2. \tag{3.4}$$

It is natural to consider whether there exists a similar result, if meromorphic function f is replaced by E-valued meromorphic function f. In this section, we extend the above theorem to E-valued meromorphic function in $\mathbb{C}_R, 0 < R \leq +\infty$.

Theorem 3.4. *Let $f(z)$ $(z \in \mathbb{C}_R, 0 < R \leq +\infty)$ be an E-valued meromorphic function with finite order $\lambda > 0$, $a^{[j]} (j = 1, 2, \ldots, q)$ any system of distinct elements in $E \cup \{\widehat{\infty}\}$, and k_j $(j = 1, 2, \ldots, q)$ any system such that k_j is a positive integer or $+\infty$. If $a^{[j]}$ is an E-valued evB for f for distinct zeros of order $\leq k_j$ $(j = 1, 2, \ldots, q)$, then*

$$\sum_{j=1}^{q} \left(1 - \frac{1}{k_j + 1}\right) \leq 2. \tag{3.5}$$

Proof. By Theorem 1.4, we have

$$(q - 2)T(r, f) \leq \sum_{j=1}^{q} \left[V\left(r, a^{[j]}\right) + \overline{N}\left(r, a^{[j]}\right)\right] + S(r, f) \tag{3.6}$$

holds for $0 < r < R$. For any $j = 1, 2, \ldots, q$, we have

$$\begin{gathered} \overline{N}\left(r, a^{[j]}\right) \leq \frac{1}{k_j + 1}\left\{k_j \overline{N}_{k_j}\left(r, a^{[j]}\right) + N\left(r, a^{[j]}\right)\right\}, \\ N\left(r, a^{[j]}\right) \leq T(r, f) - V\left(r, a^{[j]}\right) + O(1). \end{gathered} \tag{3.7}$$

Using (3.7) and (7) in (3.6), we get

$$\begin{aligned}(q-2)T(r,f) &\leq \sum_{j=1}^{q}\left(V\left(r,a^{[j]}\right)+\frac{1}{k_j+1}\left\{k_j\overline{N}_{k_j}\left(r,a^{[j]}\right)+N\left(r,a^{[j]}\right)\right\}\right)+S(r,f)\\ &=\sum_{j=1}^{q}\left(V\left(r,a^{[j]}\right)+\frac{k_j}{k_j+1}\overline{N}_{k_j}\left(r,a^{[j]}\right)+\frac{1}{k_j+1}N\left(r,a^{[j]}\right)\right)+S(r,f)\\ &\leq \sum_{j=1}^{q}\frac{k_j}{k_j+1}\left(V\left(r,a^{[j]}\right)+\overline{N}_{k_j}\left(r,a^{[j]}\right)\right)+\sum_{j=1}^{q}\frac{1}{k_j+1}T(r,f)+S(r,f).\end{aligned} \tag{3.8}$$

Therefore, we have

$$\left[\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2\right]T(r,f)\leq \sum_{j=1}^{q}\frac{k_j}{k_j+1}\left(V\left(r,a^{[j]}\right)+\overline{N}_{k_j}\left(r,a^{[j]}\right)\right)+S(r,f). \tag{3.9}$$

By hypothesis, we have

$$\overline{\rho}_{k_j}\left(a^{[j]},f\right)<\lambda,\quad j=1,2,\ldots,q. \tag{3.10}$$

If $R=+\infty$, then there is a positive number $\rho<\lambda$, such that for $j=1,2,\ldots,q$, we can get

$$V\left(r,a^{[j]}\right)+\overline{N}_{k_j}\left(r,a^{[j]}\leq r^{\rho}\right). \tag{3.11}$$

Using (3.11) to (3.9), we have

$$\left[\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2\right]T(r,f)\leq \sum_{j=1}^{q}\frac{k_j}{k_j+1}r^{\rho}+S(r,f). \tag{3.12}$$

If $\sum_{j=1}^{q}(1-(1/(k_j+1)))>2$, then by Theorem 1.4 and (3.12), we can get a contradiction $\lambda\leq\rho$. So

$$\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)\leq 2. \tag{3.13}$$

If $R<+\infty$, then there is a positive number $\rho<\lambda$, such that for $j=1,2,\ldots,q$, we can get

$$V\left(r,a^{[j]}\right)+\overline{N}_{k_j}\left(r,a^{[j]}\right)\leq\left(\frac{1}{R-r}\right)^{\rho}. \tag{3.14}$$

Using (3.14) to (3.9), we have

$$\left[\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2\right]T(r,f)\leq\sum_{j=1}^{q}\frac{k_j}{k_j+1}\left(\frac{1}{R-r}\right)^{\rho}+S(r,f). \tag{3.15}$$

If $\sum_{j=1}^{q}(1-(1/(k_j+1)))>2$, then by Theorem 1.4 and (3.15), we can get a contradiction $\lambda\leq\rho$. So

$$\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)\leq 2. \tag{3.16}$$

□

From the proof of Theorem 3.4, we can get the following.

Corollary 3.5. *Let $f(z)$ $(z\in\mathbb{C}_R, 0<R\leq+\infty)$ be a nonconstant E-valued meromorphic function. Then for any system $a^{[j]}$ $(j=1,2,\ldots t)$ of distinct elements in $E\cup\{\widehat{\infty}\}$ and any system k_j $(j=1,2,\ldots,t)$ such that k_j is a positive integer or $+\infty$, we have the following:*

(1) *if all of $a^{[j]}$ $(j=1,2,\ldots,q)$ in E, then*

$$\left(q-\sum_{j=1}^{q}\frac{1}{k_j+1}-2\right)T(r,f)\leq\sum_{j=1}^{q}\frac{k_j}{k_j+1}\left(V\left(r,a^{[j]},f\right)+\overline{N}_{k_j}\left(r,a^{[j]},f\right)\right)+S(r,f), \tag{3.17}$$

(2) *if one of $a^{[j]}$ $(j=1,2,\ldots,q)$ is $\widehat{\infty}$, say $a^{[q]}=\widehat{\infty}$. Then,*

$$\begin{aligned}\left(q-\sum_{j=1}^{q}\frac{1}{k_j+1}-2\right)T(r,f)&\leq\sum_{j=1}^{q-1}\frac{k_j}{k_j+1}\left(V\left(r,a^{[j]},f\right)+\overline{N}_{k_j}\left(r,a^{[j]},f\right)\right)\\&\quad+\frac{k_q}{k_q+1}\overline{N}_{k_q}(r,f)+S(r,f).\end{aligned} \tag{3.18}$$

Remark 3.6. If $R=+\infty$, let $q=r+t+s$ and $k_j\equiv k$ $(j=1,2,\ldots,r)$, $k_j\equiv l$ $(j=r+1,\ldots,r+t)$ and $k_j\equiv m$ $(j=r+t+1,\ldots,r+t+s)$ in Theorem 3.4. We can get the following result by Bhoosnurmath and Pujari [8].

Theorem E. *Let $f(z)$ $(z\in\mathbb{C}_R, 0<R\leq+\infty)$ be an E-valued meromorphic function of order $\lambda(f)$, $0<\lambda(f)\leq+\infty$. If there exist distinct elements*

$$a^{[1]},a^{[2]},\ldots,a^{[r]};\qquad b^{[1]},b^{[2]},\ldots,b^{[t]};\qquad c^{[1]},c^{[2]},\ldots,c^{[s]} \tag{3.19}$$

in $E \cup \{\widehat{\infty}\}$ *such that* $a^{[1]}, a^{[2]}, \ldots, a^{[r]}$ *are E-valued evB for* f *for distinct zeros of order* $\leq k$, $b^{[1]}, b^{[2]}, \ldots, b^{[t]}$ *are E-valued evB for* f *for distinct zeros of order* $\leq l$, $c^{[1]}, c^{[2]}, \ldots, c^{[s]}$ *are E-valued evB for* f *for distinct zeros of order* $\leq m$, *where* k, l, *and* m *are positive integers, then*

$$\frac{rk}{k+1} + \frac{tl}{l+1} + \frac{sm}{m+1} \leq 2. \tag{3.20}$$

Bhoosnurmath and Pujari [8] pointed out that Theorem E is valid for $0 \leq \lambda(f) \leq +\infty$. In fact, Definition 3.3 is not well in the case of $\lambda(f) = 0$. In the case of $\lambda(f) = +\infty$, a is an E-valued evB for f if and only if $\overline{\rho}_k(a, f)$ is finite. When $\overline{\rho}_k(a, f)$ is infinite, we shall give the following definitions.

Definition 3.7. Let $f(z)$ $(z \in \mathbb{C})$ be an E-valued meromorphic function of infinite order and $\rho(r)$ is a proximate order of f and $a \in E \cup \{\widehat{\infty}\}$. We say that a is an

(i) E-valued evB (exceptional value in the sense of Borel) for f for distinct zeros of order $\leq k$ if

$$\limsup_{r \to +\infty} \frac{\log^+ \left[V(a, f) + \overline{N}_k(r, a)\right]}{\log U(r)} < 1; \tag{3.21}$$

(ii) E-valued evB for f for distinct zeros if

$$\limsup_{r \to +\infty} \frac{\log^+ \left[V(a, f) + \overline{N}(r, a)\right]}{\log U(r)} < 1; \tag{3.22}$$

(iii) E-valued evB for f (for the whole aggregate of zeros) if

$$\limsup_{r \to +\infty} \frac{\log^+ \left[V(a, f) + N(r, a)\right]}{\log U(r)} < 1. \tag{3.23}$$

Theorem 3.8. *Let* $f(z)$ $(z \in \mathbb{C})$ *be an E-valued meromorphic function of infinite order and* $\rho(r)$ *is a proximate order of* f, $a^{[j]}(j = 1, 2, \ldots, q)$ *any system of distinct elements in* $E \cup \{\widehat{\infty}\}$, *and* $k_j(j = 1, 2, \ldots, q)$ *any system such that* k_j *is a positive integer or* $+\infty$. *If* $a^{[j]}$ *is an E-valued evB for* f *for distinct zeros of order* $\leq k_j (j = 1, 2 \ldots, q)$, *then*

$$\sum_{j=1}^{q} \left(1 - \frac{1}{k_j + 1}\right) \leq 2. \tag{3.24}$$

Proof. By Corollary 3.5, we have

$$\left(q - \sum_{j=1}^{q} \frac{1}{k_j + 1} - 2\right) T(r, f) \leq \sum_{j=1}^{q} \frac{k_j}{k_j + 1} \left(V\left(r, a^{[j]}\right) + \overline{N}_{k_j}\left(r, a^{[j]}\right)\right) + S(r, f). \tag{3.25}$$

By hypothesis, there exists a positive number $\eta < 1$ such that

$$V\left(r, a^{[j]}\right) + \overline{N}_{k_j}\left(r, a^{[j]}\right) < U^{\eta}(r), \quad j = 1, 2, \ldots, q. \tag{3.26}$$

Using (3.25) to (3.26), we have

$$\left[\sum_{j=1}^{q}\left(1 - \frac{1}{k_j + 1}\right) - 2\right] T(r, f) \leq \sum_{j=1}^{q} \frac{k_j}{k_j + 1} U^{\eta}(r) + S(r, f). \tag{3.27}$$

If $\sum_{j=1}^{q}(1 - (1/(k_j + 1))) > 2$, then by Theorem 1.4 and (3.27), we can get a contradiction. So

$$\sum_{j=1}^{q}\left(1 - \frac{1}{k_j + 1}\right) \leq 2. \tag{3.28}$$

□

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References

[1] H. J. W. Ziegler, *Vector Valued Nevanlinna Theory*, vol. 73 of *Research Notes in Mathematics*, Pitman (Advanced Publishing Program), Boston, Mass, USA, 1982.

[2] C.-G. Hu and C.-C. Yang, "Some remarks on Nevanlinna's theory in a Hilbert space," *Bulletin of the Hong Kong Mathematical Society*, vol. 1, no. 2, pp. 267–271, 1997.

[3] C.-G. Hu and Q. Hu, "The Nevanlinna's theorems for a class," *Complex Variables and Elliptic Equations*, vol. 51, no. 8–11, pp. 777–791, 2006.

[4] Z.-X. Xuan and N. Wu, "On the Nevanlinna's theory for vector-valued mappings," *Abstract and Applied Analysis*, vol. 2010, Article ID 864539, 15 pages, 2010.

[5] L. Yang, *Value Distribution Theory*, Springer, Berlin, Germany, Science Press Beijing, Beijing, China, 1993, Translated and revised from the 1982 Chinese original.

[6] H. S. Gopalakrishna and S. S. Bhoosnurmath, "Exceptional values of meromorphic functions," *Annales Polonici Mathematici*, vol. 32, no. 1, pp. 83–93, 1976.

[7] C.-T. Chuang, "Sur les fonctions-types," *Scientia Sinica*, vol. 10, pp. 171–181, 1961 (French).

[8] S. S. Bhoosnurmath and V. L. Pujari, "*E*-valued Borel exceptional values of meromorphic functions," *International Journal of Mathematical Analysis*, vol. 4, no. 41–44, pp. 2089–2099, 2010.