Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 285807, 12 pages
doi:10.1155/2012/285807

Research Article

Nearly Quadratic Mappings over p-Adic Fields

M. Eshaghi Gordji,[1] H. Khodaei,[1] and Gwang Hui Kim[2]

[1] *Department of Mathematics, Semnan University, P.O. Box 35195-363, Semnan, Iran*
[2] *Department of Mathematics, Kangnam University, Yongin, Gyeonggi 446-702, Republic of Korea*

Correspondence should be addressed to Gwang Hui Kim, ghkim@kangnam.ac.kr

Received 30 October 2011; Revised 20 November 2011; Accepted 21 November 2011

Academic Editor: John Rassias

We establish some stability results over p-adic fields for the generalized quadratic functional equation $\sum_{k=2}^{n}\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n} f(\sum_{i=1, i\neq i_1,\ldots,i_{n-k+1}}^{n} x_i - \sum_{r=1}^{n-k+1} x_{i_r}) + f(\sum_{i=1}^{n} x_i) = 2^{n-1}\sum_{i=1}^{n} f(x_i)$, where $n \in \mathbb{N}$ and $n \geq 2$.

1. Introduction and Preliminaries

In 1899, Hensel [1] discovered the p-adic numbers as a number of theoretical analogue of power series in complex analysis. Fix a prime number p. For any nonzero rational number x, there exists a unique integer n_x such that $x = (a/b)p^{n_x}$, where a and b are integers not divisible by p. Then, p-adic absolute value $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x,y) = |x-y|_p$ is denoted by $\mathbb{Q}_p$, and it is called the p-adic number field. In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum_{k\geq n_x}^{\infty} a_k p^k$, where $|a_k| \leq p-1$ are integers (see, e.g., [2, 3]). Note that if $p > 2$, then $|2^n|_p = 1$ for each integer n.

During the last three decades, p-adic numbers have gained the interest of physicists for their research, in particular, in problems coming from quantum physics, p-adic strings, and superstrings [4, 5]. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: For $x, y > 0$, there exists $n \in \mathbb{N}$ such that $x < ny$.

Let $\mathbb{K}$ denote a field and function (valuation absolute) $|\cdot|$ from $\mathbb{K}$ into $[0,\infty)$. A non-Archimedean valuation is a function $|\cdot|$ that satisfies the strong triangle inequality; namely, $|x+y| \leq \max\{|x|,|y|\} \leq |x|+|y|$ for all $x, y \in \mathbb{K}$. The associated field $\mathbb{K}$ is referred to as a non-Archimedean field. Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \geq 1$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except 0 into 1 and $|0| = 0$. We always assume in addition that $|\cdot|$ is nontrivial, that is, there is a $z \in \mathbb{K}$ such that $|z| \neq 0, 1$.

Let X be a linear space over a field $\mathbb{K}$ with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : X \to [0,\infty)$ is said to be a non-Archimedean norm if it is a norm over $\mathbb{K}$ with the strong triangle inequality (ultrametric); namely, $\|x+y\| \le \max\{\|x\|,\|y\|\}$ for all $x, y \in X$. Then, $(X, \|\cdot\|)$ is called a non-Archimedean space. In any such a space, a sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy if and only if $\{x_{n+1} - x_n\}_{n\in\mathbb{N}}$ converges to zero. By a complete non-Archimedean space, we mean one in which every Cauchy sequence is convergent.

The study of stability problems for functional equations is related to a question of Ulam [6] concerning the stability of group homomorphisms, which was affirmatively answered for Banach spaces by Hyers [7]. Subsequently, the result of Hyers was generalized by Aoki [8] for additive mappings and by Rassias [9] for linear mappings by considering an unbounded Cauchy difference. The paper by Rassias has provided a lot of influences in the development of what we now call the generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. Rassias [10] considered the Cauchy difference controlled by a product of different powers of norm. The above results have been generalized by Forti [11] and Găvruţa [12] who permitted the Cauchy difference to become arbitrary unbounded (see also [13–22]). Arriola and Beyer [23] investigated stability of approximate additive functions $f : \mathbb{Q}_p \to \mathbb{R}$. They showed that if $f : \mathbb{Q}_p \to \mathbb{R}$ is a continuous function for which there exists a fixed ε such that $|f(x+y) - f(x) - f(y)| \le \varepsilon$ for all $x, y \in Q_p$, then there exists a unique additive function $T : \mathbb{Q}_p \to \mathbb{R}$ such that $|f(x) - T(x)| \le \varepsilon$ for all $x \in \mathbb{Q}_p$. For more details about the results concerning such problems, the reader is referred to [24–45].

Recently, Khodaei and Rassias [46] introduced the generalized additive functional equation

$$\sum_{k=2}^{n}\left(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n}\right)f\left(\sum_{i=1,\, i\neq i_1,\dots,i_{n-k+1}}^{n} a_i x_i - \sum_{r=1}^{n-k+1} a_{i_r} x_{i_r}\right) + f\left(\sum_{i=1}^{n} a_i x_i\right) = 2^{n-1} a_1 f(x_1) \tag{1.1}$$

and proved the generalized Hyers-Ulam stability of the above functional equation. The functional equation

$$f(x_1 + x_2) + f(x_1 - x_2) = 2f(x_1) + 2f(x_2) \tag{1.2}$$

is related to symmetric biadditive function and is called a quadratic functional equation [47, 48]. Every solution of the quadratic equation (1.2) is said to be a quadratic function.

Now, we introduce the generalized quadratic functional equation in n-variables as follows:

$$\sum_{k=2}^{n}\left(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n}\right)f\left(\sum_{i=1,\, i\neq i_1,\dots,i_{n-k+1}}^{n} x_i - \sum_{r=1}^{n-k+1} x_{i_r}\right) + f\left(\sum_{i=1}^{n} x_i\right) = 2^{n-1}\sum_{i=1}^{n} f(x_i), \tag{1.3}$$

where $n \ge 2$. Moreover, we investigate the generalized Hyers-Ulam stability of functional equation (1.3) over the p-adic field $\mathbb{Q}_p$.

As a special case, if $n = 2$ in (1.3), then we have the functional equation (1.2). Also, if $n = 3$ in (1.3), we obtain

$$\sum_{i_1=2}^{2}\sum_{i_2=i_1+1}^{3} f\left(\sum_{i=1, i\neq i_1,i_2}^{3} x_i - \sum_{r=1}^{2} x_{i_r}\right) + \sum_{i_1=2}^{3} f\left(\sum_{i=1, i\neq i_1}^{3} x_i - x_{i_1}\right) + f\left(\sum_{i=1}^{3} x_i\right) = 2^2 \sum_{i=1}^{3} f(x_i), \quad (1.4)$$

that is,

$$f(x_1 - x_2 - x_3) + f(x_1 - x_2 + x_3) + f(x_1 + x_2 - x_3) + f(x_1 + x_2 + x_3) = 4f(x_1) + 4f(x_2) + 4f(x_3). \quad (1.5)$$

2. Stability of Quadratic Functional Equation (1.3) over p-Adic Fields

We will use the following lemma.

Lemma 2.1. *Let X and Y be real vector spaces. A function $f : X \to Y$ satisfies the functional equation* (1.3) *if and only if the function f is quadratic.*

Proof. Let f satisfy the functional equation (1.3). Setting $x_i = 0$ $(i = 1, \dots, n)$ in (1.3), we have

$$\sum_{k=2}^{n}\left(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n}\right) f(0) + f(0) = 2^{n-1}\sum_{i=1}^{n} f(0), \quad (2.1)$$

that is,

$$\sum_{i_1=2}^{2}\sum_{i_2=i_1+1}^{3}\cdots\sum_{i_{n-1}=i_{n-2}+1}^{n} f(0) + \sum_{i_1=2}^{3}\sum_{i_2=i_1+1}^{4}\cdots\sum_{i_{n-2}=i_{n-3}+1}^{n} f(0) + \cdots + \sum_{i_1=2}^{n} f(0) + f(0) = 2^{n-1}\sum_{i=1}^{n} f(0), \quad (2.2)$$

or

$$\left(\binom{n-1}{n-1} + \binom{n-1}{n-2} + \cdots + \binom{n-1}{1} + 1\right) f(0) = 2^{n-1}\sum_{i=1}^{n} f(0), \quad (2.3)$$

but $1 + \sum_{j=1}^{n-j}\binom{n-j}{j} = \sum_{j=0}^{n-j}\binom{n-j}{j} = 2^{n-j}$, and also $n > j \geq 1$ so $2^{n-1}(n-1)f(0) = 0$.

Putting $x_i = 0$ $(i = 2, \dots, n-1)$ in (1.3) and then using $f(0) = 0$, we get

$$f(x_1 - x_n) + \left(\binom{n-2}{1} f(x_1 - x_n) + \binom{n-2}{n-2} f(x_1 + x_n)\right) + \cdots$$
$$+ \left(\binom{n-2}{n-3} f(x_1 - x_n) + \binom{n-2}{2} f(x_1 + x_n)\right)$$

$$+\left(\binom{n-2}{n-2}f(x_1-x_n)+\binom{n-2}{1}f(x_1+x_n)\right)+f(x_1+x_n)$$
$$=2^{n-1}f(x_1)+2^{n-1}f(x_n), \tag{2.4}$$

that is,

$$\left(1+\sum_{j=1}^{n-2}\binom{n-2}{j}\right)\left(f(x_1+x_n)+f(x_1-x_n)\right)=2^{n-1}f(x_1)+2^{n-1}f(x_n), \tag{2.5}$$

for all $x_1, x_n \in X$, this shows that f satisfies the functional equation (1.2). So the function f is quadratic.

Conversely, suppose that f is quadratic, thus f satisfies the functional equation (1.2). Hence, we have $f(0)=0$ and f is even.

We are going to prove our assumption by induction on $n \geq 2$. It holds on $n=2$. Assume that it holds on the case where $n=t$; that is, we have

$$\sum_{k=2}^{t}\left(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{t-k+1}=i_{t-k}+1}^{t}\right)f\left(\sum_{i=1,\, i\neq i_1,\ldots,i_{t-k+1}}^{t}x_i-\sum_{r=1}^{t-k+1}x_{i_r}\right)+f\left(\sum_{i=1}^{t}x_i\right)=2^{t-1}\sum_{i=1}^{t}f(x_i) \tag{2.6}$$

for all $x_1, \ldots, x_t \in X$. It follows from (1.2) that

$$f\left(\sum_{i=1}^{t}x_i+x_{t+1}\right)+f\left(\sum_{i=1}^{t}x_i-x_{t+1}\right)=2f\left(\sum_{i=1}^{t}x_i\right)+2f(x_{t+1}) \tag{2.7}$$

for all $x_1, \ldots, x_{t+1} \in X$. Replacing x_t by $-x_t$ in (2.7), we obtain

$$f\left(\sum_{i=1}^{t-1}x_i-x_t+x_{t+1}\right)+f\left(\sum_{i=1}^{t-1}x_i-x_t-x_{t+1}\right)=2f\left(\sum_{i=1}^{t-1}x_i-x_t\right)+2f(x_{t+1}) \tag{2.8}$$

for all $x_1, \ldots, x_{t+1} \in X$. Adding (2.7) to (2.8), we have

$$f\left(\sum_{i=1}^{t-1}x_i-x_t-x_{t+1}\right)+f\left(\sum_{i=1}^{t-1}x_i-x_t+x_{t+1}\right)+f\left(\sum_{i=1}^{t-1}x_i+x_t-x_{t+1}\right)+f\left(\sum_{i=1}^{t-1}x_i+x_t+x_{t+1}\right)$$
$$=2\left[f\left(\sum_{i=1}^{t-1}x_i-x_t\right)+f\left(\sum_{i=1}^{t-1}x_i+x_t\right)\right]+4f(x_{t+1}) \tag{2.9}$$

for all $x_1, \ldots, x_{t+1} \in X$. Replacing x_{t-1} by $-x_{t-1}$ in (2.9), we get

$$f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} - x_t - x_{t+1}\right) + f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} - x_t + x_{t+1}\right) + f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} + x_t - x_{t+1}\right)$$
$$+ f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} + x_t + x_{t+1}\right) = 2\left[f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} - x_t\right) + f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} + x_t\right)\right] + 4f(x_{t+1}) \quad (2.10)$$

for all $x_1, \ldots, x_{t+1} \in X$. Adding (2.9) to (2.10), one gets

$$f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} - x_t - x_{t+1}\right) + f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} - x_t + x_{t+1}\right) + f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} + x_t - x_{t+1}\right)$$
$$+ f\left(\sum_{i=1}^{t-2} x_i + x_{t-1} - x_t - x_{t+1}\right) + f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} + x_t + x_{t+1}\right) + f\left(\sum_{i=1}^{t-2} x_i + x_{t-1} - x_t + x_{t+1}\right)$$
$$+ f\left(\sum_{i=1}^{t-2} x_i + x_{t-1} + x_t - x_{t+1}\right) + f\left(\sum_{i=1}^{t+1} x_i\right)$$
$$= 2\left[f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} - x_t\right) + f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} + x_t\right) + f\left(\sum_{i=1}^{t-2} x_i + x_{t-1} - x_t\right)\right.$$
$$\left. + f\left(\sum_{i=1}^{t-2} x_i + x_{t-1} + x_t\right)\right] + 8f(x_{t+1}) \quad (2.11)$$

for all $x_1, \ldots, x_{t+1} \in X$. By using the above method, for x_{t-2} until x_2, we infer that

$$\sum_{k=2}^{t+1}\left(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{t-k+2}=i_{t-k+1}+1}^{t+1}\right) f\left(\sum_{i=1,\, i\neq i_1,\ldots,i_{t-k+2}}^{t+1} x_i - \sum_{r=1}^{t-k+2} x_{i_r}\right) + f\left(\sum_{i=1}^{t+1} x_i\right)$$
$$= 2\left[\sum_{k=2}^{t}\left(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{t-k+1}=i_{t-k}+1}^{t}\right) f\left(\sum_{i=1,\, i\neq i_1,\ldots,i_{t-k+1}}^{t} x_i - \sum_{r=1}^{t-k+1} x_{i_r}\right) + f\left(\sum_{i=1}^{t} x_i\right)\right] + 2^t f(x_{t+1}) \quad (2.12)$$

for all $x_1, \ldots, x_{t+1} \in X$. Now, by the case $n = t$, we lead to

$$\sum_{k=2}^{t+1}\left(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{t-k+2}=i_{t-k+1}+1}^{t+1}\right) f\left(\sum_{i=1,\, i\neq i_1,\ldots,i_{t-k+2}}^{t+1} x_i - \sum_{r=1}^{t-k+2} x_{i_r}\right) + f\left(\sum_{i=1}^{t+1} x_i\right)$$
$$= 2\left[2^{t-1}\sum_{i=1}^{t} f(x_i)\right] + 2^t f(x_{t+1}) \quad (2.13)$$

for all $x_1, \ldots, x_{t+1} \in X$, so (1.3) holds for $n = t + 1$. This completes the proof of the lemma. □

Corollary 2.2. *A function $f : X \to Y$ satisfies the functional equation* (1.3) *if and only if there exists a symmetric biadditive function $B_1 : X \times X \to Y$ such that $f(x) = B_1(x,x)$ for all $x \in X$.*

Now, we investigate the stability of the functional equation (1.3) from a Banach space B into p-adic field $\mathbb{Q}_p$. For convenience, we define the difference operator D_f for a given function f:

$$D_f(x_1,\ldots,x_n) := \sum_{k=2}^{n}\left(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n}\right) f\left(\sum_{i=1,\, i\neq i_1,\ldots,i_{n-k+1}}^{n} x_i - \sum_{r=1}^{n-k+1} x_{i_r}\right) + f\left(\sum_{i=1}^{n} x_i\right) - 2^{n-1}\sum_{i=1}^{n} f(x_i). \tag{2.14}$$

Theorem 2.3. *Let B be a Banach space and let $\varepsilon > 0$, λ be real numbers. Suppose that a function $f : \mathbb{Q}_p \to B$ with $f(0) = 0$ satisfies the inequality*

$$\left\| D_f(x_1,\ldots,x_n)\right\| \leq \varepsilon \sum_{i=1}^{n} |x_i|_p^{\lambda} \tag{2.15}$$

for all $x_1,\ldots,x_n \in \mathbb{Q}_p$. Then there exists a unique quadratic function $Q : \mathbb{Q}_p \to B$ such that

$$\left\| f(x) - Q(x)\right\| \leq \begin{cases} \dfrac{\varepsilon}{2^{n-1} - 2^{n-\lambda-3}}|x|_p^{\lambda}, & p = 2,\ \lambda > -2; \\ \dfrac{\varepsilon}{3.2^{n-3}}|x|_p^{\lambda}, & p > 2; \end{cases} \tag{2.16}$$

for all nonzero $x \in \mathbb{Q}_p$.

Proof. Letting $x_1 = x_2 = x \neq 0$ and $x_i = 0$ $(i = 3,\ldots,n)$ in (2.15), we obtain

$$\left\| f(x) - \frac{1}{4} f(2x)\right\| \leq \frac{\varepsilon}{2^{n-1}}|x|_p^{\lambda} \tag{2.17}$$

for all $x \in \mathbb{Q}_p$. Hence,

$$\left\| \frac{1}{2^{2l}} f\left(2^l x\right) - \frac{1}{2^{2m}} f(2^m x)\right\| \leq \frac{\varepsilon}{2^{n-1}} \sum_{j=l}^{m-1} \frac{|2|_p^{\lambda j}}{2^{2j}} |x|_p^{\lambda} \tag{2.18}$$

for all nonnegative integers m and l with $m > l$ and for all $x \in \mathbb{Q}_p$. It follows from (2.18) that the sequence $\{(1/2^{2m}) f(2^m x)\}$ is a Cauchy sequence for all $x \in \mathbb{Q}_p$. Since B is complete, the sequence $\{(1/2^{2m}) f(2^m x)\}$ converges. Therefore, one can define the function $Q : \mathbb{Q}_p \to B$ by

$$Q(x) := \lim_{m\to\infty} \frac{1}{2^{2m}} f(2^m x) \tag{2.19}$$

for all $x \in \mathbb{Q}_p$. It follows from (2.15) and (2.19) that

$$\left\|D_Q(x_1,\ldots,x_n)\right\| = \lim_{m\to\infty}\frac{1}{2^{2m}}\left\|D_f(2^m x_1,\ldots,2^m x_n)\right\| \le \lim_{m\to\infty}\frac{|2|_p^{\lambda m}}{2^{2m}}\sum_{i=1}^{n}\varepsilon|x_i|_p^{\lambda} = 0 \tag{2.20}$$

for all $x_1,\ldots,x_n \in \mathbb{Q}_p$. So $D_Q(x_1,\ldots,x_n) = 0$. By Lemma 2.1, the function $Q : \mathbb{Q}_p \to B$ is quadratic.

Taking the limit $m \to \infty$ in (2.18) with $l = 0$, we find that the function Q is quadratic function satisfying the inequality (2.16) near the approximate function $f : \mathbb{Q}_p \to B$ of (1.3).

To prove the aforementioned uniqueness, we assume now that there is another additive function $Q' : \mathbb{Q}_p \to B$ which satisfies (1.3) and the inequality (2.16). So

$$\begin{aligned}
\left\|Q(x) - Q'(x)\right\| &= \frac{1}{2^{2m}}\left\|Q(2^m x) - Q'(2^m x)\right\| \\
&\le \frac{1}{2^{2m}}\left(\left\|Q(2^m x) - f(2^m x)\right\| + \left\|f(2^m x) - Q'(2^m x)\right\|\right) \\
&\le \begin{cases} \dfrac{\varepsilon}{2^{2m+\lambda m}\left(2^{n-2} - 2^{n-\lambda-4}\right)}|x|_p^{\lambda}, & p = 2,\ \lambda > -2; \\ \dfrac{\varepsilon}{3.2^{2m+n-4}}|x|_p^{\lambda}, & p > 2; \end{cases}
\end{aligned} \tag{2.21}$$

which tends to zero as $m \to \infty$ for all nonzero $x \in \mathbb{Q}_p$. This proves the uniqueness of Q, completing the proof of uniqueness. □

The following example shows that the above result is not valid over p-adic fields.

Example 2.4. Let $p > 2$ be a prime number and define $f : \mathbb{Q}_p \to \mathbb{Q}_p$ by $f(x) = x^2 - 2x$. Since $|2^n|_p = 1$,

$$\left|D_f(x_1,\ldots,x_n)\right|_p = \left|2^n\sum_{i=2}^{n}x_i\right|_p = \left|\sum_{i=2}^{n}x_i\right|_p \le \sum_{i=1}^{n}|x_i|_p \tag{2.22}$$

for all $x_1,\ldots,x_n \in \mathbb{Q}_p$. Hence, the conditions of Theorem 2.3 for $\varepsilon = 1$ and $\lambda = 1$ hold. However for each $n \in \mathbb{N}$, we have

$$\left|\frac{1}{2^{2(m+1)}}f(2^{m+1}x) - \frac{1}{2^{2m}}f(2^m x)\right|_p = \frac{|x|_p}{|2^m|_p} = |x|_p \tag{2.23}$$

for all $x \in \mathbb{Q}_p$. Hence $\{(1/2^{2m})f(2^m x)\}$ is not convergent for all nonzero $x \in \mathbb{Q}_p$.

In the next result, which can be compared with Theorem 2.3, we will show that the stability of the functional equation (1.3) in non-Archimedean spaces over p-adic fields.

Theorem 2.5. *Let $\ell \in \{-1, 1\}$ be fixed. Let $\mathcal{V}$ be a non-Archimedean space and $\mathcal{W}$ be a complete non-Archimedean space over $\mathbb{Q}_p$, where $p > 2$ is a prime number. Suppose that a function $f : \mathcal{V} \to \mathcal{W}$ satisfies the inequality*

$$\|D_f(x_1, \dots, x_n)\|_{\mathcal{W}} \leq \begin{cases} \varepsilon \sum_{i=1}^{n} \|x_i\|_{\mathcal{V}}^{\lambda}, & \lambda\ell > 2\ell; \\ \varepsilon \sum_{i=2}^{n} \|x_1\|_{\mathcal{V}}^{\lambda_1} \|x_i\|_{\mathcal{V}}^{\lambda_i}, & (\lambda_1 + \lambda_i)\ell > 2\ell; \\ \varepsilon \max\left\{ \|x_i\|_{\mathcal{V}}^{\lambda}; 1 \leq i \leq n \right\}, & \lambda\ell > 2\ell; \end{cases} \tag{2.24}$$

for all $x_1, \dots, x_n \in \mathcal{V}$, where $\varepsilon, \lambda_1, \dots, \lambda_n$ and λ are nonnegative real numbers. Then, the limit

$$Q(x) := \lim_{m \to \infty} \frac{1}{p^{2\ell m}} f\left(p^{\ell m} x\right) \tag{2.25}$$

exists for all $x \in \mathcal{V}$ and $Q : \mathcal{V} \to \mathcal{W}$ is a unique quadratic function satisfying

$$\|f(x) - Q(x)\|_{\mathcal{W}} \leq \begin{cases} 2p^{1+\ell+(1-\ell)\lambda/2} \varepsilon \|x\|_{\mathcal{V}}^{\lambda}, \\ p^{1+\ell+((1-\ell)(\lambda_1+\lambda_2)/2)} \varepsilon \|x\|_{\mathcal{V}}^{\lambda_1+\lambda_2}, \\ p^{1+\ell+(1-\ell)\lambda/2} \varepsilon \|x\|_{\mathcal{V}}^{\lambda}, \end{cases} \tag{2.26}$$

for all $x \in \mathcal{V}$.

Proof. By (2.24),

$$\|D_f(x_1, \dots, x_n)\|_{\mathcal{W}} \leq \varepsilon \sum_{i=1}^{n} \|x_i\|_{\mathcal{V}}^{\lambda} \tag{2.27}$$

for all $x_1, \dots, x_n \in \mathcal{V}$, where $\lambda\ell > 2\ell$. Putting $x_i = 0$ $(i = 1, \dots, n)$ in (2.27) to obtain $f(0) = 0$, setting $x_i = 0$ $(i = 3, \dots, n)$ in (2.27), we obtain

$$\left\|2^{n-2} f(x_1 + x_2) + 2^{n-2} f(x_1 - x_2) - 2^{n-1} f(x_1) - 2^{n-1} f(x_2)\right\|_{\mathcal{W}} \leq \varepsilon\left(\|x_1\|_{\mathcal{V}}^{\lambda} + \|x_2\|_{\mathcal{V}}^{\lambda}\right) \tag{2.28}$$

for all $x_1, x_2 \in \mathcal{V}$. So

$$\|f(x_1 + x_2) + f(x_1 - x_2) - 2f(x_1) - 2f(x_2)\|_{\mathcal{W}} \leq \varepsilon\left(\|x_1\|_{\mathcal{V}}^{\lambda} + \|x_2\|_{\mathcal{V}}^{\lambda}\right) \tag{2.29}$$

for all $x_1, x_2 \in \mathcal{V}$. Letting $x_1 = x_2 = x$ in (2.29), we have

$$\|f(2x) - 4f(x)\|_{\mathcal{W}} \leq 2\varepsilon \|x\|_{\mathcal{V}}^{\lambda} \tag{2.30}$$

for all $x \in \mathcal{V}$. By induction on j, we will show that for each $j \geq 2$,

$$\left\| f(jx) - j^2 f(x) \right\|_{\mathcal{W}} \leq 2\varepsilon \|x\|_{\mathcal{V}}^{\lambda} \tag{2.31}$$

for all $x \in \mathcal{V}$. It holds on $j = 2$; see (2.30). Let (2.31) hold for $j = 2, \ldots, k$. Replacing x_1 and x_2 by kx and x in (2.29), respectively, we get

$$\| f((k+1)x) + f((k-1)x) - 2f(kx) - 2f(x) \|_{\mathcal{W}} \leq \varepsilon \left(1 + |k|_p^{\lambda}\right) \|x\|_{\mathcal{V}}^{\lambda} \tag{2.32}$$

for all $x \in \mathcal{V}$. It follows from (2.32) and our induction hypothesis that

$$\begin{aligned} \left\| f((k+1)x) - (k+1)^2 f(x) \right\|_{\mathcal{W}} &= \| f((k+1)x) + f((k-1)x) - 2f(kx) - 2f(x) \\ &\quad - f((k-1)x) + (k-1)^2 f(x) - 2\left(f(kx) - k^2 f(x)\right) \Big\|_{\mathcal{W}} \\ &\leq \max\left\{ 2\varepsilon \|x\|_{\mathcal{V}}^{\lambda}, \varepsilon\left(1 + |k|_p^{\lambda}\right) \|x\|_{\mathcal{V}}^{\lambda} \right\} = 2\varepsilon \|x\|_{\mathcal{V}}^{\lambda} \end{aligned} \tag{2.33}$$

for all $x \in \mathcal{V}$. This proves (2.31) for each $j \geq 2$. In particular,

$$\left\| f(px) - p^2 f(x) \right\|_{\mathcal{W}} \leq 2\varepsilon \|x\|_{\mathcal{V}}^{\lambda} \tag{2.34}$$

for all $x \in \mathcal{V}$. So

$$\begin{aligned} \left\| f(x) - \frac{1}{p^2} f(px) \right\|_{\mathcal{W}} &\leq 2p^2 \varepsilon \|x\|_{\mathcal{V}}^{\lambda}, \\ \left\| f(x) - p^2 f\left(\frac{x}{p}\right) \right\|_{\mathcal{W}} &\leq 2p^{\lambda} \varepsilon \|x\|_{\mathcal{V}}^{\lambda} \end{aligned} \tag{2.35}$$

for all $x \in \mathcal{V}$. Hence,

$$\left\| \frac{1}{p^{2\ell j}} f\left(p^{\ell j} x\right) - \frac{1}{p^{2\ell(j+1)}} f\left(p^{\ell(j+1)} x\right) \right\|_{\mathcal{W}} \leq \frac{2p^{2\ell j + (1-\ell)\lambda/2 + 1 + \ell}}{p^{\lambda \ell j}} \varepsilon \|x\|_{\mathcal{V}}^{\lambda} \tag{2.36}$$

for all $x \in \mathcal{V}$. Since the right side of the above inequality tends to zero as $j \to \infty$, $\{(1/p^{2\ell m}) f(p^{\ell m} x)\}$ is a Cauchy sequence in complete non-Archimedean space $\mathcal{W}$, thus it converges to some function $Q(x) = \lim_{m\to\infty}(1/p^{2\ell m})f(p^{\ell m}x)$ for all $x \in \mathcal{V}$. Using (2.35) and induction, one can show that for any $m \in \mathbb{N}$, we have

$$\begin{aligned} \left\| f(x) - \frac{1}{p^{2\ell m}} f(p^{\ell m}x) \right\|_{\mathcal{W}} &= \left\| \sum_{j=0}^{m-1} \frac{1}{p^{2\ell j}} f(p^{\ell j}x) - \frac{1}{p^{2\ell(j+1)}} f(p^{\ell(j+1)}x) \right\|_{\mathcal{W}} \\ &\le \max\left\{ \left\| \frac{1}{p^{2\ell j}} f(p^{\ell j}x) - \frac{1}{p^{2\ell(j+1)}} f\left(p^{\ell(j+1)}x\right) \right\|_{\mathcal{W}} ; 0 \le j < m \right\} \\ &\le \max\left\{ 2p^{1+\ell+(1-\ell)\lambda/2+\ell j(2-\lambda)} \varepsilon \|x\|_{\mathcal{V}}^{\lambda}; 0 \le j < m \right\} \end{aligned} \tag{2.37}$$

for all $x \in \mathcal{V}$. Letting $m \to \infty$ in this inequality, we see that

$$\|f(x) - Q(x)\|_{\mathcal{W}} \le 2p^{1+\ell+(1-\ell)\lambda/2} \varepsilon \|x\|_{\mathcal{V}}^{\lambda} \tag{2.38}$$

for all $x \in \mathcal{V}$. Moreover,

$$\|D_Q(x_1, \ldots, x_n)\|_{\mathcal{W}} = \lim_{m\to\infty} \left\| \frac{1}{p^{2\ell m}} D_f(p^{\ell m}x_1, \ldots, p^{\ell m}x_n) \right\|_{\mathcal{W}} \le \lim_{m\to\infty} \frac{p^{2\ell m}}{p^{\lambda \ell m}} \sum_{i=1}^{n} \varepsilon \|x_i\|_{\mathcal{V}}^{\lambda} = 0 \tag{2.39}$$

for all $x_1, \ldots, x_n \in \mathcal{V}$. So $D_Q(x_1, \ldots, x_n) = 0$. By Lemma 2.1, the function $Q : \mathcal{V} \to \mathcal{W}$ is quadratic.

Now, let $Q' : \mathcal{V} \to \mathcal{W}$ be another quadratic function satisfying (1.3) and (2.38). So

$$\begin{aligned} \|Q(x) - Q'(x)\|_{\mathcal{W}} &\le p^{2\ell m} \max\left\{ \left\| Q\left(p^{\ell m}x\right) - f\left(p^{\ell m}x\right) \right\|_{\mathcal{W}'} \left\| f\left(p^{\ell m}x\right) - Q'\left(p^{\ell m}x\right) \right\|_{\mathcal{W}} \right\} \\ &\le \frac{2p^{2\ell m+(1-\ell)\lambda/2+1+\ell}}{p^{\lambda \ell m}} \varepsilon \|x\|_{\mathcal{V}}^{\lambda}, \end{aligned} \tag{2.40}$$

which tends to zero as $m \to \infty$ for all $x \in \mathcal{V}$. This proves the uniqueness of Q.

The rest of the proof is similar to the above proof, hence it is omitted. □

Acknowledgments

The third author of this work was partially supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (Grant number: 2011-0005197).

References

[1] K. Hensel, "Uber eine neue Begrundung der theorie der algebraischen Zahlen," *Jahresbericht der Deutschen Mathematiker Vereinigung*, vol. 6, pp. 83–88, 1897.

[2] V. S. Vladimirov, I. V. Volovich, and E. I. Zelenov, p-*Adic Analysis and Mathematical Physics*, vol. 1 of *Series on Soviet and East European Mathematics*, World Scientific, River Edge, NJ, USA, 1994.
[3] F. Q. Gouvêa, p-*Adic Numbers*, Springer, Berlin, Germany, 2nd edition, 1997.
[4] A. Khrennikov, p-*Adic Valued Distributions in Mathematical Physics*, vol. 309 of *Mathematics and its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1994.
[5] A. Khrennikov, *Non-Archimedean Analysis: Quantum Paradoxes, Dynamical Systems and Biological Models*, vol. 427 of *Mathematics and its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1997.
[6] S. M. Ulam, *Problems in Modern Mathematics*, John Wiley & Sons, New York, NY, USA, 1964.
[7] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.
[8] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.
[9] T. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.
[10] J. M. Rassias, "On approximation of approximately linear mappings by linear mappings," *Journal of Functional Analysis*, vol. 46, no. 1, pp. 126–130, 1982.
[11] G. L. Forti, "The stability of homomorphisms and amenability, with applications to functional equations," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 57, pp. 215–226, 1987.
[12] P. Găvruţa, "A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings," *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.
[13] M. Eshaghi Gordji, A. Ebadian, and S. Zolfaghari, "Stability of a functional equation deriving from cubic and quartic functions," *Abstract and Applied Analysis*, vol. 2008, Article ID 801904, 17 pages, 2008.
[14] M. Eshaghi Gordji, M. B. Ghaemi, and H. Majani, "Generalized Hyers-Ulam-Rassias theorem in Menger probabilistic normed spaces," *Discrete Dynamics in Nature and Society*, vol. 2010, Article ID 162371, 11 pages, 2010.
[15] M. Eshaghi Gordji, S. Kaboli Gharetapeh, J. M. Rassias, and S. Zolfaghari, "Solution and stability of a mixed type additive, quadratic, and cubic functional equation," *Advances in Difference Equations*, vol. 2009, Article ID 826130, 17 pages, 2009.
[16] M. Eshaghi Gordji, H. Khodaei, and Th. M. Rassias, "Fixed points and stability for quadratic mappings in β-normed left Banach modules on Banach algebras," *Results in Mathematics*. In press.
[17] M. Eshaghi Gordji, S. Zolfaghari, J. M. Rassias, and M. B. Savadkouhi, "Solution and stability of a mixed type cubic and quartic functional equation in quasi-Banach spaces," *Abstract and Applied Analysis*, vol. 2009, Article ID 417473, 14 pages, 2009.
[18] J. M. Rassias, "Solution of a problem of Ulam," *Journal of Approximation Theory*, vol. 57, no. 3, pp. 268–273, 1989.
[19] K. Ravi, M. Arunkumar, and J. M. Rassias, "Ulam stability for the orthogonally general Euler-Lagrange type functional equation," *International Journal of Mathematics and Statistics*, vol. 3, no. A08, pp. 36–46, 2008.
[20] H.-X. Cao, J.-R. Lv, and J. M. Rassias, "Superstability for generalized module left derivations and generalized module derivations on a Banach module. II," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 10, no. 3, pp. 1–8, 2009.
[21] H.-X. Cao, J.-R. Lv, and J. M. Rassias, "Superstability for generalized module left derivations and generalized module derivations on a Banach module. I," *Journal of Inequalities and Applications*, vol. 2009, Article ID 718020, 10 pages, 2009.
[22] J. M. Rassias and H.-M. Kim, "Approximate homomorphisms and derivations between C^*-ternary algebras," *Journal of Mathematical Physics*, vol. 49, no. 6, article 063507, 10 pages, 2008.
[23] L. M. Arriola and W. A. Beyer, "Stability of the Cauchy functional equation over p-adic fields," *Real Analysis Exchange*, vol. 31, no. 1, pp. 125–132, 2005/06.
[24] Y. J. Cho, C. Park, and R. Saadati, "Functional inequalities in non-Archimedean Banach spaces," *Applied Mathematics Letters*, vol. 23, no. 10, pp. 1238–1242, 2010.
[25] M. B. Savadkouhi, M. E. Gordji, J. M. Rassias, and N. Ghobadipour, "Approximate ternary Jordan derivations on Banach ternary algebras," *Journal of Mathematical Physics*, vol. 50, no. 4, article 042303, 9 pages, 2009.
[26] A. Ebadian, N. Ghobadipour, and M. E. Gordji, "A fixed point method for perturbation of bimultipliers and Jordan bimultipliers in C^*-ternary algebras," *Journal of Mathematical Physics*, vol. 51, no. 1, 10 pages, 2010.

[27] M. Eshaghi Gordji and Z. Alizadeh, "Stability and superstability of ring homomorphisms on non-Archimedean Banach algebras," *Abstract and Applied Analysis*, vol. 2011, Article ID 123656, 10 pages, 2011.

[28] M. S. Moslehian and T. M. Rassias, "Stability of functional equations in non-Archimedean spaces," *Applicable Analysis and Discrete Mathematics*, vol. 1, no. 2, pp. 325–334, 2007.

[29] M. Eshaghi Gordji, M. B. Ghaemi, S. Kaboli Gharetapeh, S. Shams, and A. Ebadian, "On the stability of J^*-derivations," *Journal of Geometry and Physics*, vol. 60, no. 3, pp. 454–459, 2010.

[30] M. Eshaghi Gordji and A. Najati, "Approximately J^*-homomorphisms: a fixed point approach," *Journal of Geometry and Physics*, vol. 60, no. 5, pp. 809–814, 2010.

[31] M. E. Gordji and M. S. Moslehian, "A trick for investigation of approximate derivations," *Mathematical Communications*, vol. 15, no. 1, pp. 99–105, 2010.

[32] M. Eshaghi Gordji, J. M. Rassias, and N. Ghobadipour, "Generalized Hyers-Ulam stability of generalized (n, k)-derivations," *Abstract and Applied Analysis*, vol. 2009, Article ID 437931, 8 pages, 2009.

[33] M. Eshaghi Gordji, H. Khodaei, and R. Khodabakhsh, "General quartic-cubic-quadratic functional equation in non-Archimedean normed spaces," *"Politehnica" University of Bucharest Scientific Bulletin Series A*, vol. 72, no. 3, pp. 69–84, 2010.

[34] S. Czerwik, "On the stability of the quadratic mapping in normed spaces," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 62, pp. 59–64, 1992.

[35] M. Eshaghi Gordji, "Nearly ring homomorphisms and nearly ring derivations on non-Archimedean Banach algebras," *Abstract and Applied Analysis*, vol. 2010, Article ID 393247, 12 pages, 2010.

[36] M. Eshaghi Gordji and H. Khodaei, *Stability of Functional Equations*, Lap Lambert Academic Publishing, 2010.

[37] M. Eshaghi Gordji and H. Khodaei, "On the generalized Hyers-Ulam-Rassias stability of quadratic functional equations," *Abstract and Applied Analysis*, vol. 2009, Article ID 923476, 11 pages, 2009.

[38] M. Eshaghi Gordji and H. Khodaei, "Solution and stability of generalized mixed type cubic, quadratic and additive functional equation in quasi-Banach spaces," *Nonlinear Analysis, Theory, Methods & Applications*, vol. 71, no. 11, pp. 5629–5643, 2009.

[39] D. H. Hyers, G. Isac, and T. M. Rassias, *Stability of Functional Equations in Several Variables*, Progress in Nonlinear Differential Equations and their Applications, Birkhäuser, Basel, Switzerland, 1998.

[40] S.-M. Jung, "On the Hyers-Ulam-Rassias stability of a quadratic functional equation," *Journal of Mathematical Analysis and Applications*, vol. 232, no. 2, pp. 384–393, 1999.

[41] S.-M. Jung and P. K. Sahoo, "Stability of a functional equation for square root spirals," *Applied Mathematics Letters*, vol. 15, no. 4, pp. 435–438, 2002.

[42] A. Najati and F. Moradlou, "Hyers-Ulam-Rassias stability of the Apollonius type quadratic mapping in non-Archimedean spaces," *Tamsui Oxford Journal of Mathematical Sciences*, vol. 24, no. 4, pp. 367–380, 2008.

[43] C.-G. Park, "On an approximate automorphism on a C^*-algebra," *Proceedings of the American Mathematical Society*, vol. 132, no. 6, pp. 1739–1745, 2004.

[44] R. Saadati, Y. J. Cho, and J. Vahidi, "The stability of the quartic functional equation in various spaces," *Computers & Mathematics with Applications*, vol. 60, no. 7, pp. 1994–2002, 2010.

[45] R. Saadati and C. Park, "Non-Archimedian $\mathcal{L}$-fuzzy normed spaces and stability of functional equations," *Computers & Mathematics with Applications*, vol. 60, no. 8, pp. 2488–2496, 2010.

[46] H. Khodaei and T. M. Rassias, "Approximately generalized additive functions in several variables," *International Journal of Nonlinear Analysis and Applications*, vol. 1, pp. 22–41, 2010.

[47] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, vol. 31 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, UK, 1989.

[48] P. Kannappan, "Quadratic functional equation and inner product spaces," *Results in Mathematics*, vol. 27, no. 3-4, pp. 368–372, 1995.