Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 376464, 13 pages
doi:10.1155/2012/376464

Research Article

New Results on Global Exponential Stability of Impulsive Functional Differential Systems with Delayed Impulses

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Received 20 June 2012; Accepted 7 August 2012

Academic Editor: Elena Braverman

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By using the Lyapunov functions and the Razumikhin techniques, the exponential stability of impulsive functional differential systems with delayed impulses is investigated. The obtained results have shown that the system will stable if the impulses' frequency and amplitude are suitably related to the increase or decrease of the continuous flows, and they improve and complement ones from some recent works. An example is provided to illustrate the effectiveness and the advantages of the results obtained.

1. Introduction

There has been a growing interest in the theory of impulsive dynamical systems in the past decades because of their applications to various problems arising in communications, control technology, impact mechanics, electrical engineering, medicine, biology and so forth; see the monographs [1, 2] and the papers [3–8] and the references therein. In particular, special attention has been focused on stability and impulsive stabilization of impulsive functional differential systems (IFDSs) (see, e.g., [9–26]).

However, in these previous works on stability of IFDSs, the authors always suppose that the state variables on the impulses are only related to the present state variables. But in most cases, it is more applicable that the state variables on the impulses that we add are also related to the past ones. For example, it is more realistic in practice if the impulsive control depends on a past state due to a time lag between the time when the observation of the state is made and the time when the feedback control reaches the system.

In fact, there have been several attempts in the literature to study the stability and control problems of a particular class of IFDSs with delayed impulses (see, e.g., [27–36]).

Lian et al. [27] investigated the optimal control problem of linear continuous-time systems possessing delayed discrete-time controllers in networked control systems. For nonlinear impulsive systems, Khadra et al. studied the impulsive synchronization problem coupled by linear delayed impulses in [28]. In addition, in [29–34], the authors investigate the uniform asymptotic stability and global exponential stability of general IFDSs:

$$\begin{aligned} \dot{x}(t) &= f(t, x_t), \quad t \neq t_k,\ t \geqslant t_0, \\ \Delta x(t_k) &= I_k\left(t_k, x_{t_k^-}\right), \quad k \in \mathbb{Z}_+. \end{aligned} \tag{1.1}$$

But in these stability analyses, the effects of time delay on the impulses have been ignored. For example in [31–34], the Lyapunov function was assumed to be satisfied $V(t_k, \varphi(0) + I_k(t_k, \varphi)) \leqslant (1 + d_k)V(t_k^-, \varphi(0))$.

Very recently, in [35], Zhang and Sun established some sufficient conditions for uniform stability, uniform asymptotical stability, and practical stability of a particular class of IFDSs with delayed impulses:

$$\begin{aligned} \dot{x}(t) &= f(t, x_t), \quad t \neq t_k,\ t \geqslant t_0, \\ \Delta x(t_k) &= I_k\left(x\left(t_k^-\right)\right) + J_k\left(x\left(t_k^- - \tau\right)\right), \quad k \in \mathbb{Z}_+. \end{aligned} \tag{1.2}$$

However, their results are only valid for some specific systems due to the restrictive requirements on the continuous flows and impulsive gain. Lin et al. [36] investigated the exponential stability and uniform stability of the following more generalized IFDSs with delayed impulses:

$$\begin{aligned} \dot{x}(t) &= f(t, x_t), \quad t \neq t_k,\ t \geqslant t_0, \\ \Delta x(t_k) &= I_k\left(x\left(t_k^-\right)\right) + J_k\left(x_{t_k^-}\right), \quad k \in \mathbb{Z}_+. \end{aligned} \tag{1.3}$$

But those results can only been applied to the systems with stable discrete dynamics since their results need the strong condition of impulsive gain $d_k + e_k < 1$.

Motivated by the above discussions, in this paper, we further study the exponential stability of IFDSs with delayed impulses. Different from the previous works on exponential stability of IFDSs with/without delayed impulses [18, 31, 34, 36], we will divide the systems into two classes: the system with stable continuous dynamics and unstable discrete dynamics, the systems with unstable continuous dynamics and stable discrete dynamics. The first class of impulsive systems corresponds to the case when the continuous dynamics are subjected to impulsive perturbations, while the second class of impulsive systems corresponds to the case when impulses are employed to stabilize the unstable continuous dynamics. This idea is enlightened in part by the works Chen and Zheng [37] about the uncertain impulsive systems. By using the Lyapunov functions and the Razumikhin techniques, some global exponential stability criteria are derived. The results obtained improve and complement some recent works. It is worth mentioning that our results shown that the system will be stable if the impulses' frequency and amplitude are suitably related to the increase or decrease of the continuous flows. Moreover, some results obtained can be applied to IFDSs with any time delay. In the end, an example is provided to illustrate the effectiveness and the advantages of the results obtained.

2. Preliminaries

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ the set of nonnegative real numbers, $\mathbb{Z}_+$ the set of positive integers, and $\mathbb{R}^n$ the n-dimensional real space equipped with the Euclidean norm $|\cdot|$. Let $\tau > 0$ and $\mathrm{PC}([-\tau,0];\mathbb{R}^n) = \{\varphi : [-\tau,0] \to \mathbb{R}^n | \varphi(t^+) = \varphi(t)$ for all $t \in [-\tau,0)$, $\varphi(t^-)$ exist and $\varphi(t^-) = \varphi(t)$ for all, but at most a finite number of points $t \in (-\tau,0]\}$ be with the norm $\|\varphi\| = \sup_{-\tau\leqslant\theta\leqslant 0}|\varphi(\theta)|$, where $\varphi(t^+)$ and $\varphi(t^-)$ denote the right-hand and left-hand limits of function $\varphi(t)$ at t, respectively. Denote $\mathrm{PC}([t_0-\tau,b];\mathbb{R}_+) = \{\varphi : [t_0-\tau,b] \to \mathbb{R}_+ \mid \varphi$ is piecewise continuous$\}$ for $b > t_0$, and $\mathrm{PC}([t_0-\tau,\infty);\mathbb{R}_+) = \{\varphi | \varphi|_{[t_0-\tau,b]} \in \mathrm{PC}([t_0-\tau,b];\mathbb{R}_+)$ for all $b > t_0 - \tau\}$.

Consider the IFDS in which the state variables on the impulses are related to the time delay:

$$\begin{gathered}\dot{x}(t) = f(t,x_t), \quad t \neq t_k,\ t \geqslant t_0,\\ \Delta x(t_k) = I_k(t_k, x(t_k^-)) + J_k\left(t_k, x_{t_k^-}\right), \quad k \in \mathbb{Z}_+\\ x_{t_0} = \phi(s), \quad s \in [-\tau,0],\end{gathered} \tag{2.1}$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R}_+\times\mathbb{C} \to \mathbb{R}^n$, $I_k : \mathbb{R}_+\times\mathbb{R}^n \to \mathbb{R}^n$, $J_k : \mathbb{R}_+\times\mathbb{C} \to \mathbb{R}^n$, $\phi \in \mathrm{PC}([-\tau,0];\mathbb{R}^n)$, $\mathbb{C}$ is a open set in $\mathrm{PC}([-\tau,0];\mathbb{R}^n)$. and The fixed moments of impulse times $\{t_k, k \in \mathbb{Z}_+\}$ satisfy $0 \leqslant t_0 < t_1 < \cdots < t_k < \cdots$, $t_k \to \infty$ (as $k \to \infty$), $\Delta x(t_k) = x(t_k) - x(t_k^-)$; $x_t, x_{t^-} \in$ PC $([-\tau,0];\mathbb{R}^n)$ are defined by $x_t = x(t+\theta)$, $x_{t^-} = x(t^-+\theta)$ for $\theta \in [-\tau,0]$, respectively.

Throughout this paper, we assume that f, I_k, and J_k, $k \in \mathbb{Z}_+$, satisfy the necessary conditions for the global existence and uniqueness of solutions for all $t \geqslant t_0$, see [6, 30–33]. Then for any $\phi \in$ PC $([-\tau,0];\mathbb{R}^n)$, there exists a unique function satisfying system (2.1) denoted by $x(t;t_0,\phi)$, which is continuous on the right-hand side and limitable on the left-hand side. Moreover, we assume that $f(t,0) \equiv 0$, $I_k(t_k,0) \equiv 0$ and $J_k(t_k,0) \equiv 0$, $k \in \mathbb{Z}_+$, which imply that $x(t) \equiv 0$ is a solution of (2.1), which is called the trivial solution.

At the end of this section, let us introduce the following definitions.

Definition 2.1. A function $V : [t_0-\tau,\infty)\times\mathbb{R}^n \to \mathbb{R}_+$ belongs to class v_0 if

(i) V is continuous on each of the sets $[t_{k-1},t_k)\times\mathbb{R}^n$, and for each $x \in \mathbb{R}^n$, $t \in [t_{k-1},t_k)$, $k \in \mathbb{Z}_+$, $\lim_{(t,y)\to(t_k^-,x)} V(t,y) = V(t_k^-,x)$ exists;

(ii) $V(t,x)$ is locally Lipschitz in $x \in \mathbb{R}^n$, and $V(t,0) \equiv 0$ for all $t \geqslant t_0$.

Definition 2.2. Given a function $V \in v_0$, the upper right-hand Dini derivative of V with respect to system (2.1) is defined by

$$D^+V(t,\varphi(0)) = \limsup_{h\to 0^+}\frac{1}{h}\left[V(t+h,\varphi(0)+hf(t,\varphi)) - V(t,\varphi(0))\right], \tag{2.2}$$

for $(t,\varphi) \in [t_0,\infty)\times\mathrm{PC}([-\tau,0];\mathbb{R}^n)$.

Definition 2.3. The trivial solution of system (2.1) or, simply, system (2.1) is said to be globally exponentially stable if there exist positive constants α and C such that for any initial data $x_{t_0} = \phi \in \mathrm{PC}([-\tau, 0]; \mathbb{R}^n)$, the solution $x(t; t_0, \phi)$ satisfies

$$\left| x(t; t_0, \phi) \right| \leqslant C \|\phi\| e^{-\alpha(t-t_0)}, \quad t \geqslant t_0. \tag{2.3}$$

3. Main Results

In this section, we shall analyze the global exponential stability of system (2.1) by employing the Razumikhin techniques and the Lyapunov functions.

Theorem 3.1. *Assume that there exist functions $V \in v_0$, $c \in$ PC $([t_0 - \tau, \infty); \mathbb{R}_+)$, several positive constants c_1, c_2, $\tilde{c}$, p, q, and nonnegative constants ρ_1, ρ_2, $\rho_1 + \rho_2 \geqslant 1$ such that*

(i) $c_1|x|^p \leqslant V(t, x) \leqslant c_2|x|^p$, *for all* $(t, x) \in [t_0 - \tau, \infty) \times \mathbb{R}^n$;

(ii) $V(t_k, \varphi(0)) \leqslant \rho_1(1+\mu_k)V(t_k^-, \varphi(0)) + \rho_2(1+\mu_k)\sup_{\theta\in[-\tau,0]} V(t_k^- + \theta, \varphi(\theta))$, *for each* $k \in \mathbb{Z}_+$ *and* $\varphi \in PC([-\tau, 0]; \mathbb{R}^n)$, *where* μ_k, $k \in \mathbb{Z}_+$, *are nonnegative constants with* $\Sigma_{k=1}^{\infty} \mu_k < \infty$;

(iii) $D^+V(t, \varphi(0)) \leqslant -c(t)V(t, \varphi(0))$, *for all* $t \geqslant t_0$, $t \neq t_k$, $k \in \mathbb{Z}_+$, $\varphi \in$ PC $([-\tau, 0]; \mathbb{R}^n)$, *whenever* $V(t + \theta, \varphi) < qV(t, \varphi(0))$, $\theta \in [-\tau, 0]$;

(iv) $\rho_1 + \rho_2 e^{\tilde{c}\tau} < q < e^{\tilde{c}\varrho}$, $\inf_{t\geqslant t_0} c(t) \geqslant \tilde{c}$, *where* $\varrho = \inf_{k\in\mathbb{Z}_+}\{t_k - t_{k-1}\}$.

Then the trivial solution of system (2.1) *is globally exponentially stable and the convergence rate should not be greater than* $(1/p)(\tilde{c} - (\ln q/\varrho))$.

Proof. Set $L = \prod_{k=1}^{\infty}(1 + \mu_k)$; from the condition $\Sigma_{k=1}^{\infty}\mu_k < \infty$, we known that $1 \leqslant L < \infty$. Fix any initial data $\phi \in$ PC $([-\tau, 0]; \mathbb{R}^n)$ and write $x(t; t_0, \phi) = x(t)$, $V(t, x(t)) = V(t)$ simply. From condition (iv), we can choose a small enough constant $\gamma > 0$ such that

$$e^{\gamma\tau}\left(\rho_1 + \rho_2 e^{\tilde{c}\tau}\right) < q < e^{(\tilde{c}-\gamma)\varrho}, \quad \gamma < \tilde{c}. \tag{3.1}$$

Set $\tilde{q} = qe^{-\gamma\tau} > 1$, choose $M > 0$ such that $\tilde{q}c_2 < M$. Define $W(t) = e^{\gamma(t-t_0)}V(t)$. In the following, we shall show that

$$W(t) \leqslant LM\|\phi\|^p, \quad t \geqslant t_0. \tag{3.2}$$

In order to do so, we first prove that

$$W(t) < M\|\phi\|^p, \quad t \in [t_0 - \tau, t_1). \tag{3.3}$$

It is noted that

$$W(t_0 + \theta) \leqslant c_2\|\phi\|^p < \frac{1}{\tilde{q}}M\|\phi\|^p < M\|\phi\|^p, \quad \theta \in [-\tau, 0]. \tag{3.4}$$

So it only needs to prove

$$W(t) < M\|\phi\|^p, \quad t \in (t_0, t_1). \tag{3.5}$$

We assume, on the contrary, there exist some $t \in (t_0, t_1)$ such that $W(t) \geqslant M\|\phi\|^p$. Set

$$t^* = \inf\{t \in [t_0, t_1) : W(t) \geqslant M\|\phi\|^p\}. \tag{3.6}$$

Note that $W(t)$ is continuous on $t \in [t_0, t_1)$, then $t^* \in (t_0, t_1)$ and

$$W(t^*) = M\|\phi\|^p, \quad W(t) < M\|\phi\|^p, \quad t \in [t_0 - \tau, t^*). \tag{3.7}$$

Define

$$t^{**} = \sup\left\{t \in [t_0, t^*] : W(t) \leqslant \frac{1}{\tilde{q}} M\|\phi\|^p\right\}, \tag{3.8}$$

then $t^{**} \in (t_0, t^*)$ and

$$W(t^{**}) = \frac{1}{\tilde{q}} M\|\phi\|^p, \quad W(t) > \frac{1}{\tilde{q}} M\|\phi\|^p, \quad t \in (t^{**}, t^*]. \tag{3.9}$$

Consequently, for all $t \in [t^{**}, t^*]$,

$$W(t+\theta) \leqslant M\|\phi\|^p \leqslant \tilde{q} W(t), \quad \theta \in [-\tau, 0], \tag{3.10}$$

which implies that

$$V(t+\theta) = e^{-\gamma(t+\theta-t_0)} W(t+\theta) \leqslant \tilde{q} e^{-\gamma(t+\theta-t_0)} W(t) \leqslant \tilde{q} e^{\gamma\tau} V(t) = qV(t), \quad \theta \in [-\tau, 0]. \tag{3.11}$$

Then it follows from condition (iii) that one has that

$$D^+W(t) = e^{\gamma(t-t_0)}\left[\gamma V(t) + D^+V(t)\right] \leqslant \left(\gamma - c(t)\right) W(t), \quad t \in [t^{**}, t^*], \tag{3.12}$$

which leads to

$$\begin{aligned} W(t^*) &\leqslant W(t^{**}) e^{\int_{t^{**}}^{t^*}(\gamma - c(s))\mathrm{d}s} \leqslant W(t^{**}) e^{(\gamma - \tilde{c})(t^* - t^{**})} \\ &\leqslant \frac{1}{\tilde{q}} M\|\phi\|^p < M\|\phi\|^p, \end{aligned} \tag{3.13}$$

this is a contradiction. Thus (3.5) holds.

Now we assume that for some $m \in \mathbb{Z}_+,\ m \geqslant 1$,

$$W(t) < M_m \|\phi\|^p, \quad t \in [t_0 - \tau, t_m), \tag{3.14}$$

where $M_1 = M$, $M_m = M\prod_{1 \leqslant i \leqslant m-1}(1 + \mu_i)$ for $m \geqslant 2$. We will prove that

$$W(t) < M_{m+1} \|\phi\|^p, \quad t \in [t_m, t_{m+1}). \tag{3.15}$$

To do this, we first claim

$$W(t_m^- + \theta) \leqslant \frac{e^{(\tilde{c}-\gamma)\tau}}{\tilde{q}} M_m \|\phi\|^p, \quad \theta \in [-\tau, 0). \tag{3.16}$$

Suppose not, then there exists $\tilde{\theta} \in [-\tau, 0)$ such that $W(t_m^- + \tilde{\theta}) > (e^{(\tilde{c}-\gamma)\tau}/\tilde{q}) M_m \|\phi\|^p$. Without lose generality, we assume $t_m + \tilde{\theta} \in (t_{l-1}, t_l]$, $l \in \mathbb{Z}_+, l \leqslant m$.

There are two cases to be considered.

Case 1. $W(t) > (e^{(\tilde{c}-\gamma)\tau}/\tilde{q}) M_m \|\phi\|^p$ over $t \in [t_{l-1}, t_m + \tilde{\theta})$.

By assumption (3.14), for all $t \in [t_{l-1}, t_m + \tilde{\theta})$, we get

$$W(t + \theta) < M_m \|\phi\|^p < e^{(\tilde{c}-\gamma)\tau} M_m \|\phi\|^p < \tilde{q} W(t), \quad \theta \in [-\tau, 0]. \tag{3.17}$$

Thus, by conditions (iii)-(iv) and inequalities (3.10)–(3.13), we have

$$\begin{aligned} W\left(t_m^- + \tilde{\theta}\right) &\leqslant W(t_{l-1}) e^{(\gamma - \tilde{c})(t_m^- + \tilde{\theta} - t_{l-1})} \\ &< M_m \|\phi\|^p e^{(\tilde{c}-\gamma)\tau} e^{(\gamma - \tilde{c})(t_m - t_{l-1})} \\ &\leqslant \frac{e^{(\tilde{c}-\gamma)\tau}}{q^{m-l+1}} M_m \|\phi\|^p \\ &< \frac{e^{(\tilde{c}-\gamma)\tau}}{\tilde{q}} M_m \|\phi\|^p. \end{aligned} \tag{3.18}$$

This is a contradiction.

Case 2. There are some $t \in [t_{l-1}, t_m + \tilde{\theta})$ such that $W(t) > (e^{(\tilde{c}-\gamma)\tau}/\tilde{q}) M_m \|\phi\|^p$.

In this case, define

$$\bar{t} = \sup\left\{ t \in \left[t_{l-1}, t_m + \tilde{\theta}\right) : W(t) \leqslant \frac{e^{(\tilde{c}-\gamma)\tau}}{\tilde{q}} M_m \|\phi\|^p \right\}. \tag{3.19}$$

Then $\bar{t} \in [t_{l-1}, t_m + \tilde{\theta})$ and

$$W\left(\bar{t}\right) = \frac{e^{(\tilde{c}-\gamma)\tau}}{\tilde{q}} M_m \|\phi\|^p, \quad W(t) > \frac{e^{(\tilde{c}-\gamma)\tau}}{\tilde{q}} M_m \|\phi\|^p, \quad t \in \left(\bar{t}, t_m + \tilde{\theta}\right). \tag{3.20}$$

So from assumption (3.14), for any $t \in [\bar{t}, t_m + \widetilde{\theta})$, we have

$$W(t+\theta) < M_m \|\phi\|^p < e^{(\widetilde{c}-\gamma)\tau} M_m \|\phi\|^p \leqslant \widetilde{q} W(t), \quad \theta \in [-\tau, 0]. \tag{3.21}$$

It follows from condition (iii) that

$$W\left(t_m^- + \widetilde{\theta}\right) \leqslant W\left(\bar{t}\right) = \frac{e^{(\widetilde{c}-\gamma)\tau}}{\widetilde{q}} M_m \|\phi\|^p. \tag{3.22}$$

This is also a contradiction. Hence, inequality (3.16) holds.

Similarly, we can prove

$$W(t_m^-) \leqslant \frac{1}{\widetilde{q}} M_m \|\phi\|^p. \tag{3.23}$$

Then it follows from (3.16), (3.23), and condition (ii) that we obtain

$$\begin{aligned} W(t_m) &\leqslant \rho_1(1+\mu_m) W(t_m^-) + \rho_2(1+\mu_m) e^{\gamma\tau} \sup_{\theta\in[-\tau,0]} W(t_m^- + \theta) \\ &\leqslant \frac{\rho_1 + \rho_2 e^{\widetilde{c}\tau}}{\widetilde{q}} M_{m+1} \|\phi\|^p < M_{m+1} \|\phi\|^p. \end{aligned} \tag{3.24}$$

Now we suppose that (3.15) is not true, let

$$t^* = \inf\{t \in [t_m, t_{m+1}) : W(t) \geqslant M_{m+1} \|\phi\|^p\}. \tag{3.25}$$

Then $t^* \in (t_m, t_{m+1})$ and

$$W(t^*) = M_{m+1} \|\phi\|^p, \quad W(t) < M_{m+1} \|\phi\|^p, \quad t \in [t_m, t^*). \tag{3.26}$$

If $W(t) > (1/\widetilde{q}) M_{m+1} \|\phi\|^p$ for all $t \in [t_m, t^*]$, set $t^{**} = t_m$; otherwise, let

$$t^{**} = \sup\left\{t \in [t_m, t^*] : W(t) \leqslant \frac{1}{\widetilde{q}} M_{m+1} \|\phi\|^p\right\}. \tag{3.27}$$

Thus for all $t \in [t^{**}, t^*]$, we have

$$W(t+\theta) \leqslant M_{m+1} \|\phi\|^p \leqslant \widetilde{q} W(t), \quad \theta \in [-\tau, 0]. \tag{3.28}$$

It follows from condition (iii) that

$$D^+ W(t) = e^{\gamma(t-t_0)} \left[\gamma V(t) + D^+ V(t)\right] \leqslant (\gamma - c(t)) W(t), \quad t \in [t^{**}, t^*], \tag{3.29}$$

which implies

$$W(t^*) \leqslant W(t^{**}) e^{(\gamma-\tilde{c})(t^*-t^{**})} \leqslant W(t^{**}) < M_{m+1} \|\phi\|^p. \quad (3.30)$$

This is a contradiction. Therefore, (3.15) holds.

By mathematical induction, (3.15) holds for any $m \in \mathbb{Z}_+$. That is, (3.2) holds, which implies that

$$|x(t)| \leqslant C\|\phi\| e^{(-\gamma/p)(t-t_0)}, \quad t \geqslant t_0, \quad (3.31)$$

where $C = (\mathrm{LM}/c_1)^{1/p}$. This completes the proof. □

Remark 3.2. The parameters ρ_1 and ρ_2 in condition (ii) describe the influence of impulses on the stability of the underlying continuous systems. When $\rho_1 + \rho_2 \geqslant 1$, the Lyapunov function V may jump up along the state trajectories of system (2.1) at impulsive time instant t_k. Thus the impulses may be viewed as disturbances, that is, they potentially destroy the stability of continuous system. In this case, it is required that the impulses do not occur too frequently. Theorem 3.1 tells us to what extent we can relax the restriction on the impulses to keep the exponential stability property of the original continuous system.

Theorem 3.3. *Assume that there exist functions $V \in v_0$, $c \in$ PC $([t_0-\tau,\infty);\mathbb{R}_+)$, several positive constants c_1, c_2, $\tilde{c}$, p, q, and nonnegative constants ρ_1, ρ_2, $\rho_1+\rho_2 < 1$ such that*

(i) $c_1|x|^p \leqslant V(t,x) \leqslant c_2|x|^p$, *for all* $(x,t) \in \mathbb{R}^n \times [t_0-\tau,\infty)$;

(ii) $V(t_k,\varphi(0)) \leqslant \rho_1(1+\mu_k)V(t_k^-,\varphi(0))+\rho_2(1+\mu_k)\sup_{\theta\in[-\tau,0]} V(t_k^-+\theta,\varphi(\theta))$, *for each* $k \in \mathbb{Z}_+$ *and* $\varphi \in P([-\tau,0];\mathbb{R}^n)$, *where* μ_k, $k \in \mathbb{Z}_+$, *are nonnegative constants with* $\sum_{k=1}^{\infty} \mu_k < \infty$;

(iii) $D^+V(t,\varphi(0)) \leqslant c(t)V(t,\varphi(0))$, *for all* $t \geqslant t_0$, $t \neq t_k$, $k \in \mathbb{Z}_+$, $\varphi \in$ PC $([-\tau,0];\mathbb{R}^n)$, *whenever* $V(t+\theta,\varphi) < qV(t,\varphi(0))$, $\theta \in [-\tau,0]$;

(iv) $q > 1/(\rho_1+\rho_2) > e^{\tilde{c}\varrho}$, $\tilde{c}\varrho \geqslant \sup_{t\geqslant t_0}\int_t^{t+\varrho} c(s)ds$, *where* $\varrho = \sup_{k\in\mathbb{Z}_+}\{t_k - t_{k-1}\}$.

Then the trivial solution of system (2.1) *is globally exponentially stable for any time delay* $\tau \in (0,\infty)$ *and the convergence rate should not be greater than* $(1/p)((\ln q/\varrho) - \tilde{c})$.

Proof. From condition (iv), we can choose a small enough constant $\gamma > 0$ such that

$$q > \frac{e^{\gamma\tau}}{\rho_1 + \rho_2 e^{\gamma\tau}} > \frac{1}{\rho_1 + \rho_2 e^{\gamma\tau}} > e^{(\tilde{c}+\gamma)\varrho}, \quad qe^{-\gamma\tau} > 1. \quad (3.32)$$

Set $\tilde{q} = qe^{-\gamma\tau}$. The following proof can be completed by using the similar arguments as in the proof of Theorem 3.1, so it is omitted. □

Remark 3.4. When $\rho_1 + \rho_2 < 1$, the Lyapunov function V may jump down along the state trajectories of system (2.1) at impulsive time instant t_k. Thus the impulses may be viewed impulsive stabilizing, that is, they may be used to stabilize the continuous system if the original continuous system is not stable. In this case, the impulses must be frequent and their amplitude must be suitably related the growth rate of V.

Remark 3.5. If $c(t) \equiv c$, then Theorem 3.3 becomes Theorem 3.1 in [36] with $d_k = \rho_1(1+\mu_k)$, $e_k = \rho_2(1+\mu_k)$, $d_k + e_k < 1$. Obviously, Theorem 3.3 in this paper has a wider adaptive range than those in [36].

Let $J_k \equiv 0$ in system (2.1), then we have the following IFDS (see [9–23, 26]):

$$\begin{gathered} \dot{x}(t) = f(x_t, t), \quad t \neq t_k,\ t \geqslant t_0, \\ \Delta x(t_k) = I_k\left(x(t_k^-), t_k\right), \quad k \in \mathbb{Z}_+, \\ x_{t_0} = \phi(s), \quad s \in [-\tau, 0]. \end{gathered} \tag{3.33}$$

For system (3.33), we have the following results by Theorems 3.1 and 3.3, respectively.

Corollary 3.6. *Assume that there exist functions $V \in v_0$, $c \in$ PC $([t_0-\tau,\infty);\mathbb{R}_+)$, and several positive constants c_1, c_2, $\widetilde{c}$, p, q, and a constant $\rho \geqslant 1$ such that*

(i) $c_1|x|^p \leqslant V(t,x) \leqslant c_2|x|^p$, *for all* $(x,t) \in \mathbb{R}^n \times [t_0-\tau,\infty)$;

(ii) $V(t_k,\varphi(0)) \leqslant \rho(1+\mu_k)V(t_k^-,\varphi(0))$, *for each* $k \in \mathbb{Z}_+$ *and* $\varphi \in P([-\tau,0];\mathbb{R}^n)$, *where* μ_k, $k \in \mathbb{Z}_+$, *are nonnegative constants with* $\sum_{k=1}^{\infty}\mu_k < \infty$;

(iii) $D^+V(t,\varphi(0)) \leqslant -c(t)V(t,\varphi(0))$, *for all* $t \geqslant t_0$, $t \neq t_k$, $k \in \mathbb{Z}_+$, $\varphi \in$ PC $([-\tau,0];\mathbb{R}^n)$, *whenever* $V(t+\theta,\varphi) < qV(t,\varphi(0))$, $\theta \in [-\tau,0]$;

(iv) $\rho < q < e^{\widetilde{c}\varrho}$, $\inf_{t\geqslant t_0} c(t) \geqslant \widetilde{c}$, *where* $\varrho = \inf_{k\in\mathbb{Z}_+}\{t_k - t_{k-1}\}$.

Then the trivial solution of system (3.33) *is globally exponentially stable for any time delay* $\tau \in (0,\infty)$ *and the convergence rate should not be greater than* $(1/p)(\widetilde{c} - (\ln q/\varrho))$.

Corollary 3.7. *Assume that there exist functions $V \in v_0$, $c \in$ PC $([t_0-\tau,\infty);\mathbb{R}_+)$ and several positive constants c_1, c_2, $\widetilde{c}$, p, q, and a constant $\rho < 1$ such that*

(i) $c_1|x|^p \leqslant V(t,x) \leqslant c_2|x|^p$, *for all* $(x,t) \in \mathbb{R}^n \times [t_0-\tau,\infty)$;

(ii) $V(t_k,\varphi(0)) \leqslant \rho(1+\mu_k)V(t_k^-,\varphi(0))$, *for each* $k \in \mathbb{Z}_+$, $\varphi \in P([-\tau,0];\mathbb{R}^n)$, *where* μ_k, $k \in \mathbb{Z}_+$, *are nonnegative constants with* $\sum_{k=1}^{\infty}\mu_k < \infty$;

(iii) $D^+V(t,\varphi(0)) \leqslant c(t)V(t,\varphi(0))$, *for all* $t \geqslant t_0$, $t \neq t_k$, $k \in \mathbb{Z}_+$, $\varphi \in$ PC $([-\tau,0];\mathbb{R}^n)$, *whenever* $V(t+\theta,\varphi) < qV(t,\varphi(0))$, $\theta \in [-\tau,0]$;

(iv) $q > 1/\rho > e^{\widetilde{c}\varrho}$, $\widetilde{c}\varrho \geqslant \sup_{t\geqslant t_0}\int_t^{t+\varrho} c(s)ds$, *where* $\varrho = \sup_{k\in\mathbb{Z}_+}\{t_k - t_{k-1}\}$.

Then the trivial solution of system (3.33) *is globally exponentially stable for any time delay* $\tau \in (0,\infty)$ *and the convergence rate should not be greater than* $(1/p)((\ln q/\varrho) - \widetilde{c})$.

Remark 3.8. If $c(t) \equiv c > 0$, $\mu_k \equiv 0$, $k \in \mathbb{Z}_+$, then Theorems 3.1 and 3.2 in [25] follow from Corollaries 3.6 and 3.7, respectively.

4. Example

In this section, an example is given to show the effectiveness and advantages of our results.

Example 4.1. Consider the following IFDS (see [35, 36]):

$$\begin{aligned} \dot{x}(t) &= ax(t) + bx(t-\tau), \quad t \neq t_k,\ t > 0, \\ x(t_k) &= cx\left(t_k^-\right) + dx\left(t_k^- - \tau\right), \quad k \in \mathbb{Z}_+, \end{aligned} \tag{4.1}$$

where $x \in \mathbb{R}$, $\tau > 0$.

In the following, we will divide the system (4.1) into two classes to consider.

Case 1. $a \geqslant 0$ and $0 < |c| + |d| < 1$.

Property 1. The trivial solution of system (4.1) is globally exponentially stable with impulse time sequences that satisfy

$$\sup_{k \in \mathbb{Z}_+} \{t_k - t_{k-1}\} < -\frac{(|c| + |d|)\ln(|c| + |d|)}{a(|c| + |d|) + |b|}. \tag{4.2}$$

Proof . From equality (4.2), one can choose a small enough constant $h > 0$ such that

$$\begin{gathered} |c| + |d| - h > 0, \\ \sup_{k \in \mathbb{Z}_+} \{t_k - t_{k-1}\} < -\frac{(|c| + |d| - h)\ln(|c| + |d|)}{a(|c| + |d| - h) + |b|}. \end{gathered} \tag{4.3}$$

Let $V(t, x) = |x|$. By calculation, we have

$$D^+V(t, \varphi(0)) \leqslant a|\varphi(0)| + |b||\varphi(-\tau)| = aV(t, \varphi(0)) + |b|V(t, \varphi), \tag{4.4}$$

for all $t \neq t_k$, $k \in \mathbb{Z}_+$ and $\varphi \in \mathrm{PC}([-\tau, 0]; \mathbb{R})$. By taking $p = 1$, $c_1 = c_2 = 1$, $\rho_1 = |c|$, $\rho_2 = |d|$, $q = 1/(|c| + |d| - h)$, $\widetilde{c} = c(t) \equiv a + (|b|/(|c| + |d| - h))$, and $\mu_k \equiv 0$, $k \in \mathbb{Z}_+$ in Theorem 3.3, it is easy to obtain Property 1. □

Remark 4.2. In this case, the impulses are used to stabilize the unstable original continuous system. In [35], under assumption that $a, b, c, d > 0$, and $c + d < 1$, Zhang and Sun obtained that system (4.1) is uniformly stable if the impulses' instances satisfy

$$\sup_{k \in \mathbb{Z}_+} \{t_k - t_{k-1}\} < \frac{-2(c+d)^2 \ln(c+d)}{(2a+b)(c+d)^2 + b}; \tag{4.5}$$

Lin et al. [36] derived that system (4.1) is exponentially stable if

$$\sup_{k \in \mathbb{Z}_+} \{t_k - t_{k-1}\} < -\frac{1}{2}\frac{(c+d)\ln(c+d)}{a(c+d) + b}. \tag{4.6}$$

Obviously, under condition $a, b, c, d > 0$, and $c + d < 1$, we get

$$-\frac{(|c| + |d|)\ln(|c| + |d|)}{a(|c| + |d|) + |b|} = -\frac{(c + d)\ln(c + d)}{a(c + d) + b} > -\frac{1}{2}\frac{(c + d)\ln(c + d)}{a(c + d) + b}, \tag{4.7}$$

and one can also verify that

$$-\frac{(c + d)\ln(c + d)}{a(c + d) + b} > \frac{-2(c + d)^2\ln(c + d)}{(2a + b)(c + d)^2 + b}. \tag{4.8}$$

So our results are less conservative than those in [35, 36].

Case 2. $a < 0$ and $|c| + |d| \geqslant 1$.

Property 2. Suppose that system's parameters a, b, c, d and time delay τ satisfy

$$(|c| + |d|)^2 e^{-2a\tau} < -\frac{2a + |b|}{|b|}. \tag{4.9}$$

Then the trivial solution of system (4.1) is globally exponentially stable with impulse time sequences that satisfy

$$\inf_{k\in\mathbb{Z}_+}\{t_k - t_{k-1}\} > \frac{2a\tau - 2\ln(|c| + |d|)}{2a + |b| + |b|(|c| + |d|)^2 e^{-2a\tau}}. \tag{4.10}$$

Proof. From equalities (4.9) and (4.10), we can choose a small enough constant $h > 0$ such that

$$\begin{gathered}(|c| + |d|)^2 e^{-2a\tau} < (|c| + |d|)^2 e^{-2a\tau} + h < -\frac{2a + |b|}{|b|},\\ \inf_{k\in\mathbb{Z}_+}\{t_k - t_{k-1}\} > -\frac{\ln\left[(|c| + |d|)^2 e^{-2a\tau} + 2h\right]}{2a + |b| + |b|(|c| + |d|)^2 e^{-2a\tau} + h}.\end{gathered} \tag{4.11}$$

Set $q = (|c| + |d|)^2 e^{-2a\tau} + h$, then one can conclude that

$$2a + |b| + q|b| < 0, \qquad (|c| + |d|)^2 e^{-(2a+|b|+q|b|)\tau} < q < -\frac{2a + |b|}{|b|}. \tag{4.12}$$

Let $V(t, x) = (1/2)x^2$. By calculation, we have

$$D^+V(t,\varphi(0)) \leqslant \left(a + \frac{1}{2}|b|\right)\varphi^2(0) + \frac{1}{2}|b|\varphi^2(-\tau) = (2a + |b|)V(t,\varphi(0)) + |b|V(t,\varphi), \tag{4.13}$$

for all $t \neq t_k$, $k \in \mathbb{Z}_+$, and $\varphi \in PC([-\tau, 0]; \mathbb{R})$. By taking $p = 2$, $c_1 = c_2 = 2$, $\rho_1 = |c|(|c| + |d|)$, $\rho_2 = |d|(|c| + |d|)$, $\tilde{c} = c(t) \equiv -(2a + |b| + q|b|)$, and $\mu_k \equiv 0$, $k \in \mathbb{Z}_+$ in Theorem 3.3, we can obtain Property 2. □

Remark 4.3. In this case, the underlying continuous system is stable, the impulses are disturbances, which potentially destroy the stability of continuous system. So the existing results in [35, 36] are invalid for this case.

5. Conclusions

This paper has studied the exponential stability of IFDSs in which the state variables on the impulses are related to the time delay. By using the Razumikhin techniques and the Lyapunov functions, some criteria on the global exponential stability are established. The obtained results improve and complement some recent works. An example has been given to illustrate the effectiveness and the advantages of the results obtained.

Acknowledgments

This work was supported by the 211 Project of the Anhui University (32030018/33010205/KJTD002B), the Research Fund for Doctor Station of the Ministry of Education of China (20113401110001), the Key Natural Science Foundation (KJ2009A49), the Foundation of Anhui Education Bureau (KJ2012A019), and the National Natural Science Foundation of China (11126179).

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