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Research Article

Power Increasing Sequences and Their Some New Applications

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In the work of Bor (2008), we have proved a result dealing with $|\overline{N}, p_n, \theta_n|_k$ summability factors by using a quasi-β-power increasing sequence. In this paper, we prove that result under less and more weaker conditions. Some new results have also been obtained.

1. Introduction

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). We write $\mathcal{BV}_{\mathcal{O}} = \mathcal{BV} \cap \mathcal{C}_{\mathcal{O}}$, where $\mathcal{C}_{\mathcal{O}} = \{x = (x_k) \in \Omega : \lim_k |x_k| = 0\}$, $\mathcal{BV} = \{x = (x_k) \in \Omega : \sum_k |x_k - x_{k+1}| < \infty\}$ and Ω being the space of all real or complex-valued sequences. A positive sequence $X = (X_n)$ is said to be a quasi-β-power increasing sequence if there exists a constant $K = K(\beta, X) \geq 1$ such that $Kn^\beta X_n \geq m^\beta X_m$ holds for all $n \geq m \geq 1$. It should be noted that every almost increasing sequence is a quasi-β-power increasing sequence for any nonnegative β, but the converse is not true for $\beta > 0$. Moreover, for any positive β there exists a quasi-β-power increasing sequence tending to infinity, but it is not almost increasing (see [2]). Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \longrightarrow \infty \quad \text{as } n \longrightarrow \infty, \quad (P_{-i} = p_{-i} = 0, \ i \geq 1). \tag{1.1}$$

Let (θ_n) be any sequence of positive real constants. The series $\sum a_n$ is said to be summable $|\overline{N}, p_n|_k$, $k \geq 1$, if (see [3])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |V_n - V_{n-1}|^k < \infty, \tag{1.2}$$

and it is said to be summable $|\overline{N}, p_n, \theta_n|_k,\ k \geq 1$, if (see [4])

$$\sum_{n=1}^{\infty} \theta_n^{k-1} |V_n - V_{n-1}|^k < \infty, \tag{1.3}$$

where

$$V_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v. \tag{1.4}$$

If we take $\theta_n = P_n/p_n$, then $|\overline{N}, p_n, \theta_n|_k$ summability reduces to $|\overline{N}, p_n|_k$ summability. Also if we take $\theta_n = n$ and $p_n = 1$ for all values of n, then we get $|C, 1|_k$ summability (see [5]). Furthermore, if we take $\theta_n = n$, then $|\overline{N}, p_n, \theta_n|_k$ summability reduces to $|R, p_n|_k$ summability (see [6]).

2. Known Result

In [7], we have proved the following theorem dealing with $|\overline{N}, p_n, \theta_n|_k$ summability factors of infinite series.

Theorem 2.1. *Let $(\lambda_n) \in \mathcal{BV}_{\mathcal{O}}$, (X_n) be a quasi-β-power increasing sequence for some β $(0 < \beta < 1)$, and let $(\theta_n p_n/P_n)$ be a nonincreasing sequence. Suppose also there exists sequences (λ_n) and (p_n) such that*

$$\begin{gathered} |\lambda_m| X_m = O(1) \quad \text{as } m \longrightarrow \infty, \\ \sum_{n=1}^{m} n X_n \left| \Delta^2 \lambda_n \right| = O(1), \\ \sum_{n=1}^{m} \frac{P_n}{n} = O(P_m) \quad \text{as } m \longrightarrow \infty. \end{gathered} \tag{2.1}$$

If

$$\sum_{n=1}^{m} \frac{|t_n|^k}{n} = O(X_m) \quad \text{as } m \longrightarrow \infty, \tag{2.2}$$

$$\sum_{n=1}^{m} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k |t_n|^k = O(X_m) \quad \text{as } m \longrightarrow \infty, \tag{2.3}$$

are satisfied, then the series $\sum a_n \lambda_n$ is summable $|\overline{N}, p_n, \theta_n|_k,\ k \geq 1$, where (t_n) is the nth $(C, 1)$ mean of the sequence $(n a_n)$.

Remark 2.2. It should be noticed that, if we take (X_n) as an almost increasing sequence and $\theta_n = P_n/p_n$, then we obtain a theorem of Mazhar (see [8]), in this case the condition "$(\lambda_n) \in \mathcal{BV}_{\mathcal{O}}$" is not needed.

3. The Main Result

The aim of this paper is to prove Theorem 2.1 under less and more weaker conditions. Now, we prove the following theorem.

Theorem 3.1. *Let (X_n) be a quasi-β-power increasing sequence for some β $(0 < \beta < 1)$, and let $(\theta_n p_n/P_n)$ be a nonincreasing sequence. Suppose also there exists sequences (λ_n) and (p_n) such that conditions* (2.1) *of Theorem 2.1 are satisfied. If*

$$\sum_{n=1}^{m} \frac{|t_n|^k}{nX_n^{k-1}} = O(X_m) \quad \text{as } m \longrightarrow \infty, \tag{3.1}$$

$$\sum_{n=1}^{m} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{|t_n|^k}{X_n^{k-1}} = O(X_m) \quad \text{as } m \longrightarrow \infty \tag{3.2}$$

are satisfied, then the series $\sum a_n \lambda_n$ is summable $|\overline{N}, p_n, \theta_n|_k$, $k \geq 1$.

Remark 3.2. It should be noted that conditions (3.1) and (3.2) are the same as conditions (2.2) and (2.3), respectively, when $k = 1$. When $k > 1$, conditions (3.1) and (3.2) are weaker than conditions (2.2) and (2.3), respectively. But the converses are not true. In fact, if (2.2) is satisfied, then we get that

$$\sum_{n=1}^{m} \frac{|t_n|^k}{nX_n^{k-1}} = O\left(\frac{1}{X_1^{k-1}}\right) \sum_{n=1}^{m} \frac{|t_n|^k}{n} = O(X_m). \tag{3.3}$$

If (3.1) is satisfied, then for $k > 1$, we obtain that

$$\sum_{n=1}^{m} \frac{|t_n|^k}{n} = \sum_{n=1}^{m} X_n^{k-1} \frac{|t_n|^k}{nX_n^{k-1}} = O\left(X_m^{k-1}\right) \sum_{n=1}^{m} \frac{|t_n|^k}{nX_n^{k-1}} = O\left(X_m^k\right) \neq O(X_m). \tag{3.4}$$

The similar argument is also valid for the conditions (2.3) and (3.2). Also it should be noted that condition "$(\lambda_n) \in \mathcal{BV}_{\mathcal{O}}$" has been removed.

We need following lemma for the proof of our theorem.

Lemma 3.3 (see [9]). *Under the conditions on the sequences (X_n) and (λ_n) as expressed in the statement of the theorem, one has the following:*

$$\begin{gathered} nX_n|\Delta\lambda_n| = O(1), \\ \sum_{n=1}^{\infty} X_n|\Delta\lambda_n| < \infty. \end{gathered} \tag{3.5}$$

4. Proof of the Theorem

Let (T_n) denote the $(\overline{N}, p_n)$ mean of the series $\sum a_n \lambda_n$. Then, for $n \geq 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \lambda_v = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1}\lambda_v}{v} v a_v. \tag{4.1}$$

By Abel's transformation, we have

$$
\begin{aligned}
T_n - T_{n-1} &= \frac{n+1}{nP_n} p_n t_n \lambda_n - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v t_v \lambda_v \frac{v+1}{v} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \Delta\lambda_v t_v \frac{v+1}{v} \\
&\quad + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v t_v \lambda_{v+1} \frac{1}{v} = T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}.
\end{aligned}
\tag{4.2}
$$

To complete the proof of the theorem, by Minkowski's inequality, it is enough to show that

$$
\sum_{n=1}^{\infty} \theta_n^{k-1} |T_{n,r}|^k < \infty, \quad \text{for } r = 1, 2, 3, 4. \tag{4.3}
$$

Firstly, we have that

$$
\begin{aligned}
\sum_{n=1}^{m} \theta_n^{k-1} |T_{n,1}|^k &= \sum_{n=1}^{m} \theta_n^{k-1} |\lambda_n|^{k-1} |\lambda_n| \left(\frac{p_n}{P_n}\right)^k |t_n|^k \\
&= O(1) \sum_{n=1}^{m} |\lambda_n| \theta_n^{k-1} \left(\frac{1}{X_v}\right)^{k-1} \left(\frac{p_n}{P_n}\right)^k |t_n|^k \\
&= O(1) \sum_{n=1}^{m-1} \Delta|\lambda_n| \sum_{v=1}^{n} \theta_v^{k-1} \left(\frac{p_v}{P_v}\right)^k \frac{|t_v|^k}{X_v^{k-1}} + O(1) |\lambda_m| \sum_{n=1}^{m} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{|t_n|^k}{X_n^{k-1}} \\
&= O(1) \sum_{n=1}^{m-1} |\Delta\lambda_n| X_n + O(1) |\lambda_m| X_m \\
&= O(1) \quad \text{as } m \longrightarrow \infty,
\end{aligned}
\tag{4.4}
$$

by virtue of the hypotheses of the theorem and lemma. Now, when $k > 1$ applying Hölder's inequality with indices k and k', where $(1/k) + (1/k') = 1$, as in $T_{n,1}$, we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1} \theta_n^{k-1} |T_{n,2}|^k &= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}} \left\{ \sum_{v=1}^{n-1} p_v |\lambda_v|^k |t_v|^k \right\} \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} p_v |\lambda_v|^{k-1} |\lambda_v| |t_v|^k \sum_{n=v+1}^{m+1} \left(\frac{\theta_n p_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} p_v |t_v|^k |\lambda_v| \left(\frac{1}{X_v}\right)^{k-1} \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} \theta_v^{k-1} \left(\frac{p_v}{P_v}\right)^k |\lambda_v| \left(\frac{1}{X_v}\right)^{k-1} |t_v|^k \\
&= O(1) \sum_{v=1}^{m} |\lambda_v| \theta_v^{k-1} \left(\frac{p_v}{P_v}\right)^k \frac{|t_v|^k}{X_v^{k-1}} = O(1) \quad \text{as } m \longrightarrow \infty.
\end{aligned}
\tag{4.5}
$$

Again we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\theta_n^{k-1}|T_{n,3}|^k &= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^k\frac{1}{P_{n-1}}\left\{\sum_{v=1}^{n-1}\frac{P_v}{v}|\Delta\lambda_v|^k v^k|t_v|^k\right\}\times\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}\frac{P_v}{v}\right\}^{k-1}\\
&= O(1)\sum_{v=1}^{m}\frac{P_v}{v}|t_v|^k v^k|\Delta\lambda_v|^{k-1}|\Delta\lambda_v|\sum_{n=v+1}^{m+1}\left(\frac{\theta_n p_n}{P_n}\right)^{k-1}\frac{p_n}{P_nP_{n-1}}\\
&= O(1)\sum_{v=1}^{m}\left(\frac{\theta_v p_v}{P_v}\right)^{k-1}v^{k-1}\left(\frac{1}{vX_v}\right)^{k-1}|\Delta\lambda_v||t_v|^k\\
&= O(1)\left(\frac{\theta_1 p_1}{P_1}\right)^{k-1}\sum_{v=1}^{m}v|\Delta\lambda_v|\frac{|t_v|^k}{vX_v^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}\Delta(v|\Delta\lambda_v|)\sum_{i=1}^{v}\frac{|t_i|^k}{iX_i^{k-1}}+O(1)m|\Delta\lambda_m|\sum_{v=1}^{m}\frac{|t_v|^k}{vX_v^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}|\Delta(v|\Delta\lambda_v|)|X_v+O(1)m|\Delta\lambda_m|X_m\\
&= O(1)\sum_{v=1}^{m-1}\left|(v+1)\left|\Delta^2\lambda_v\right|-|\Delta\lambda_v|\right|X_v+O(1)m|\Delta\lambda_m\left|X_m\right.\\
&= O(1)\sum_{v=1}^{m-1}v\left|\Delta^2\lambda_v\right|X_v+O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v\\
&\quad+O(1)m|\Delta\lambda_m|X_m=O(1)\quad\text{as } m\longrightarrow\infty,
\end{aligned}\tag{4.6}$$

by virtue of the hypotheses of the theorem and lemma. Finally, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\theta_n^{k-1}|T_{n,4}|^k &\le \sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^k\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}P_v|\lambda_{v+1}|^k|t_v|^k\frac{1}{v}\times\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}\frac{P_v}{v}\right\}^{k-1}\\
&= O(1)\sum_{v=1}^{m}P_v|\lambda_{v+1}|^{k-1}|\lambda_{v+1}||t_v|^k\frac{1}{v}\sum_{n=v+1}^{m+1}\left(\frac{\theta_n p_n}{P_n}\right)^{k-1}\frac{p_n}{P_nP_{n-1}}\\
&= O(1)\sum_{v=1}^{m}P_v\left(\frac{1}{X_v}\right)^{k-1}|\lambda_{v+1}||t_v|^k\frac{1}{v}\sum_{n=v+1}^{m+1}\left(\frac{\theta_n p_n}{P_n}\right)^{k-1}\frac{p_n}{P_nP_{n-1}}\\
&= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|\left(\frac{\theta_v p_v}{P_v}\right)^{k-1}\frac{|t_v|^k}{vX_v^{k-1}}\\
&= O(1)\left(\frac{\theta_1 p_1}{P_1}\right)^{k-1}\sum_{v=1}^{m}|\lambda_{v+1}|\frac{|t_v|^k}{vX_v^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_{v+1}|\sum_{r=1}^{v}\frac{|t_r|^k}{rX_r^{k-1}}+O(1)|\lambda_{m+1}|\sum_{v=1}^{m}\frac{|t_v|^k}{vX_v^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_{v+1}|X_{v+1}+O(1)|\lambda_{m+1}|X_{m+1}\\
&= O(1)\quad\text{as } m\longrightarrow\infty,
\end{aligned}\tag{4.7}$$

by virtue of the hypotheses of the theorem and lemma. This completes the proof of the theorem. If we take $p_n = 1$ for all values of n and $\theta_n = n$, then we get a result dealing with $|C,1|_k$ summability factors. Also, if we take $p_n = 1$ for all values of n, then we have a new result for $|C,1,\theta_n|_k$ summability. Finally, if we take $\theta_n = n$, then we have another new result for $|R,p_n|_k$ summability factors.

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