Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 475956, 24 pages
doi:10.1155/2012/475956

Research Article

Periodic Solutions of Second-Order Differential Inclusions Systems with $p(t)$-Laplacian

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Received 19 June 2012; Accepted 26 September 2012

Academic Editor: Toka Diagana

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The existence of periodic solutions for nonautonomous second-order differential inclusion systems with $p(t)$-Laplacian is considered. We get some existence results of periodic solutions for system, $(d/dt)(|\dot{u}(t)|^{p(t)-2}\dot{u}(t)) \in \partial F(t,u(t))$ a.e. $t \in [0,T]$, $u(0)-u(T)=\dot{u}(0)-\dot{u}(T)=0$, by using nonsmooth critical point theory. Our results generalize and improve some theorems in the literature.

1. Introduction

Consider the second-order system with $p(t)$-Laplacian

$$\begin{aligned} &\frac{d}{dt}\left(|\dot{u}(t)|^{p(t)-2}\dot{u}(t)\right) \in \partial F(t,u(t)) \quad \text{a.e. } t \in [0,T], \\ &u(0)-u(T)=\dot{u}(0)-\dot{u}(T)=0, \end{aligned} \tag{1.1}$$

where $T>0$, ∂ denotes the Clarke subdifferential, and $p(t) \in C([0,T],\mathbb{R}^+)$ satisfies the following assumption:

(A) $p(0)=p(T)$ and $p^- := \min_{0\le t\le T} p(t) > 1$, where $q^+ > 1$ which satisfies $1/p^- + 1/q^+ = 1$.

Moreover, we suppose that $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumption:

(A′) $F(t,x)$ is measurable in t for every $x \in \mathbb{R}^N$ and locally Lipschitz in x for a.e. $t \in [0,T]$, $F(t,0) \in L^1(0,T)$ and there exist positive constants C, C_0, and $\alpha \in [0,\infty)$ such that

$$\zeta \in \partial F(t,x) \Longrightarrow |\zeta| \le C|x|^{\alpha} + C_0, \tag{1.2}$$

for a.e. $t \in [0,T]$ and all $x \in \mathbb{R}^N$.

If $F \in C^1$ and $p(t) \equiv p > 1$, system (1.1) reduces to the ordinary p-Laplacian system:

$$\begin{gathered} \frac{d}{dt}\left(|\dot{u}(t)|^{p-2}\dot{u}(t)\right) = \nabla F(t,u(t)) \quad \text{a.e. } t \in [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{gathered} \tag{1.3}$$

Especially, when $p = 2$, then system (1.3) reduces to

$$\begin{gathered} \ddot{u}(t) = \nabla F(t,u(t)) \quad \text{a.e. } t \in [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{gathered} \tag{1.4}$$

The corresponding functional φ on H_T^1 given by

$$\varphi(u) := \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt + \int_0^T F(t,u(t))dt \tag{1.5}$$

is continuously differentiable and weakly lower semicontinuous on H_T^1 (see [1]), where

$$H_T^1 := \left\{u : [0,T] \to \mathbb{R}^{\mathbb{N}}, u \text{ is absolutely continuous, } u(0) = u(T), \dot{u} \in L^2\left([0,T];\mathbb{R}^{\mathbb{N}}\right)\right\} \tag{1.6}$$

is a Hilbert space with a norm defined by

$$\|u\|_{H_T^1} := \left(\int_0^T |u(t)|^2 dt + \int_0^T |\dot{u}(t)|^2 dt\right)^{1/2}, \tag{1.7}$$

for $u \in H_T^1$.

Since Mawhin and Willem studied the periodic solutions of Hamilton system and obtained a series of results (see [2]). Considerable attention has been paid to the existence of periodic and subharmonic solutions for system (1.3) and (1.4) by the use of critical point theory in variational methods. Many solvability conditions are given, such as Ambrosetti-Rabinowitz conditions, coercivity condition, the convexity condition, the boundedness condition, the subadditive condition, the sublinear condition, and the periodicity condition, see [3–7] and the references therein.

The classical critical point theory was developed in the sixties and seventies for C^1 functionals. The needs of specific applications (such as nonsmooth mechanics and nonsmooth gradient systems) and the impressive progress in nonsmooth analysis and multivalued analysis led to extensions of the critical point theory to nondifferentiable functions, in particular locally Lipschitz functions. The nonsmooth critical point theory for locally Lipschitz functions started with the work of Chang (see [8]). He was able to construct a substitute for the pseudogradient vector field of the smooth theory and use it to obtain nonsmooth versions of the Mountain Pass Theorem of Ambrosetti and Rabinowitz (see [9]) and of the Saddle Point Theorem of Rabinowitz (see [10]). Chang used his theory to study semilinear elliptic boundary value problems with a discontinuous nonlinearity. Later, in 2000, Kourogenis and Papageorgiou (see [11]) obtained some nonsmooth critical point theories and applied these to nonlinear elliptic equations at resonance, involving the p-Laplacian with discontinuous nonlinearities. Subsequently, many authors also studied the nonsmooth critical point theory (see [2, 11–17]), then the nonsmooth critical point theory is also widely used to deal with the nonlinear boundary value problems (see [11, 14, 15, 17–26]). A good survey for nonsmooth critical point theory and nonlinear boundary value problems is the book of Gasinski and Papageorgiou [22].

The operator $(d/dt)(|\dot{u}(t)|^{p(t)-2}\dot{u}(t))$ is said to be $p(t)$-Laplacian and becomes p-Laplacian when $p(t) \equiv p$ (a constant). The $p(t)$-Laplacian possesses more complicated nonlinearity than p-Laplacian, for example, it is inhomogeneous. The study of various mathematical problems with variable exponent growth conditions has received considerable attention in recent years. These problems are interesting in applications and raise many mathematical problems. One of the most studied models leading to problem of this type is the model of motion of electrorheological fluids, which are characterized by their ability to drastically change the mechanical properties under the influence of an exterior electromagnetic field. Another field of application of equations with variable exponent growth conditions is image processing (see [12, 27]). The variable nonlinearity is used to outline the borders of the true image and to eliminate possible noise. We refer the reader to [12, 28–33] for an overview on this subject.

In 2003, X. L. Fan and X. Fan (see [34]) studied the ordinary $p(t)$-Laplacian system and introduced a generalized Orlicz-Sobolev space $W_T^{1,p(t)}$, which is different from the usual space $W_T^{1,p}$, then Wang and Yuan (see [35]) obtained the existence and mulplicity of periodic solutions for ordinary $p(t)$-Laplacian system (1.1) with a smooth potential F under the generalized Ambrosetti-Rabinowitz conditions. In recent years, there are some papers discussing existence and multiplicity of periodic solutions and subharmonic solutions for problem (1.3) and (1.4) when the potential F is just locally Lipschitz in the second variable x not continuously differentiable. Some results were obtained based on various hypotheses on the potential F. Here we only mention [7, 20, 21, 25, 26], and it should be noted that the abstract result of Clarke (see [1, Theorem 2.7.5]) plays an important role in the establishment of corresponding variational structure. However, to the best of our knowledge, few papers investigated the existence of solutions for problem (1.1), because the main difficulty is the abstract result of Clarke cannot be applied to system (1.1) directly. So we have to find a new approach to solve this problem, and our main idea of the new approach comes from the inspiration of the Theorem 2.7.3 and Theorem 2.7.5 in [1].

The main purpose of this paper is to establish the corresponding variational structure for system (1.1), and we get some existence results of periodic solutions for system (1.1) by using nonsmooth critical point theory. Our results are extensions of the results presented

in [4, 26], and our results are new even in the case $F \in C^1$ for system (1.1). The paper is divided into four sections. Basic definitions and preliminary results are collected in Section 2. We give the main results and proofs in Section 3. In Section 4, three examples are presented to illustrate our results.

In this paper, we denote by $p^+ := \max_{0\le t\le T} p(t) > 1$ throughout this paper, and we use $\langle\cdot,\cdot\rangle$ and $|\cdot|$ to denote the usual inner product and norm in $\mathbb{R}^N$, respectively.

2. Basic Definitions and Preliminary Results

In this section, we recall some known results in nonsmooth critical point theory, and the properties of space $W_T^{1,p(t)}$ are listed for the convenience of readers.

Definition 2.1 (see [35]). Let $p(t)$ satisfy the condition (A), define

$$L^{p(t)}\left([0,T],\mathbb{R}^N\right) = \left\{u \in L^1\left([0,T],\mathbb{R}^N\right); \int_0^T |u|^{p(t)}dt < \infty\right\}, \tag{2.1}$$

with the norm

$$|u|_{p(t)} := \inf\left\{\lambda > 0; \int_0^T \left|\frac{u}{\lambda}\right|^{p(t)} dt \le 1\right\}. \tag{2.2}$$

For $u \in L^1_{\mathrm{loc}}([0,T],\mathbb{R}^N)$, let u' denote the weak derivative of u, if $u' \in L^1_{\mathrm{loc}}([0,T],\mathbb{R}^N)$ and satisfies

$$\int_0^T u'\phi\, dt = -\int_0^T u\phi' dt, \quad \forall \phi \in C_0^\infty\left([0,T],\mathbb{R}^N\right). \tag{2.3}$$

Define

$$W^{1,p(t)}\left([0,T],\mathbb{R}^N\right) = \left\{u \in L^{p(t)}\left([0,T],\mathbb{R}^N\right);\ u' \in L^{p(t)}\left([0,T],\mathbb{R}^N\right)\right\}, \tag{2.4}$$

with the norm $\|u\|_{W^{1,p(t)}} := |u|_{p(t)} + |u'|_{p(t)}$.

Remark 2.2. If $p(t) = p$, where $p \in [1,\infty)$ is a constant, by the definition of $|u|_{p(t)}$, it is easy to get $|u|_p = (\int_0^T |u(t)|^p dt)^{1/p}$, which is the same with the usual norm in space L^p.

The space $L^{p(t)}$ is a generalized Lebesgue space, and the space $W^{1,p(t)}$ is a generalized Sobolev space. Because most of the following Lemmas have appeared in [1, 7, 24, 34], we omitted their proofs.

Lemma 2.3 (see [34]). *$L^{p(t)}$ and $W^{1,p(t)}$ are both Banach spaces with the norms defined above, when $p^- > 1$, they are reflexive.*

Lemma 2.4 (see [34]). *The space $L^{p(t)}$ is a separable, uniform convex Banach space, its conjugate space is $L^{q(t)}$, for any $u \in L^{p(t)}$ and $v \in L^{q(t)}$, we have*

$$\left| \int_0^T uv \, dt \right| \leq 2|u|_{p(t)}|v|_{q(t)}, \tag{2.5}$$

where $1/p(t) + 1/q(t) = 1$.

Lemma 2.5 (see [35]). *If we denote $\rho(u) = \int_0^T |u|^{p(t)} dt$, for all $u \in L^{p(t)}$, then*

(i) $|u|_{p(t)} < 1$ *(=1;>1)*$\Leftrightarrow \rho(u) < 1$ *(=1;>1)*

(ii) $|u|_{p(t)} > 1 \Rightarrow |u|^{p^-}_{p(t)} \leq \rho(u) \leq |u|^{p^+}_{p(t)}$, $|u|_{p(t)} < 1 \Rightarrow |u|^{p^+}_{p(t)} \leq \rho(u) \leq |u|^{p^-}_{p(t)}$

(iii) $|u|_{p(t)} \to 0 \Leftrightarrow \rho(u) \to 0$; $|u|_{p(t)} \to \infty \Leftrightarrow \rho(u) \to \infty$

(iv) *For $u \in L^{p(t)}$ and $u \neq 0$, $|u|_{p(t)} = \lambda \Leftrightarrow \rho(u/\lambda) = 1$.*

Definition 2.6 (see [2]).

$$C_T^\infty = C_T^\infty\left(\mathbb{R}, \mathbb{R}^N\right) := \left\{ u \in C^\infty\left(\mathbb{R}, \mathbb{R}^N\right);\ u \text{ is } T\text{-periodic} \right\}, \tag{2.6}$$

with the norm $\|u\|_\infty := \max_{t\in[0,T]}|u(t)|$.

For a constant $p \in [1, \infty)$, using another conception of weak derivative which is called T-weak derivative, Mawhin and Willem gave the definition of the space $W_T^{1,p}$ by the following way.

Definition 2.7 (see [2]). Let $u \in L^1([0,T], \mathbb{R}^N)$ and $v \in L^1([0,T], \mathbb{R}^N)$, if

$$\int_0^T v\phi \, dt = -\int_0^T u\phi' dt, \quad \forall \phi \in C_T^\infty, \tag{2.7}$$

then v is called a T-weak derivative of u and is denoted by $\dot{u}$.

Definition 2.8 (see [2]). Define

$$W_T^{1,p}\left([0,T], \mathbb{R}^N\right) = \left\{ u \in L^p\left([0,T], \mathbb{R}^N\right);\ \dot{u} \in L^p\left([0,T], \mathbb{R}^N\right) \right\}, \tag{2.8}$$

with the norm $\|u\|_{W_T^{1,p}} = (|u|_p^p + |\dot{u}|_p^p)^{1/p}$.

Definition 2.9 (see [34]). Define

$$W_T^{1,p(t)}\left([0,T], \mathbb{R}^N\right) = \left\{ u \in L^{p(t)}\left([0,T], \mathbb{R}^N\right);\ \dot{u} \in L^{p(t)}\left([0,T], \mathbb{R}^N\right) \right\}, \tag{2.9}$$

and let $H_T^{1,p(t)}([0,T], \mathbb{R}^N)$ be the closure of C_T^∞ in $W^{1,p(t)}([0,T], \mathbb{R}^N)$.

Remark 2.10. From Definition 2.8, if $u \in W_T^{1,p(t)}([0,T],\mathbb{R}^N)$, it is easy to conclude that $u \in W_T^{1,p^-}([0,T],\mathbb{R}^N)$.

Lemma 2.11 (see [34]).

(i) $C_T^\infty([0,T],\mathbb{R}^N)$ *is dense in* $W_T^{1,p(t)}([0,T],\mathbb{R}^N)$,

(ii) $W_T^{1,p(t)}([0,T],\mathbb{R}^N) = H_T^{1,p(t)}([0,T],\mathbb{R}^N) := \{u \in W^{1,p(t)}([0,T],\mathbb{R}^N); u(0) = u(T)\}$,

(iii) *If* $u \in H_T^{1,1}$, *then the derivative* u' *is also the T-weak derivative* $\dot{u}$, *that is,* $u' = \dot{u}$.

Remark 2.12. In the following paper, we use $\|u\|$ instead of $\|u\|_{W_T^{1,p(t)}}$ for convenience without clear indications.

Lemma 2.13 (see [24]). *Assume that* $u \in W_T^{1,1}$, *then*

(i) $\int_0^T \dot{u}\,dt = 0$,

(ii) u *has its continuous representation, which is still denoted by* $u(t) = \int_0^t \dot{u}(s)ds + u(0)$, $u(0) = u(T)$,

(iii) $\dot{u}$ *is the classical derivative of* u, *if* $\dot{u} \in C([0,T],\mathbb{R}^N)$.

Since every closed linear subspace of a reflexive Banach space is also reflexive, we have

Lemma 2.14 (see [34]). $H_T^{1,p(t)}([0,T],\mathbb{R}^N)$ *is a reflexive Banach space if* $p^- > 1$.

Obviously, there are continuous embeddings $L^{p(t)} \hookrightarrow L^{p^-}$, $W^{1,p(t)} \hookrightarrow W^{1,p^-}$, and $H_T^{1,p(t)} \hookrightarrow H_T^{1,p^-}$. By the classical Sobolev embedding theorem we obtain.

Lemma 2.15 (see [34]). *There is a continuous embedding*

$$W^{1,p(t)}\left(or\ H_T^{1,p(t)}\right) \hookrightarrow C\left([0,T],\mathbb{R}^N\right), \tag{2.10}$$

when $p^- > 1$, *the embedding is compact.*

In order to establish the variational structure for system (1.1), it is necessary to construct some appropriate function spaces. The Cartesian product space W is defined by

$$W = L^{p(t)}\left([0,T],\mathbb{R}^N\right) \times L^{p(t)}\left([0,T],\mathbb{R}^N\right) \tag{2.11}$$

and is also a reflexive and separable Banach space with respect to the norm

$$\|v\|_W = |v_1|_{p(t)} + |v_2|_{p(t)}, \tag{2.12}$$

where $v = (v_1, v_2) \in W$.

Lemma 2.16. *Define the operator* $A: W_T^{1,p(t)}([0,T],\mathbb{R}^N) \to W$ *as follows:*

$$Au = (u, \dot{u}), \quad \forall u \in W_T^{1,p(t)}, \tag{2.13}$$

then $W_{p(t)} := \{(u, \dot{u}) : \forall u \in W_T^{1,p(t)}\}$ *is also a reflexive and separable Banach space with respect to the norm defined in* (2.12).

Proof. Let $(u_n, \dot{u}_n)$ be a Cauchy sequence in $W_{p(t)}$, then there exists (u, v) in W such that $(u_n, \dot{u}_n)$ converge to (u, v) in W. We have

$$\int_0^T \dot{u}_n \varphi \, dt = -\int_0^T u_n \varphi' dt, \quad \forall \varphi \in C_T^\infty, \tag{2.14}$$

by Definition 2.7, then by Lemma 2.4, we conclude

$$\int_0^T v\varphi \, dt = -\int_0^T u\varphi' dt, \quad \forall \varphi \in C_T^\infty, \tag{2.15}$$

as $n \to \infty$ in (2.14). In view of (2.15), v is the T-weak derivative of u, that is, (u, v) is also in $W_{p(t)}$, so $W_{p(t)}$ is a complete subspace of W, which implies $W_{p(t)}$ is also a reflexive and separable Banach space. □

Remark 2.17. We use $\|(u, \dot{u})\|_{W_{p(t)}}$ to denote the norm in $W_{p(t)}$ defined by (2.12).

Definition 2.18. Let $L^\infty([0,T], \mathbb{R}^N \times \mathbb{R}^N)$ denote the space of essentially bounded measurable functions from $[0,T]$ into $\mathbb{R}^N \times \mathbb{R}^N$ under the norm

$$\|(u, v)\|_{L^\infty} := \operatorname{ess\,sup}\{|u(t)| + |v(t)| : t \in [0,T]\}, \tag{2.16}$$

it is obvious that $L^\infty([0,T], \mathbb{R}^N \times \mathbb{R}^N)$ is a Banach space under the norm defined above.

Remark 2.19. We use W_∞ and $\|(u, v)\|_{W_\infty}$ to denote $L^\infty([0,T], \mathbb{R}^N \times \mathbb{R}^N)$ and $\|(u, v)\|_{L^\infty}$, respectively.

Lemma 2.20. $W_{p(t)} \cap W_\infty$ *is a closed subspace of* W_∞.

Proof. Let $(u_n, \dot{u}_n)$ be a Cauchy sequence in $W_{p(t)} \cap W_\infty$ with respect to the norm defined in (2.16). Then there exists (u, v) in W_∞ such that $(u_n, \dot{u}_n)$ converge to (u, v) in W_∞. By Definition 2.7, we have

$$\int_0^T \dot{u}_n \varphi \, dt = -\int_0^T u_n \varphi' dt, \quad \forall \varphi \in C_T^\infty, \tag{2.17}$$

we conclude that

$$\int_0^T v\varphi \, dt = -\int_0^T u\varphi' dt, \quad \forall \varphi \in C_T^\infty, \tag{2.18}$$

by Lemma 2.4 as $n \to \infty$ in (2.17). In view of (2.18), v is the T-weak derivative of u, that is, (u, v) is also in $W_{p(t)} \cap W_\infty$, so $W_{p(t)} \cap W_\infty$ is a complete subspace of W_∞, which implies $W_{p(t)} \cap W_\infty$ is a closed subspace of W_∞. □

Lemma 2.21. *Suppose f is a bounded linear functional on $W_{p(t)}$, if restricted to the space $W_{p(t)} \cap W_\infty$, denoted by f', that is,*

$$f'(u,\dot{u}) = f(u,\dot{u}), \quad \forall (u,\dot{u}) \in W_{p(t)} \cap W_\infty, \tag{2.19}$$

then f' is a bounded linear functional on $W_{p(t)} \cap W_\infty$.

Proof. It is obvious that f' is a linear functional on $W_{p(t)} \cap W_\infty$, so we only to show the f' is continuous on $W_{p(t)} \cap W_\infty$.

Let $|u|_{p(t)} = \lambda$ and $|\dot{u}|_{p(t)} = \mu$, that is,

$$\int_0^T \left| \frac{u(t)}{\lambda} \right|^{p(t)} dt = 1, \qquad \int_0^T \left| \frac{\dot{u}(t)}{\mu} \right|^{p(t)} dt = 1, \tag{2.20}$$

where $(u,\dot{u}) \in W_{p(t)} \cap W_\infty$ and $(u,\dot{u}) \neq (0,0)$, then

$$\int_0^T \left| \frac{u(t)}{C_1 \|(u,\dot{u})\|_{W_\infty}} \right|^{p(t)} dt \leq 1, \qquad \int_0^T \left| \frac{\dot{u}(t)}{C_1 \|(u,\dot{u})\|_{W_\infty}} \right|^{p(t)} dt \leq 1, \tag{2.21}$$

by Definition 2.1 and (2.16), where $C_1 := |1|_{p(t)}$. Then we conclude

$$|u|_{p(t)} \leq C_1 \|(u,\dot{u})\|_{W_\infty}, \qquad |\dot{u}|_{p(t)} \leq C_1 \|(u,\dot{u})\|_{W_\infty}, \tag{2.22}$$

by (2.21) and Definition 2.1.

Furthermore, the norm of f in $W_{p(t)}$ is

$$\|f\| = \sup_{W_{p(t)} \setminus 0} \frac{|f(u,\dot{u})|}{\|(u,\dot{u})\|_{W_{p(t)}}} = \sup_{W_{p(t)} \setminus 0} \frac{|f(u,\dot{u})|}{\left(|u|_{p(t)} + |\dot{u}|_{p(t)} \right)}, \tag{2.23}$$

therefore, combining (2.22) and (2.23), we get the bound

$$\|f'\| \leq 2C_1 \|f\|, \tag{2.24}$$

and this completes the proof. □

Lemma 2.22. *The space $W_T^{1,p(t)} = \widetilde{W}_T^{1,p(t)} \oplus \mathbb{R}^N$, where*

$$\widetilde{W}_T^{1,p(t)} := \left\{ u \in W_T^{1,p(t)}; \int_0^T u(t)dt = 0 \right\}, \tag{2.25}$$

there exists $C_2 > 0$, *if* $u \in \widetilde{W}_T^{1,p(t)}$, *such that*

$$\|u\|_\infty \le 2C_2\left(\int_0^T |\dot{u}(t)|^{p(t)}dt\right)^{1/p^-} + 2C_2T^{1/p^-}. \tag{2.26}$$

Proof. Let $E = \{t \in [0,T] \mid |\dot{u}(t)| \ge 1\}$, from Remark 2.10, $u \in W_T^{1,p^-}$, from the inequality in classical Sobolev space, there exists a positive constant $C_2 > 0$, such that

$$\begin{aligned}
\|u\|_\infty &\le C_2\left(\int_0^T |\dot{u}(t)|^{p^-}dt\right)^{1/p^-} \\
&= C_2\left(\int_E |\dot{u}(t)|^{p^-}dt + \int_{[0,T]\setminus E} |\dot{u}(t)|^{p^-}dt\right)^{1/p^-} \\
&\le C_2\left(\int_E |\dot{u}(t)|^{p(t)}dt + \operatorname{meas}[0,T]\setminus E\right)^{1/p^-} \\
&\le C_2\left(\int_0^T |\dot{u}(t)|^{p(t)}dt + T\right)^{1/p^-} \\
&\le 2C_2\left(\int_0^T |\dot{u}(t)|^{p(t)}dt\right)^{1/p^-} + 2C_2T^{1/p^-},
\end{aligned} \tag{2.27}$$

and this completes the proof. □

Lemma 2.23 (see [34]). *Each of the following two norms is equivalent to the norm in* $W_T^{1,p(t)}$:

(i) $|\dot{u}|_{p(t)} + |u|_q$, $1 \le q \le \infty$,

(ii) $|\dot{u}|_{p(t)} + |\overline{u}|$, *where* $\overline{u} = (1/T)\int_0^T u(t)dt$.

Proposition 2.24. *In space* $W_T^{1,p(t)}$, $\|u\| \to \infty \Rightarrow \int_0^T |\dot{u}|^{p(t)}dt + |\overline{u}| \to \infty$.

Proof. From Lemma 2.23, there exists a positive constant C_3, such that

$$\|u\| \le C_3\left(|\dot{u}|_{p(t)} + |\overline{u}|\right), \tag{2.28}$$

if $|\dot{u}|_{p(t)} < 1$, it is easy to get

$$|\dot{u}|_{p(t)} < \int_0^T |\dot{u}|^{p(t)}dt + 1. \tag{2.29}$$

When $|\dot{u}|_{p(t)} \geq 1$, we conclude that

$$|\dot{u}|_{p(t)} \leq \left(\int_0^T |\dot{u}|^{p(t)} dt \right)^{1/p^-}, \tag{2.30}$$

by Lemma 2.5, it follows (2.29) and (2.30) that

$$\|u\| \leq C_3 \left(\left(\int_0^T |\dot{u}|^{p(t)} dt \right)^{1/p^-} + 1 + |\overline{u}| \right), \tag{2.31}$$

which implies that

$$\|u\| \longrightarrow \infty \Longrightarrow \int_0^T |\dot{u}|^{p(t)} dt + |\overline{u}| \longrightarrow \infty. \tag{2.32}$$

The proof is completed. □

Lemma 2.25 (see [34]). *If $u, u_n \in L^{p(t)}$ $(n = 1, 2, \ldots)$, then the following statements are equivalent to each other*

(i) $\lim_{n\to\infty} |u_n - u|_{p(t)} = 0$,

(ii) $\lim_{n\to\infty} \rho(u_n - u) = 0$,

(iii) $u_n \to u$ *in measure in* $[0, T]$ *and* $\lim_{n\to\infty} \rho(u_n) = \rho(u)$.

Definition 2.26 (see [1]). Let f be Lipschitz near a given point x in a Banach space X, and let v be any other vector in X. The generalized directional derivative of f at x in the direction v, denoted by $f^0(x; v)$, is defined as follows:

$$f^0(x; v) = \limsup_{y \to x,\ \lambda \downarrow 0} \frac{f(y + \lambda v) - f(y)}{\lambda}, \tag{2.33}$$

where y is also a vector in X and λ is a positive scalar, and we denote by

$$\partial f(x) := \left\{ x^* \in X^* : f^0(x; v) \geq \langle x^*, v \rangle, \forall v \text{ in } X \right\}, \tag{2.34}$$

the generalized gradient of f at x (the Clarke subdifferential).

Lemma 2.27 (see [1]). *Let x and y be points in a Banach space X, and suppose that f is Lipschitz on open set containing the line segment $[x, y]$. Then there exists a point u in (x, y) such that*

$$f(y) - f(x) \in \langle \partial f(u), y - x \rangle. \tag{2.35}$$

Definition 2.28 (see [8]). A point $u \in X$ is said to be a critical point of a locally Lipschitz f if $\theta \in \partial f(u)$, namely, $f^0(u;v) \geq 0$ for all every $v \in X$. A real number c is called a critical value of f if there is a critical point $u \in X$ such that $f(u) = c$.

Definition 2.29 (see [8]). If f is a locally Lipschitz function, we say that f satisfies the Palais-Smale condition if each sequence (x_n) in X such that $(f(x_n))$ is bounded and $\lim_{n\to\infty} \lambda(x_n) = 0$ has a convergent subsequence. We define $\lambda(x) = \min_{x^* \in \partial f(x)} \|x^*\|$, where the minimum exists from the fact that $\partial f(x)$ is a w*-weakly compact convex subset.

Lemma 2.30 (see [8]). *Let X be a real Banach space, and let f be a locally Lipschitz function defined on X satisfying the (PS) condition. Suppose $X = X_1 \oplus X_2$ with a finite dimensional subspace X_1, and there exist constants $b_1 < b_2$ and a bounded neighborhood N of θ in X_1 such that*

$$f|_{X_2} \geq b_2, \qquad f|_{\partial N} \leq b_1. \tag{2.36}$$

Then f has a critical point.

Lemma 2.31. *The functional given by*

$$\varphi(u) = \int_0^T \frac{1}{p(t)} |\dot{u}(t)|^{p(t)} dt + \int_0^T F(t, u(t)) dt \tag{2.37}$$

is weakly lower semicontinuous on $W_T^{1,p(t)}$.

Proof. We divide φ into two parts, $\varphi(u) := J(u) + H(u)$, where

$$J(u) := \int_0^T \frac{1}{p(t)} |\dot{u}(t)|^{p(t)} dt, \qquad H(u) := \int_0^T F(t, u(t)) dt, \tag{2.38}$$

it is obvious that J is convex and continuous by Lemma 2.25, then J is weakly lower semicontinuous by Theorem 1.2 in [2], and H is weakly continuous, that is, φ is the sum of two weakly lower semicontinuous functionals, which implies that φ is weakly lower semicontinuous. □

Lemma 2.32 (see [35]). *The functional J defined in Lemma 2.31 is continuously differentiable on $W_T^{1,p(t)}$ and J' is given by*

$$\left\langle J'(u), v \right\rangle = \int_0^T \left(|\dot{u}(t)|^{p(t)-2} \dot{u}(t), \dot{v}(t) \right) dt, \tag{2.39}$$

and J' is a mapping of (S_+), that is, if $u_n \rightharpoonup u$ weakly in $W_T^{1,p(t)}$ and

$$\limsup_{n\to\infty} \left(J'(u_n) - J'(u), u_n - u \right) \leq 0, \tag{2.40}$$

then u_n has a convergent subsequence on $W_T^{1,p(t)}$.

Clarke considered the following abstract framework in [1]:

(i) let $(T, \mathcal{T}, \mu)$ be a σ-finite positive measure space, and let Y be a separable Banach space,

(ii) let Z be a closed subspace of $L^\infty(T, Y)$, where $L^\infty(T, Y)$ denotes the space of measure essentially bounded functions mapping T to Y, equipped with the usual supremum norm,

(iii) define a functional f on Z via

$$f(x) = \int_T f_t(x(t))\mu(dt), \tag{2.41}$$

where $f_t : Y \to \mathbb{R}$ $(t \in T)$ is a given family of functions,

(iv) suppose that the mapping $t \to f_t(v)$ is measurable for each v in Y, and that x is a point at which $f(x)$ is defined (finitely),

(v) suppose that there exist $\varepsilon > 0$ and a function $k(t)$ in $L^1(T, \mathbb{R})$ such that

$$\left|f_t(v_1) - f_t(v_2)\right| \le k(t)\|v_1 - v_2\|_Y, \tag{2.42}$$

for all $t \in T$ and all v_1 and v_2 in $x(t) + \varepsilon B_Y$.

Under this conditions described above, f is Lipschitz in a neighborhood of x and one has

$$\partial f(x) \subset \int_T \partial f_t(x(t))\mu(dt). \tag{2.43}$$

Further, if each f_t is regular at $x(t)$ for each t, then f is regular at x and equality holds.

We give an example to illustrate Clarke's abstract framework with the following cast of characters:

(i) $(T, \mathcal{T}, \mu) := [0, T]$ with Lebesgue measure, and let $Y := \mathbb{R}^N$, which is a separable Banach space,

(ii) let $Z := C([0, T], \mathbb{R}^N)$, which is a closed subspace of $L^\infty([0, T], \mathbb{R}^N)$,

(iii) define a functional f on Z via

$$f(x) = \int_0^T F(t, x(t))dt, \tag{2.44}$$

(iv) $F(t, x)$ satisfies the condition (A′).

Under the hypothesis above, we only need to justify the condition (2.42), in fact, by Lebourg's mean value theorem,

$$|F(t, x_1) - F(t, x_2)| = |\langle \zeta_t, x_1 - x_2 \rangle|, \tag{2.45}$$

where $\lambda_t \in (x_1, x_2)$ and $\zeta_t \in \partial F(t, \lambda_t)$ for a.e. $t \in [0,T]$ and all x_1 and x_2 in $x(t) + \varepsilon B_Y$, where $x(t) \in Z$ and ε is a positive constant.

In view of (A′), we get

$$\begin{aligned} |F(t,x_1) - F(t,x_2)| &\le \left[2^{\alpha} C\left(|x_1|^{\alpha} + |x_2|^{\alpha}\right) + C_0\right]|x_1 - x_2| \\ &\le \left[2^{2\alpha+1} C\left(|x(t)|^{\alpha} + \varepsilon^{\alpha}\right) + C_0\right]|x_1 - x_2|, \end{aligned} \tag{2.46}$$

for a.e. $t \in [0,T]$ and all x_1, x_2 in $x(t) + \varepsilon B_Y$.

We can apply Clarke's abstract framework to our example, that is, for any $\zeta \in \partial f(u)$ such that

$$\langle \zeta, v \rangle = \int_0^T (q(t), v(t))\, dt, \tag{2.47}$$

for all $v \in Z$, where $q(t)$ is a measurable selection of $\partial F(t, x(t))$.

Now we can prove the following result which is fundamental in our paper.

Lemma 2.33. *Suppose $L : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is given by*

$$L(t,x,y) = L_1(t,x,y) + L_2(t,x,y), \tag{2.48}$$

where

$$L_1(t,x,y) = \frac{1}{p(t)}|y|^{p(t)}, \qquad L_2(t,x,y) = F(t,x), \tag{2.49}$$

and $F(t,x)$ satisfies the condition (A′). The corresponding functionals f_1 and f_2 on $W_{p(t)}$ are given by

$$\begin{aligned} f_1(u,\dot{u}) &= \int_0^T L_1(t,u(t),\dot{u}(t))\, dt = \int_0^T \frac{1}{p(t)}|\dot{u}(t)|^{p(t)}\, dt, \\ f_2(u,\dot{u}) &= \int_0^T L_2(t,u(t),\dot{u}(t))\, dt = \int_0^T F(t,u(t))\, dt. \end{aligned} \tag{2.50}$$

Then $f = f_1 + f_2$ is Lipschitz on $W_{p(t)}$ and one has

$$\partial f(u,\dot{u}) \subset \int_0^T \left\{ |\dot{u}(t)|^{p(t)-2}\dot{u}(t) \right\} \times \{\partial F(t,u(t))\}\, dt. \tag{2.51}$$

Proof. Take an arbitrary element $(u_0, \dot{u}_0)$ in $W_{p(t)}$, and it suffices to prove f is Lipschitz on $(u_0, \dot{u}_0)$ and (2.51) holds for $(u_0, \dot{u}_0)$.

f_1 is continuously differentiable on $(u_0, \dot{u}_0)$, that is,

$$\left\langle f_1'(u_0,\dot{u}_0), (v,\dot{v}) \right\rangle = \int_0^T \left(|\dot{u}_0(t)|^{p(t)-2}\dot{u}_0(t), \dot{v}(t) \right) dt, \tag{2.52}$$

for any $(v, \dot{v})$ in $W_{p(t)}$, so f_1 is Lipschitz on $(u_0, \dot{u}_0)$.

When $\|(u_i, \dot{u}_i) - (u_0, \dot{u}_0)\|_{W_{p(t)}} \leq \varepsilon$ $(i = 1, 2)$, we conclude

$$\|u_i(t) - u_0(t)\|_\infty \leq C_4 \varepsilon, \tag{2.53}$$

by Lemma 2.15, where C_4 is a positive constant. Arguing as in (2.46),

$$|F(t, u_1(t)) - F(t, u_2(t))| \leq k'(t)|u_1(t) - u_2(t)|, \tag{2.54}$$

for a.e. $t \in [0, T]$, where $k'(t) \in L^1([0, T]; \mathbb{R}^+)$.

By Lemma 2.15 and (2.54), we have

$$\begin{aligned} \left|f_2(u_1, \dot{u}_1) - f_2(u_2, \dot{u}_2)\right| &= \left|\int_0^T F(t, u_1(t)) - F(t, u_2(t))dt\right| \\ &\leq \int_0^T k'(t)dt \|u_1(t) - u_2(t)\|_\infty \\ &\leq C_4 \int_0^T k'(t)dt \|(u_1, \dot{u}_1) - (u_2, \dot{u}_2)\|_{W_{p(t)}}, \end{aligned} \tag{2.55}$$

so f_2 is also Lipschitz on $(u_0, \dot{u}_0)$, which implies that f as the sum of two Lipschitz functionals is also Lipschitz on $(u_0, \dot{u}_0)$.

For any ζ in $\partial f_2(u_0, \dot{u}_0)$, one has

$$\int_0^T F^0(t, u_0(t); v(t))dt \geq f_2^0((u_0, \dot{u}_0); (v, \dot{v})) \geq \langle \zeta, (v, \dot{v}) \rangle, \tag{2.56}$$

for any $(v, \dot{v})$ in $W_{p(t)}$ by Fatou Lemma and it is obvious that

$$L_2^0(t, u_0(t), \dot{u}_0(t); v_1, v_2) = F^0(t, u_0(t); v_1), \tag{2.57}$$

for a.e. $t \in [0, T]$ and all (v_1, v_2) in $\mathbb{R}^N \times \mathbb{R}^N$, then we conclude

$$\int_0^T L_2^0(t, u_0(t), \dot{u}_0(t); v(t), \dot{v}(t))dt \geq f_2^0((u_0, \dot{u}_0); (v, \dot{v})) \geq \langle \zeta, (v, \dot{v}) \rangle, \tag{2.58}$$

by (2.56) for any $(v, \dot{v})$ in $W_{p(t)}$ and (2.58) remains true if we restrict $(v, \dot{v})$ to $W_{p(t)} \cap W_\infty$, which is a closed subspace of W_∞ by Lemma 2.20.

By Lemma 2.21, we conclude the bounded linear functional ζ on $W_{p(t)}$ restricted to $W_{p(t)} \cap W_\infty$ is also a bounded linear functional, and we use ζ' to denote the functional restricted on $W_{p(t)} \cap W_\infty$.

We interpret (2.58) as saying that ζ' belongs to the subgradient at (0, 0) of the convex functional

$$\widehat{f}_2(v, \dot{v}) := \int_0^T \widehat{f}_t(v(t), \dot{v}(t))dt, \tag{2.59}$$

which is defined in $W_{p(t)} \cap W_\infty$, where $\widehat{f}_t(v_1, v_2) := L_2^0(t, u_0(t), \dot{u}_0(t); v_1, v_2)$ for all (v_1, v_2) in $\mathbb{R}^N \times \mathbb{R}^N$.

In view of condition (A′) and (2.57), we have

$$\begin{aligned}
&\left|L_2^0(t, u_0(t), \dot{u}_0(t); v_1, v_2) - L_2^0(t, u_0(t), \dot{u}_0(t); v_3, v_4)\right| \\
&= \left|F^0(t, u_0(t); v_1) - F^0(t, u_0(t); v_3)\right| \\
&\le \left(C\left(|u_0(t)|^\alpha + C_0\right)|v_1 - v_3|\right. \\
&\le \left(C\left(|u_0(t)|^\alpha + C_0\right)\left(|v_1 - v_3| + |v_2 - v_4|\right),\right.
\end{aligned} \tag{2.60}$$

for a.e. $t \in [0, T]$ and all (v_1, v_2), (v_3, v_4) in $\mathbb{R}^N \times \mathbb{R}^N$.

Now we can apply Clarke's abstract framework to $\widehat{f}_2$ with the following cast of characters:

(i) $(T, \mathcal{T}, \mu) := [0, T]$ with Lebesgue measure, and let $Y := \mathbb{R}^N \times \mathbb{R}^N$, which is a separable Banach space with the norm $|\cdot| + |\cdot|$,

(ii) let $Z := W_{p(t)} \cap W_\infty$, which is a closed subspace of W_∞, and W_∞ denotes the space of measure essentially bounded functions mapping T to Y, equipped with the usual supremum norm by Definition 2.18,

(iii) define a functional $\widehat{f}_2$ on Z by (2.59),

(iv) the mapping $t \to L_2^0(t, u_0(t), \dot{u}_0(t); v_1, v_2)$ is measurable for each (v_1, v_2) in $\mathbb{R}^N \times \mathbb{R}^N$ by (2.57), see details in [1], and that $(0, 0)$ is a point at which $\widehat{f}_2$ is defined (finitely),

(v) the condition (2.42) in Clarke's abstract framework is satisfied by (2.60).

□

By (2.57), We get

$$\partial \widehat{f}_t(0, 0) = \partial L_2(t, u_0(t), \dot{u}_0(t)) \subset \partial F(t, u_0(t)) \times \{0\}, \tag{2.61}$$

thus, every $\zeta' \in \partial \widehat{f}_2(0, 0)$ can be written as

$$\begin{aligned}
\left\langle \zeta', (v, \dot{v}) \right\rangle &= \int_0^T \left(q(t), v(t)\right) + (0, \dot{v}(t))dt \\
&= \int_0^T \left(q(t), v(t)\right)dt,
\end{aligned} \tag{2.62}$$

for any $(v, \dot{v})$ in $W_{p(t)} \cap W_\infty$, where $q(t) \in \partial F(t, u_0(t))$ for a.e. $t \in [0, T]$.

When $v \in C_T^\infty([0, T], \mathbb{R}^N)$, it is obvious that $(v, \dot{v}) \in W_{p(t)} \cap W_\infty$ and $(v, \dot{v})$ is dense in $W_{p(t)}$ by Lemma 2.11. So for each $(v, \dot{v}) \in W_{p(t)}$, we can choose $(v_n, \dot{v}_n) \in W_{p(t)} \cap W_\infty$ such that

$$\|(v_n, \dot{v}_n) - (v, \dot{v})\|_{W_{p(t)}} \longrightarrow 0, \qquad \langle \zeta, (v_n, \dot{v}_n) \rangle \longrightarrow \langle \zeta, (v, \dot{v}) \rangle. \tag{2.63}$$

Combining (2.62) and (2.63), we have

$$\langle \zeta, (v, \dot{v}) \rangle = \int_0^T (q(t), v(t)) dt, \tag{2.64}$$

for all $(v, \dot{v}) \in W_{p(t)}$.

We conclude

$$\partial f(u_0, \dot{u}_0) \subset \partial f_1(u_0, \dot{u}_0) + \partial f_2(u_0, \dot{u}_0) = \int_0^T \left\{ |\dot{u}_0(t)|^{p(t)-2} \dot{u}_0(t) \right\} \times \{\partial F(t, u_0(t))\} dt, \tag{2.65}$$

and this completes the proof.

3. Main Results and Proofs of Theorems

Theorem 3.1. *Let $F(t, x)$ satisfy the condition (A') with $\alpha \in [0, p^- - 1)$, and we suppose the following condition holds*

$$|x|^{-q^+\alpha} \int_0^T F(t, x) dt \longrightarrow +\infty \quad \text{as } |x| \longrightarrow \infty, \tag{A1}$$

where q^+ is the same in condition (A).

Then system (1.1) has at least one solution which minimizes φ in $W_T^{1,p(t)}$.

If we replace the (A1) in Theorem 3.1 by the following condition:

$$|x|^{-q^+\alpha} \int_0^T F(t, x) dt \longrightarrow -\infty \quad \text{as } |x| \longrightarrow \infty, \tag{A2}$$

we obtain the following theorem.

Theorem 3.2. *Let $F(t, x)$ satisfy the condition (A') with $\alpha \in [0, p^- - 1)$ and (A2). Then system* (1.1) *has at least one solution in $W_T^{1,p(t)}$.*

Remark 3.3. Theorems 3.1 and 3.2 generalize Theorems 1 and 2, respectively in [3].

Proof of Theorem 3.1. For $u \in W_T^{1,p(t)}$, let $\overline{u} = (1/T) \int_0^T u(t) dt$ and $\widetilde{u} = u - \overline{u}$. From Lemma 2.27, it follows that there exist $z(t)$ in $(\overline{u}, u(t))$ such that

$$F(t, u(t)) - F(t, \overline{u}) = \langle \zeta_t, \widetilde{u}(t) \rangle, \tag{3.1}$$

for a.e. $t \in [0, T]$, where $\zeta_t \in \partial F(t, z(t))$.

It follows from (A′), Young inequality and Lemma 2.22 that

$$\begin{aligned}\left|\int_0^T [F(t,u(t))-F(t,\overline{u})]dt\right| &\le \int_0^T |F(t,u(t))-F(t,\overline{u})|dt\\ &\le \int_0^T |\zeta_t||\widetilde{u}(t)|dt \le \int_0^T \left[2^\alpha C\left(|\overline{u}|^\alpha+|\widetilde{u}(t)|^\alpha\right)+C_0\right]|\widetilde{u}(t)|dt\\ &\le 2^\alpha CT\|\widetilde{u}\|_\infty^{\alpha+1}+2^\alpha CT\|\widetilde{u}\|_\infty|\overline{u}|^\alpha+C_0T\|\widetilde{u}\|_\infty\\ &\le \frac{1}{2p^+}\int_0^T |\dot{u}(t)|^{p(t)}dt+C_5\left(\int_0^T |\dot{u}(t)|^{p(t)}dt\right)^{(\alpha+1)/p^-}\\ &\quad +C_6\left(\int_0^T |\dot{u}(t)|^{p(t)}dt\right)^{1/p^-}+C_7|\overline{u}|^{q^+\alpha}+C_8,\end{aligned} \tag{3.2}$$

for all $u\in W_T^{1,p(t)}$, and some positive constants C_5, C_6, C_7, and C_8.

Hence we have

$$\begin{aligned}\varphi(u) &\ge \frac{1}{p^+}\int_0^T |\dot{u}(t)|^{p(t)}dt+\int_0^T F(t,\overline{u})dt+\int_0^T [F(t,u(t))-F(t,\overline{u})]dt\\ &\ge \frac{1}{2p^+}\int_0^T |\dot{u}(t)|^{p(t)}dt-C_5\left(\int_0^T |\dot{u}(t)|^{p(t)}dt\right)^{(\alpha+1)/p^-}-C_6\left(\int_0^T |\dot{u}(t)|^{p(t)}dt\right)^{1/p^-}\\ &\quad -C_7|\overline{u}|^{q^+\alpha}-C_8+\int_0^T F(t,\overline{u})dt\\ &\ge \frac{1}{2p^+}\int_0^T |\dot{u}(t)|^{p(t)}dt-C_5\left(\int_0^T |\dot{u}(t)|^{p(t)}dt\right)^{(\alpha+1)/p^-}-C_6\left(\int_0^T |\dot{u}(t)|^{p(t)}dt\right)^{1/p^-}\\ &\quad +|\overline{u}|^{q^+\alpha}\left(|\overline{u}|^{-q^+\alpha}\int_0^T F(t,\overline{u})dt-C_7\right)-C_8,\end{aligned} \tag{3.3}$$

for all $u\in W_T^{1,p(t)}$, which implies that

$$\varphi(u)\longrightarrow\infty \quad \text{as } \|u\|\longrightarrow\infty, \tag{3.4}$$

because of $\alpha<p^--1$ and the Proposition 2.24.

By Lemma 2.31, the functional φ is weakly lower semicontinuous on $W_T^{1,p(t)}$, and it follows that φ has a minimum u_0 on $W_T^{1,p(t)}$ by Theorem 1.1 in [2]. Proposition 2.3.2 in [1] implies that $0\in\partial\varphi(u_0)$, that is, u_0 is a critical point for φ. So, problem (1.1) has at least one solution $u_0\in W_T^{1,p(t)}$. □

Proof of Theorem 3.2. We will show that φ defined in Lemma 2.31 satisfies the (PS) condition. Let $\{u_n\}$ be a sequence in $W_T^{1,p(t)}$ such that $\varphi(u_n)$ is bounded and $\lambda(u_n) \to 0$ as $n \to \infty$. Using the definition of $\lambda(u_n)$, it results that for each $n \geq n_0$ there exists $u_n^* \in \partial\varphi(u_n)$ with

$$|\langle u_n^*, h\rangle| \leq \|h\|, \quad \forall h \in W_T^{1,p(t)}. \tag{3.5}$$

In view of Lemma 2.33, if $u_n^* \in \partial\varphi(u_n)$, it results that there exist $q_n(t) \in \partial F(t, u_n(t))$ such that

$$|\langle u_n^*, \widetilde{u}_n\rangle| = \left|\int_0^T |\dot{u}_n(t)|^{p(t)}dt + \int_0^T \big(q_n(t), \widetilde{u}_n(t)\big)dt\right| \leq \|\widetilde{u}_n\|, \quad \forall n \geq n_0. \tag{3.6}$$

It follows Lemma 2.22 and Young inequality that

$$\begin{aligned}
\left|\int_0^T \big(q_n(t), \widetilde{u}_n(t)\big)\right| &\leq C\int_0^T |\overline{u}_n + \widetilde{u}_n(t)|^{\alpha}|\widetilde{u}_n(t)|dt + C_0\int_0^T |\widetilde{u}_n(t)|dt \\
&\leq 2^{p^- -1}C\int_0^T \big(|\overline{u}_n|^{\alpha} + |\widetilde{u}_n(t)|^{\alpha}\big)|\widetilde{u}_n(t)|dt + C_0\int_0^T |\widetilde{u}_n(t)|dt \\
&\leq 2^{p^- -1}CT\big(|\overline{u}_n|^{\alpha} + \|\widetilde{u}_n\|_{\infty}^{\alpha}\big)\|\widetilde{u}_n\|_{\infty} + C_0T\|\widetilde{u}_n\|_{\infty} \\
&= \left(\left(\frac{1}{2}\right)^{1/p^-}\frac{\|\widetilde{u}_n\|_{\infty}}{4C_2}\right)\left(\left(2^{p^- +1}\right)(2)^{1/p^-}CC_2T|\overline{u}_n|^{\alpha}\right) \\
&\quad + 2^{p^- -1}CT\|\widetilde{u}_n\|_{\infty}^{1+\alpha} + C_0T\|\widetilde{u}_n\|_{\infty} \\
&\leq \frac{1}{2}\int_0^T |\dot{u}_n(t)|^{p(t)}dt + C_9|\overline{u}_n|^{q^+\alpha} + C_{10}\left(\int_0^T |\dot{u}_n(t)|^{p(t)}dt\right)^{(\alpha+1)/p^-} \\
&\quad + C_{11}\left(\int_0^T |\dot{u}_n(t)|^{p(t)}dt\right)^{1/p^-} + C_{12},
\end{aligned} \tag{3.7}$$

for all n and some positive constants C_9, C_{10}, C_{11}, and C_{12}, where C_2 is the same as in Lemma 2.22.

Hence, we have

$$\begin{aligned}\|\widetilde{u}_n\| &\geq |\langle u_n^*, \widetilde{u}_n\rangle| \\ &\geq \int_0^T |\dot{u}_n(t)|^{p(t)}dt + \int_0^T \left(q_n(t), \widetilde{u}_n(t)\right)dt \\ &\geq \frac{1}{2}\int_0^T |\dot{u}_n(t)|^{p(t)}dt - C_9|\overline{u}_n|^{q^+\alpha} - C_{10}\left(\int_0^T |\dot{u}_n(t)|^{p(t)}dt\right)^{(\alpha+1)/p^-} \\ &\quad - C_{11}\left(\int_0^T |\dot{u}_n(t)|^{p(t)}dt\right)^{1/p^-} - C_{12},\end{aligned} \tag{3.8}$$

for all $n \geq n_0$.

It follows from (2.31) that

$$\|\widetilde{u}_n\| \leq C_3\left(\left(\int_0^T |\dot{u}_n(t)|^{p(t)}dt\right)^{1/p^-} + 1\right), \tag{3.9}$$

by (3.8) and (3.9), we have

$$\left(\int_0^T |\dot{u}_n(t)|^{p(t)}dt\right)^{1/q^+} \leq C_{13}|\overline{u}_n|^{\alpha} + C_{14}, \tag{3.10}$$

for some positive constants C_{13}, C_{14}, and all $n \geq n_0$.

By the proof of (3.2) we have

$$\begin{aligned}\int_0^T [F(t, u_n(t)) - F(t, \overline{u}_n)]dt \leq \frac{1}{2p^+}\int_0^T |\dot{u}_n(t)|^{p(t)}dt + C_5\left(\int_0^T |\dot{u}(t)|_n^{p(t)}dt\right)^{(\alpha+1)/p^-} \\ + C_6\left(\int_0^T |\dot{u}_n(t)|^{p(t)}dt\right)^{1/p^-} + C_7|\overline{u}_n|^{q^+\alpha} + C_8,\end{aligned} \tag{3.11}$$

for all n.

It follows from the boundness of $\{\varphi(u_n)\}$, (3.10) and (3.11) that

$$\begin{aligned} C_9 &\le \varphi(u_n) \\ &= \int_0^T \frac{1}{p(t)}|\dot{u}_n(t)|^{p(t)}dt + \int_0^T [F(t,u_n(t)) - F(t,\overline{u}_n)]dt + \int_0^T F(t,\overline{u}_n)dt \\ &\le \left(\frac{1}{p^-} + \frac{1}{2p^+}\right)\int_0^T |\dot{u}_n(t)|^{p(t)}dt + C_7|\overline{u}_n|^{q^+\alpha} + C_5\left(\int_0^T |\dot{u}_n(t)|^{p(t)}\mathrm{d}t\right)^{(\alpha+1)/p^-} \\ &\quad + C_6\left(\int_0^T |\dot{u}_n(t)|^{p(t)}dt\right)^{1/p^-} + \int_0^T F(t,\overline{u}_n)dt + C_8 \\ &\le |\overline{u}_n|^{q^+\alpha}\left(|\overline{u}_n|^{-q^+\alpha}\int_0^T F(t,\overline{u}_n)dt + C_7\right) + C_{15}, \end{aligned} \tag{3.12}$$

for all $n \ge n_0$ and some positive constant C_{15}.

It follows (A2) and (3.12) that $(|\overline{u}_n|)$ is bounded, hence $\{\|u_n\|\}$ is bounded by (2.31) and (3.10).

The sequence $\{u_n\}$ has a subsequence, also denoted by $\{u_n\}$, such that

$$u_n \rightharpoonup u \quad \text{weakly in } W_T^{1,p(t)}, \qquad u_n \longrightarrow u \quad \text{strongly in } C\Big([0,T];\mathbb{R}^N\Big), \tag{3.13}$$

and $\|u_n\|_\infty \le C_{16}$ is bounded by Lemma 2.15, where C_{16} is a positive constant.

Therefore we have $u_n^* \in \partial\varphi(u_n)$, where u_n^* is the function from the Palais-Smale condition, and $u^* \in \partial\varphi(u)$ such that

$$\langle u_n^* - u^*, u_n - u\rangle \longrightarrow 0, \tag{3.14}$$

as $n \to \infty$, so

$$\begin{aligned} \langle u_n^* - u^*, u_n - u\rangle = &\int_0^T \big(q_n(t) - q(t), u_n(t) - u(t)\big)dt \\ &+ \int_0^T \Big(|\dot{u}_n(t)|^{p(t)-2}\dot{u}_n(t) - |\dot{u}(t)|^{p(t)-2}\dot{u}(t), \dot{u}_n(t) - \dot{u}(t)\Big)dt, \end{aligned} \tag{3.15}$$

where $q_n(t) \in \partial(F(t,u_n(t))$ and $q(t) \in \partial(F(t,u(t))$.

By (3.14) and (3.15), we get $\langle J'(u) - J'(u_n), u - u_n\rangle \to 0$, that is,

$$\int_0^T \Big(|\dot{u}_n(t)|^{p(t)-2}\dot{u}_n(t) - |\dot{u}(t)|^{p(t)-2}\dot{u}(t), \dot{u}_n(t) - \dot{u}(t)\Big)dt \longrightarrow 0, \tag{3.16}$$

so it follows from Lemma 2.32 that $\{u_n\}$ admits a convergent subsequence.

We now prove φ satisfies the other conditions of Lemma 2.30. Let $\widetilde{W}_T^{1,p(t)}$ be the subspace of $W_T^{1,p(t)}$ defined in Lemma 2.22, then we have

$$\varphi(u) \longrightarrow +\infty, \tag{3.17}$$

as $\|u\| \to \infty$ in $\widetilde{W}_T^{1,p(t)}$. In fact it follows from (3.2) that

$$\begin{aligned}\left|\int_0^T [F(t,u(t)) - F(t,0)]dt\right| \le \int_0^T |\zeta||u(t)|dt &\le C_{17}\left(\int_0^T |\dot{u}(t)|^{p(t)}dt\right)^{(\alpha+1)/p^-} \\ &+ C_{18}\left(\int_0^T |\dot{u}(t)|^{p(t)}dt\right)^{1/p^-} + C_{19},\end{aligned} \tag{3.18}$$

for all $u \in \widetilde{W}_T^{1,p(t)}$ and some positive constants C_{17}, C_{18}, and C_{19}.

$$\begin{aligned}\varphi(u) - \int_0^T F(t,0)dt &= \int_0^T \frac{1}{p(t)}|\dot{u}(t)|^{p(t)}dt + \int_0^T [F(t,u(t)) - F(t,0)]dt \\ &\ge \frac{1}{p^+}\int_0^T |\dot{u}(t)|^{p(t)}dt - C_{17}\left(\int_0^T |\dot{u}(t)|^{p(t)}dt\right)^{(\alpha+1)/p^-} \\ &- C_{18}\left(\int_0^T |\dot{u}(t)|^{p(t)}dt\right)^{1/p^-} - C_{19},\end{aligned} \tag{3.19}$$

for all $u \in \widetilde{W}_T^{1,p(t)}$, which implies (3.17) by Proposition 2.24.

Moreover, we have

$$\varphi(x) \longrightarrow -\infty, \tag{3.20}$$

as $|x| \to \infty$ in $\mathbb{R}^N$, which follows (A2).

We have proved the functional φ satisfies all the conditions of Lemma 2.30, so we know that φ has at least one critical point by Lemma 2.30, which is a periodic solution for system (1.1). The proof is complete. □

4. Example

In this section, we give three examples to illustrate our results.

Example 4.1. In system (1.1), let $F(t,x) = |x|$ and

$$p(t) = \begin{cases} 3 + t, & 0 \le t \le \dfrac{T}{2}, \\ 3 - t + T, & \dfrac{T}{2} < t \le T, \end{cases} \tag{4.1}$$

it is easy to verify that $|\zeta| \le 1$, where $\zeta \in \partial F(t,x)$ for every $t \in [0,T]$ and all $x \in \mathbb{R}^N$.

By Theorem 3.1, system (1.1) has at least one solution $u \in W_T^{1,p(t)}$, but it is obvious that the results in the reference cannot be applied to our example.

Example 4.2. In system (1.1), let

$$p(t) = \begin{cases} 2^t + 1, & 0 \le t \le \dfrac{T}{2}, \\ \dfrac{2(1-2^{T/2})}{T}\left(t - \dfrac{T}{2}\right) + 2^{T/2} + 1 & \dfrac{T}{2} < t \le T, \end{cases} \tag{4.2}$$

and $F(t,x) = -|x|$, it is easy to verify that $|\zeta| \le 1$, where $\zeta \in \partial F(t,x)$ for every $t \in [0,T]$ and all $x \in \mathbb{R}^N$.

By Theorem 3.2, system (1.1) has at least one solution $u \in W_T^{1,p(t)}$, but it is obvious that the results in the reference cannot be applied to our example.

Example 4.3. In system (1.1), let $p(t) = \sin \omega t + 5$, and

$$F(t,x) = \left(\sin \omega t - \frac{1}{2}\right)|x|^3 + x_1, \tag{4.3}$$

where ω denotes the positive constant $2\pi/T$.

It is obvious that F is continuously differentiable, then the the Clarke subdifferential set $\partial F(t,x)$ reduces to one element $\nabla F(t,x)$, then

$$|\nabla F(t,x)| \le 6\left(|x|^2 + 1\right), \quad |x|^{-8/3} \int_0^T F(t,x)dt \longrightarrow -\infty \quad \text{as } |x| \longrightarrow +\infty. \tag{4.4}$$

These show that all conditions of Theorem 3.2 are satisfied, where

$$\alpha = 2, \qquad p^- = 4, \qquad q^+ = \frac{4}{3}, \tag{4.5}$$

and by Theorem 3.2, system (1.1) has at least one periodic solution on $W_T^{1,p(t)}$. But the results in [35] cannot be applied to our example, so our results are new even in the case $F \in C^1$ for system (1.1).

Aknowledgment

This work is partially supported by the NNSF (nos. 11171351, 11261020) of China and Hunan Provincial Innovation Foundation for Postgraduate (no. CX2011B079).

References

[1] F. H. Clarke, *Optimization and Nonsmooth Analysis*, vol. 5 of *Classics in Applied Mathematics*, SIAM, Philadelphia, Pa, USA, 1990.

[2] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, vol. 74 of *Applied Mathematical Sciences*, Springer, Berlin, Germany, 1989.

[3] C. L. Tang, "Periodic solutions for nonautonomous second order systems with sublinear nonlinearity," *Proceedings of the American Mathematical Society*, vol. 126, no. 11, pp. 3263–3270, 1998.

[4] C. L. Tang, "Existence and multiplicity of periodic solutions for nonautonomous second order systems," *Nonlinear Analysis*, vol. 32, no. 3, pp. 299–304, 1998.

[5] C. L. Tang and X. P. Wu, "Periodic solutions for second order systems with not uniformly coercive potential," *Journal of Mathematical Analysis and Applications*, vol. 259, no. 2, pp. 386–397, 2001.

[6] C. L. Tang and X. P. Wu, "Notes on periodic solutions of subquadratic second order systems," *Journal of Mathematical Analysis and Applications*, vol. 285, no. 1, pp. 8–16, 2003.

[7] F. Zhao and X. Wu, "Saddle point reduction method for some non-autonomous second order systems," *Journal of Mathematical Analysis and Applications*, vol. 291, no. 2, pp. 653–665, 2004.

[8] K. C. Chang, "Variational methods for nondifferentiable functionals and their applications to partial differential equations," *Journal of Mathematical Analysis and Applications*, vol. 80, no. 1, pp. 102–129, 1981.

[9] A. Ambrosetti and P. H. Rabinowitz, "Dual variational methods in critical point theory and applications," *Journal of Functional Analysis*, vol. 14, pp. 349–381, 1973.

[10] P. H. Rabinowitz, "Periodic solutions of Hamiltonian systems," *Communications on Pure and Applied Mathematics*, vol. 31, no. 2, pp. 157–184, 1978.

[11] N. C. Kourogenis and N. S. Papageorgiou, "Nonsmooth critical point theory and nonlinear elliptic equations at resonance," *Journal of the Australian Mathematical Society*, vol. 69, no. 2, pp. 245–271, 2000.

[12] P. Harjulehto and P. Hästö, "An overview of variable exponent Lebesgue and Sobolev spaces," in *Future Trends in Geometric Function Theory RNC Work-Shop*, D. Herron, Ed., vol. 92, pp. 85–93, University of Jyväskylä, Jyväskylä, Finland, 2003.

[13] S. Hu and N. S. Papageorgiou, "Positive solutions and multiple solutions for periodic problems driven by scalar p-Laplacian," *Mathematische Nachrichten*, vol. 279, no. 12, pp. 1321–1334, 2006.

[14] J. Liu and Y. Guo, "Critical point theory for nonsmooth functionals," *Nonlinear Analysis*, vol. 66, no. 12, pp. 2731–2741, 2007.

[15] S. A. Marano and D. Motreanu, "Infinitely many critical points of non-differentiable functions and applications to a Neumann-type problem involving the p-Laplacian," *Journal of Differential Equations*, vol. 182, no. 1, pp. 108–120, 2002.

[16] M. Růžička, *Electrorheological Fluids: Modeling and Mathematical Theory*, vol. 1748 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2000.

[17] E. H. Papageorgiou and N. S. Papageorgiou, "Existence of solutions and of multiple solutions for nonlinear nonsmooth periodic systems," *Czechoslovak Mathematical Journal*, vol. 54, no. 2, pp. 347–371, 2004.

[18] G. Dai, "Three solutions for a Neumann-type differential inclusion problem involving the $p(x)$-Laplacian," *Nonlinear Analysis*, vol. 70, no. 10, pp. 3755–3760, 2009.

[19] G. Dai, "Infinitely many solutions for a differential inclusion problem in R^N involving the $p(x)$-Laplacian," *Nonlinear Analysis*, vol. 71, no. 3-4, pp. 1116–1123, 2009.

[20] M. Filippakis, L. Gasiński, and N. S. Papageorgiou, "Periodic problems with asymmetric nonlinearities and nonsmooth potentials," *Nonlinear Analysis*, vol. 58, no. 5-6, pp. 683–702, 2004.

[21] M. E. Filippakis, "Existence and multiplicity results for nonlinear nonautonomous second-order systems," *Nonlinear Analysis*, vol. 68, no. 6, pp. 1611–1626, 2008.

[22] L. Gasinski and N. S. Papageorgiou, *Nonsmooth Critical Point Theory and Nonlinear Boundary Value Problems*, vol. 8 of *Series in Mathematical Analysis and Applications*, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2005.

[23] S. A. Marano and D. Motreanu, "On a three critical points theorem for non-differentiable functions and applications to nonlinear boundary value problems," *Nonlinear Analysis*, vol. 48, no. 1, pp. 37–52, 2002.

[24] E. H. Papageorgiou and N. S. Papageorgiou, "Non-linear second-order periodic systems with non-smooth potential," *Proceedings of the Indian Academy of Sciences*, vol. 114, no. 3, pp. 269–298, 2004.

[25] D. Paşca, "Periodic solutions of a class of non-autonomous second-order differential inclusions systems," *Abstract and Applied Analysis*, vol. 6, no. 3, pp. 151–161, 2001.

[26] D. Paşca, "Periodic solutions of second-order differential inclusions systems with p-Laplacian," *Journal of Mathematical Analysis and Applications*, vol. 325, no. 1, pp. 90–100, 2007.

[27] Y. Chen, S. Levine, and M. Rao, "Variable exponent, linear growth functionals in image restoration," *SIAM Journal on Applied Mathematics*, vol. 66, no. 4, pp. 1383–1406, 2006.

[28] L. Diening, P. Hästö, and A. Nekvinda, "Open problems in variable exponent Lebesgue and Sobolev spaces," in *Proceedings, Function Spaces, Differential Operators and Nonlinear Analysis FSDONA 2004*, P. Drábek and J. Rákosník, Eds., pp. 38–58, Academy of Sciences of the Czech Republic, Prague, Czech Republic, 2004.

[29] X. Fan and D. Zhao, "The quasi-minimizer of integral functionals with $m(x)$ growth conditions," *Nonlinear Analysis*, vol. 39, no. 7, pp. 807–816, 2000.

[30] X. Fan and D. Zhao, "On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$," *Journal of Mathematical Analysis and Applications*, vol. 263, no. 2, pp. 424–446, 2001.

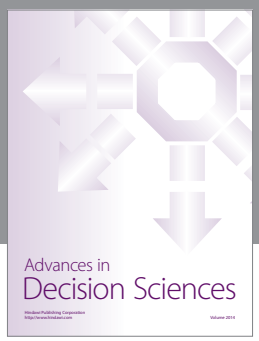
[31] X. L. Fan and Q. H. Zhang, "Existence of solutions for $p(x)$-Laplacian Dirichlet problem," *Nonlinear Analysis*, vol. 52, no. 8, pp. 1843–1852, 2003.

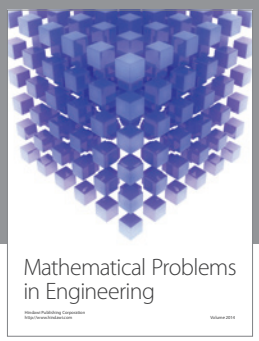
[32] S. Samko, "On a progress in the theory of Lebesgue spaces with variable exponent: maximal and singular operators," *Integral Transforms and Special Functions*, vol. 16, no. 5-6, pp. 461–482, 2005.

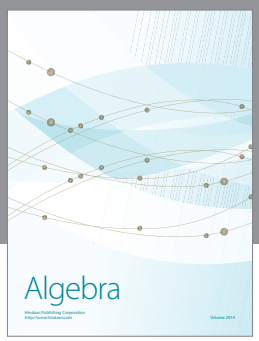
[33] V. V. Zhikov, "Averaging of functionals of the calculus of variations and elasticity theory," *Mathematics of the USSR-Izvestiya*, vol. 29, no. 1, pp. 33–66, 1987.

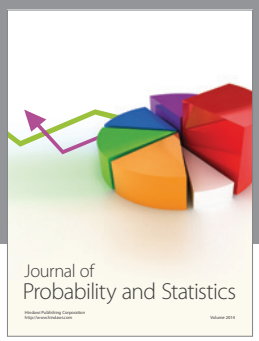
[34] X. L. Fan and X. Fan, "A Knobloch-type result for $p(t)$-Laplacian systems," *Journal of Mathematical Analysis and Applications*, vol. 282, no. 2, pp. 453–464, 2003.

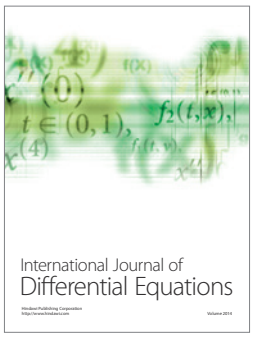
[35] X. J. Wang and R. Yuan, "Existence of periodic solutions for $p(t)$-Laplacian systems," *Nonlinear Analysis*, vol. 70, no. 2, pp. 866–880, 2009.

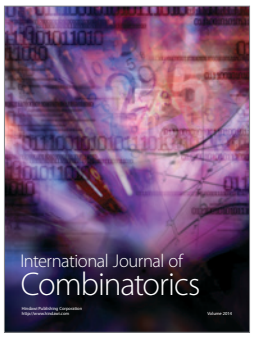


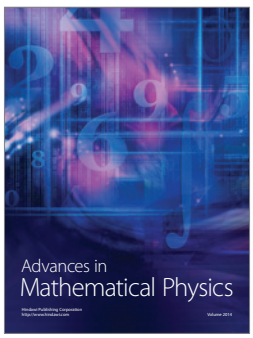


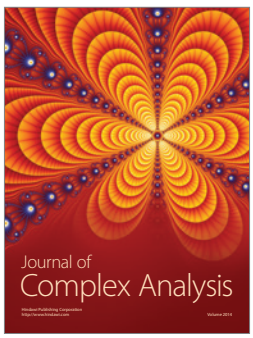


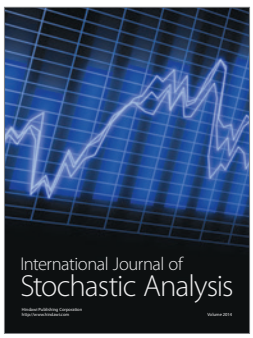


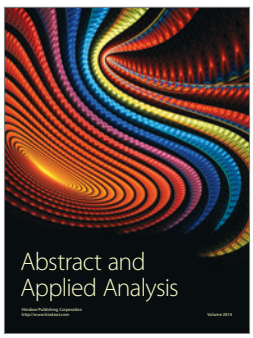


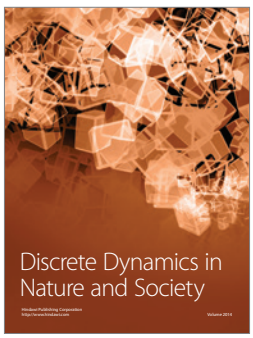


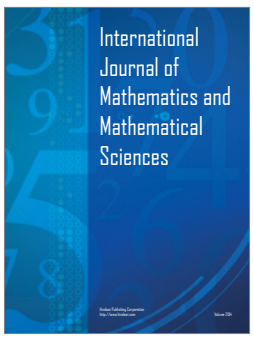


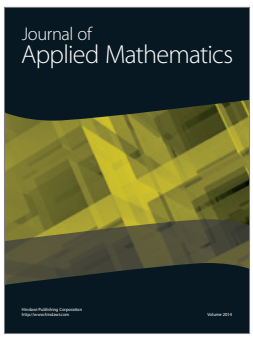












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