Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 489043, 11 pages
doi:10.1155/2012/489043

Research Article

The Bäcklund Transformations and Abundant Exact Explicit Solutions for a General Nonintegrable Nonlinear Convection-Diffusion Equation

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Received 25 October 2011; Accepted 14 November 2011

Academic Editor: Shaher M. Momani

The Bäcklund transformations and abundant exact explicit solutions for a class of nonlinear wave equation are obtained by the extended homogeneous balance method. These solutions include the solitary wave solution of rational function, the solitary wave solutions, singular solutions, and the periodic wave solutions of triangle function type. In addition to rederiving some known solutions, some entirely new exact solutions are also established. Explicit and exact particular solutions of many well-known nonlinear evolution equations which are of important physical significance, such as Kolmogorov-Petrovskii-Piskunov equation, FitzHugh-Nagumo equation, Burgers-Huxley equation, Chaffee-Infante reaction diffusion equation, Newell-Whitehead equation, Fisher equation, Fisher-Burgers equation, and an isothermal autocatalytic system, are obtained as special cases.

1. Introduction

The existence of solitary wave solutions and periodic wave solutions is an important question in the study of nonlinear evolution equations. The methods of finding such solutions for integrable equations are well known: the solitary wave solutions can be found by inverse scattering transformation [1] and the Hirota bilinear method [2], and the periodic solutions can be represented by sums of equally spaced solitons represented by sech-function [3, 4]. Weiss et al. developed the singular manifold method to introduce the Painlevé property in the theory of partial differential equations [5]. The beauty of the singular manifold method is that this expansion for a nonlinear PDE contains a lot of information about this PDE. For an equation that possesses the Painlevé property the singular manifold method leads to the Bäcklund transformation, the Lax pair, and Miura transformations and makes connections to the Hirota bilinear method, Laplace-Darboux transformations [6]. Most nonlinear nonintegrable equations do not possess the Painlevé property; that is, they are not free from “movable” critical singularities. For some nonintegrable nonlinear equations it is still possible to obtain single-value expansions by putting a constraint on the arbitrary function in the Painlevé expansion. Such equations are said to be partially integrable, and Weiss [7] conjectured that these systems can be reduced to integrable equations. Another treatment of the partially integrable systems was offered by Hietarinta [8] by the generalization of the Hirota bilinear formalism for nonintegrable systems. He conjectured that all completely integrable PDEs can be put into a bilinear form. There are also nonintegrable equations that can be put into the bilinear form and then the partial integrability is associated with the levels of integrability defined by the number of solitons that can be combined to an N-soliton solution. Partial integrability then means that the equation allows a restricted number of multisoliton solutions. In [9] Berloff and Howard suggested joining these treatments of the partial nonintegrability and using the Painlevé expansion truncated before the “constant term” level as the transform for reducing a nonintegrable PDE to a multilinear equation.

The Bäcklund transformation is not only a useful tool to obtain exact solutions of some soliton equation from a trivial “seed” but also related to infinite conservation laws and inverse scattering method [1]. In [10–12], Wang Mingliang proposed the homogeneous balance method—an effective method solving nonlinear partial differential equations. Fan and Zhang extended the homogeneous balance method and proposed an approach to obtain Bäcklund transformation for the nonlinear evolution equations [13]. In a recent paper [14], Shang obtained the Bäcklund transformation, a Lax pair, and some new explicit exact solutions of Hirota-Satsuma SWW equation (2.3) by means of the Bäcklund transformations and the extension of the hyperbolic function method presented in [15].

In this paper we investigate a general nonintegrable nonlinear convection-diffusion equation

$$u_t - u_{xx} + \alpha u u_x + \beta u + \gamma u^2 + \delta u^3 = 0, \tag{1.1}$$

where α, β, γ, and δ are arbitrary real constants. Equation (1.1) include many well-known nonlinear equations that are with applied background as special examples, such as Burgers equation, Kolmogorov-Petrovskii-Piskunov equation, FitzHugh-Nagumo equation, Burgers-Huxley equation, Chaffee-Infante reaction-diffusion equation, Newell-Whitehead equation, Fisher equation, Fisher-Burgers equation, and an isothermal autocatalytic system. The extended homogeneous balance method is applied for a reliable treatment of the nonintegrable nonlinear equation (1.1). Some Bäcklund transformations and abundant explicit exact particular solutions of the nonintegrable nonlinear equation (1.1) are obtained by means of the extended homogeneous balance method. Some explicit exact solutions obtained here have more general form than some known solutions, and some explicit exact solutions obtained here are entirely new solutions.

2. Bäcklund Transformations for the Nonintegrable Nonlinear Wave Equation

According to the extended homogeneous balance method, we suppose that the solution of (1.1) is of the form

$$u(x,t) = f'(\phi)\phi_x + u_1(x,t), \tag{2.1}$$

where f, ϕ are two functions to be determined and $u_1(x,t)$ is a solution of (1.1).

From (2.1), we have

$$u_t = f''(\phi)\phi_x\phi_t + f'(\phi)\phi_{xt} + u_{1t}, \tag{2.2}$$

$$u_x = f''(\phi)\phi_x^2 + f'(\phi)\phi_{xx} + u_{1x}, \tag{2.3}$$

$$u_{xx} = f'''(\phi)\phi_x^3 + 3f''(\phi)\phi_x\phi_{xx} + f'(\phi)\phi_{xxx} + u_{1xx}, \tag{2.4}$$

$$\begin{aligned} u^2 &= (f')^2(\phi)\phi_x^2 + 2f'\phi_x u_1(x,t) + u_1^2(x,t), \\ u^3 &= (f')^3(\phi)\phi_x^3 + 3(f')^2\phi_x^2 u_1(x,t) + 3f'\phi_x u_1^2(x,t) + u_1^3(x,t). \end{aligned} \tag{2.5}$$

Substituting (2.1)–(2.5) into the left side of (1.1) and collecting all terms with ϕ_x^3, we obtain

$$\begin{aligned} &u_t - u_{xx} + \alpha u u_x + \beta u + \gamma u^2 + \delta u^3 \\ &\quad= \left(\alpha f'' f' - f''' + \delta (f')^3\right)\phi_x^3 \\ &\qquad+ \left[f''\phi_x\phi_t - 3f''\phi_x\phi_{xx} + \alpha f''\phi_x^2 u_1 + \alpha (f')^2\phi_x\phi_{xx} + \gamma (f')^2\phi_x^2 + 3\delta (f')^2\phi_x^2 u_1(x,t)\right] \\ &\qquad+ f'\left[\phi_{xt} - \phi_{xxx} + \alpha\phi_{xx}u_1 + \alpha\phi_x u_{1x} + \beta\phi_x + 2\gamma\phi_x u_1 + 3\delta\phi_x u_1^2\right] \\ &\qquad+ \left[u_{1t} - u_{1xx} + \alpha u_1 u_{1x} + \beta u_1 + \gamma u_1^2 + \delta u_1^3\right] = 0. \end{aligned} \tag{2.6}$$

Setting the coefficient of ϕ_x^3 in (2.6) to be zero, we obtain an ordinary differential equation for f

$$\alpha f'' f' - f''' + \delta (f')^3 = 0, \tag{2.7}$$

which has a solution

$$f(\phi) = \lambda \ln(\phi), \tag{2.8}$$

where $\lambda = (\alpha \pm \sqrt{\alpha^2 + 8\delta})/2\delta$. And then

$$(f')^2 = (-\lambda) f''. \tag{2.9}$$

By virtue of (2.7)–(2.9), (2.6) becomes

$$\begin{aligned} &u_t - u_{xx} + \alpha u u_x + \beta u + \gamma u^2 + \delta u^3 \\ &\quad= f''\left[\phi_x\phi_t - 3\phi_x\phi_{xx} + \alpha\phi_x^2 u_1 - \alpha\lambda\phi_x\phi_{xx} - \gamma\lambda\phi_x^2 - 3\delta\lambda\phi_x^2 u_1(x,t)\right] \\ &\qquad+ f'\left[\phi_{xt} - \phi_{xxx} + \alpha\phi_{xx}u_1 + \alpha\phi_x u_{1x} + \beta\phi_x + 2\gamma\phi_x u_1 + 3\delta\phi_x u_1^2\right] \\ &\qquad+ \left[u_{1t} - u_{1xx} + \alpha u_1 u_{1x} + \beta u_1 + \gamma u_1^2 + \delta u_1^3\right] = 0. \end{aligned} \tag{2.10}$$

Setting the coefficients of f'', f', f^0 to be zero, respectively, it is easy to see from (2.10) that

$$\phi_t + \left(\alpha u_1 - \gamma\lambda - 3\delta\lambda u_1\right)\phi_x - (3+\alpha\lambda)\phi_{xx} = 0, \tag{2.11}$$

$$\phi_{xt} - \phi_{xxx} + \alpha\phi_{xx}u_1 + \alpha\phi_x u_{1x} + \beta\phi_x + 2\gamma\phi_x u_1 + 3\delta\phi_x u_1^2 = 0, \tag{2.12}$$

$$u_{1t} - u_{1xx} + \alpha u_1 u_{1x} + \beta u_1 + \gamma u_1^2 + \delta u_1^3 = 0. \tag{2.13}$$

Substituting (2.8) into (2.1), we obtain a Bäcklund transformation

$$u(x,t) = \lambda\frac{\phi_x}{\phi} + u_1(x,t), \tag{2.14}$$

where $\lambda = (\alpha \pm \sqrt{\alpha^2+8\delta})/2\delta, \phi, u_1$ satisfy (2.11)–(2.13). Substituting a seed solution $u_1(x,t)$ of (1.1) into linear equations (2.11) and (2.12), then solving (2.11) and (2.12), we can get a new solution of (1.1) from (2.14). Thus we can obtain infinite solutions of (1.1) by the Bäcklund transformation (2.14) and (2.11)-(2.12) from a seed solution of (1.1).

Taking $u_1 = 0$, by (2.11)–(2.14), we obtain a transformation

$$u(x,t) = \lambda\frac{\phi_x}{\phi}, \tag{2.15}$$

that transforms (1.1) into linear equations

$$\begin{gathered}\phi_t - \gamma\lambda\phi_x - (3+\alpha\lambda)\phi_{xx} = 0,\\ \phi_t - \phi_{xx} + \beta\phi = E,\end{gathered} \tag{2.16}$$

where $\lambda = (\alpha \pm \sqrt{\alpha^2+8\delta})/2\delta$, E is an arbitrary constant.

Taking $u_1 = (-\gamma \pm \sqrt{\Delta})/2\delta$, from (2.11)–(2.14) we obtain another transformation

$$u(x,t) = \frac{-\gamma \pm \sqrt{\Delta}}{2\delta} + \lambda\frac{\phi_x}{\phi}. \tag{2.17}$$

Equation (1.1) can be solved by solving two linear equations

$$\begin{gathered}\phi_t + \left(\alpha u_1 - \gamma\lambda - 3\delta\lambda u_1\right)\phi_x - (3+\alpha\lambda)\phi_{xx} = 0,\\ \phi_{xt} - \phi_{xxx} + \alpha\phi_{xx} + \beta\phi_x + 2\gamma\phi_x u_1 + 3\delta\phi_x u_1^2 = 0,\end{gathered} \tag{2.18}$$

where $u_1 = (-\gamma \pm \sqrt{\Delta})/2\delta$, $\lambda = (\alpha \pm \sqrt{\alpha^2+8\delta})/2\delta$, $\Delta = \gamma^2 - 4\beta\delta$.

3. Exact Explicit Solutions to (1.1)

In this section we want to obtain abundant exact explicit particular solutions of (1.1) from the Bäcklund transformation (2.14) and a trivial solution of (1.1).

Noting the homogeneous property of (2.16) we can expect that ϕ in (2.16) is of the form

$$\phi(x,t) = A\sinh(kx+\omega t+\xi_0) + B\cosh(kx+\omega t+\xi_0) + C \tag{3.1}$$

with A, B, C, k, ω, and ξ_0 constants to be determined. Substituting (3.1) into (2.16), one gets a set of nonlinear algebraic equation

$$\begin{gathered} A\omega - \gamma\lambda Ak - (3+\alpha\lambda)Bk^2 = 0, \\ B\omega - \gamma\lambda Bk - (3+\alpha\lambda)Ak^2 = 0, \\ A\omega - Bk^2 + \beta B = 0, \\ B\omega - Ak^2 + \beta A = 0, \\ \beta C = E. \end{gathered} \tag{3.2}$$

Solving (3.2), we have the following.

Case 1. $A = B$, $C = E/\beta$, $\omega = k^2 - \beta$, and k is a root of second-order algebraic equation $(2+\alpha\lambda)k^2 + \gamma\lambda k + \beta = 0$.

Case 2. $A = -B$, $C = E/\beta$, $\omega = \beta - k^2$, and k is a root of second-order algebraic equation $(2+\alpha\lambda)k^2 - \gamma\lambda k + \beta = 0$.

Thus we obtain the following explicit exact solutions of (1.1) given by

$$u(x,t) = \lambda k \frac{\exp(kx+\omega t+\xi_0)}{\exp(kx+\omega t+\xi_0)+C}, \tag{3.3}$$

where $\lambda = (\alpha \pm \sqrt{\alpha^2+8\delta})/2\delta$, $\omega = k^2 - \beta$, k is a root of second-order algebraic equation $(2+\alpha\lambda)k^2 + \gamma\lambda k + \beta = 0$, $C \neq 0$, and ξ_0 are arbitrary constants.

We can also obtain the following explicit exact solutions of (1.1) given by

$$u(x,t) = \lambda k \frac{1}{C\exp(kx+\omega t+\xi_0)-1}, \tag{3.4}$$

where $\lambda = (\alpha \pm \sqrt{\alpha^2+8\delta})/2\delta$, $\omega = \beta - k^2$, k is a root of second-order algebraic equation $(2+\alpha\lambda)k^2 - \gamma\lambda k + \beta = 0$, $C \neq 0$, and ξ_0 are arbitrary constants.

By direct computation, we readily obtain the following two useful formulas:

$$\frac{\exp(\xi)}{C+\exp(\xi)} = \begin{cases} 1, & \text{for } C = 0, \\ \dfrac{1}{2}\left[\tanh\dfrac{1}{2}(\xi - \ln C) + 1\right], & \text{for } C > 0, \\ \dfrac{1}{2}\left[\coth\dfrac{1}{2}(\xi - \ln(-C)) + 1\right], & \text{for } C < 0, \end{cases} \tag{3.5}$$

$$\frac{1}{C\exp(\xi)-1}=\begin{cases}-1, & \text{for } C=0,\\ \dfrac{1}{2}\left[\coth\dfrac{1}{2}(\xi+\ln C)-1\right], & \text{for } C>0,\\ \dfrac{1}{2}\left[\tanh\dfrac{1}{2}(\xi+\ln(-C))-1\right], & \text{for } C<0,\end{cases} \tag{3.6}$$

where C is arbitrary.

Thanks to the two formulas (3.5) and (3.6), we can assert.

The solutions (3.3) ((3.4), resp.) are soliton solutions of kink type in the case of $C>0$ ($C<0$, resp.).

The solutions (3.3) ((3.4), resp.) are soliton-like solutions of singular type in the csae of $C<0$ ($C>0$, resp.).

Analogously, we assume that ϕ in (2.16) is of the form

$$\phi(x,t)=A\sin(kx+\omega t+\xi_0)+B\cos(kx+\omega t+\xi_0)+C \tag{3.7}$$

with A, B, C, k, ω, and ξ_0 constants to be determined. Substituting (3.7) into (2.16), one gets a set of nonlinear algebraic equation

$$\begin{gathered}A\omega-\gamma\lambda Ak+(3+\alpha\lambda)Bk^2=0,\\ -B\omega+\gamma\lambda Bk+(3+\alpha\lambda)Ak^2=0,\\ A\omega+Bk^2+\beta B=0,\\ -B\omega+Ak^2+\beta A=0,\\ \beta C=E.\end{gathered} \tag{3.8}$$

Solving (3.8), we have the following.

Case 1. $A=Bi$, $C=E/\beta$, $\omega=(k^2+\beta)i$, and k is a root of second order algebraic equation $(2+\alpha\lambda)k^2-\gamma\lambda ki-\beta=0$, $i=\sqrt{-1}$.

Case 2. $A=-Bi$, $C=E/\beta$, $\omega=-i(k^2+\beta)$, and k is a root of second order algebraic equation $(2+\alpha\lambda)k^2+\gamma\lambda ki-\beta=0$, $i=\sqrt{-1}$.

According to the result of Case 1, from (2.15) and (3.7), we obtain the exact explicit solutions of (1.1) given by

$$u(x,t)=\lambda ki\frac{\exp(i\xi)}{\exp(i\xi)+C}, \tag{3.9}$$

where $\lambda=(\alpha\pm\sqrt{\alpha^2+8\delta})/2\delta$, $\xi=kx+\omega t+\xi_0$, $\omega=i(k^2+\beta)$, k is a root of second-order algebraic equation $(2+\alpha\lambda)k^2-\gamma\lambda ki-\beta=0$, $i=\sqrt{-1}$.

By the result of Case 2 and (2.15), (3.7), we can obtain the following exact explicit solutions of (1.1) given by

$$u(x,t) = (-\lambda k i)\frac{1}{1 + C\exp(i\xi)}, \tag{3.10}$$

where $\lambda = (\alpha \pm \sqrt{\alpha^2 + 8\delta})/2\delta$, $\xi = kx + \omega t + \xi_0$, $\omega = (-i)(k^2 + \beta)$, k is a root of second-order algebraic equation $(2 + \alpha\lambda)k^2 + \gamma\lambda k i - \beta = 0$, $i = \sqrt{-1}$.

Analogously, we have the following two useful formulas:

$$\frac{\exp i(\xi)}{C + \exp i(\xi)} = \begin{cases} 1, & \text{for } C = 0, \\ \dfrac{1}{2}\left[i\tan\dfrac{1}{2}(\xi + i\ln C) + 1\right], & \text{for } C > 0, \\ \dfrac{1}{2}\left[-i\cot\dfrac{1}{2}(\xi + i\ln(-C)) + 1\right], & \text{for } C < 0, \end{cases} \tag{3.11}$$

$$\frac{1}{C\exp i(\xi) + 1} = \begin{cases} 1, & \text{for } C = 0, \\ \dfrac{1}{2}\left[1 - i\tan\dfrac{1}{2}(\xi - i\ln C)\right], & \text{for } C > 0, \\ \dfrac{1}{2}\left[1 + i\cot\dfrac{1}{2}(\xi - i\ln(-C))\right], & \text{for } C < 0. \end{cases} \tag{3.12}$$

Due to the formula (3.11), we have from(3.9)

$$u(x,t) = \begin{cases} -\dfrac{\lambda k}{2}\tan\left[\dfrac{1}{2}(kx + \omega t + \xi_0 + i\ln(C))\right] + \dfrac{\lambda k i}{2}, & \text{for } C > 0, \\ \dfrac{\lambda k}{2}\cot\left[\dfrac{1}{2}(kx + \omega t + \xi_0 + i\ln(-C))\right] + \dfrac{\lambda k i}{2}, & \text{for } C < 0, \end{cases} \tag{3.13}$$

where $\lambda = (\alpha \pm \sqrt{\alpha^2 + 8\delta})/2\delta$, $\xi = kx + \omega t + \xi_0$, $\omega = i(k^2 + \beta)$, k is a root of second-order algebraic equation $(2 + \alpha\lambda)k^2 - \gamma\lambda k i - \beta = 0$, $i = \sqrt{-1}$.

Owing to the formula (3.12), we have from (3.10)

$$u(x,t) = \begin{cases} -\dfrac{\lambda k}{2}\tan\left[\dfrac{1}{2}(kx + \omega t + \xi_0 - i\ln(C))\right] - \dfrac{\lambda k i}{2}, & \text{for } C > 0, \\ \dfrac{\lambda k}{2}\cot\left[\dfrac{1}{2}(kx + \omega t + \xi_0 - i\ln(-C))\right] - \dfrac{\lambda k i}{2}, & \text{for } C < 0, \end{cases} \tag{3.14}$$

where $\lambda = (\alpha \pm \sqrt{\alpha^2 + 8\delta})/2\delta$, $\xi = kx + \omega t + \xi_0$, $\omega = (-i)(k^2 + \beta)$, k is a root of second-order algebraic equation $(2 + \alpha\lambda)k^2 + \gamma\lambda k i - \beta = 0$, $i = \sqrt{-1}$.

By virtue of the homogeneous property of (2.18), we can expect that ϕ is of the linear function form

$$\phi(x,t) = kx + \omega t + \xi_0, \tag{3.15}$$

with k and ω, ξ_0 constants to be determined. Substituting (3.15) into (2.18), we find that (3.15) satisfies (2.18), provided that k and ω satisfy the following algebraic equations:

$$\begin{gathered}\omega+\left(\alpha u_1-\gamma\lambda-3\delta\lambda u_1\right)k=0,\\ \beta k+2\gamma k u_1+3\delta k u_1^2=0,\end{gathered}\tag{3.16}$$

where $u_1=(-\gamma\pm\sqrt{\Delta})/2\delta$, $\lambda=(\alpha\pm\sqrt{\alpha^2+8\delta})/2\delta$, $\Delta=\gamma^2-4\beta\delta$. Solving (3.16), we obtain that

$$\omega=\frac{\alpha\mp\sqrt{\alpha^2+8\delta}}{4\delta}\gamma k,\qquad k=\text{arbitrary constant},\qquad u_1=-\frac{\gamma}{2\delta},\tag{3.17}$$

provided that coefficients β, γ, and δ of (1.1) satisfy condition $\gamma^2=4\beta\delta$.

Substituting (3.15) with (3.17) into (2.17), we obtain the exact particular solutions of (1.1)

$$u(x,t)=-\frac{\gamma}{2\delta}+\frac{\alpha\pm\sqrt{\alpha^2+8\delta}}{2\delta}\frac{1}{x+\left(\left(\alpha\mp\sqrt{\alpha^2+8\delta}\right)/4\delta\right)\gamma t+\xi_0}.\tag{3.18}$$

Now we suppose that (2.18) has solutions of the form (3.1) substituting (3.1) into (2.18), one gets a set of algebraic equations:

$$\begin{gathered}A\omega+\left(\alpha u_1-\gamma\lambda-3\delta\lambda u_1\right)Ak-(3+\alpha\lambda)Bk^2=0,\\ B\omega+\left(\alpha u_1-\gamma\lambda-3\delta\lambda u_1\right)Bk-(3+\alpha\lambda)Ak^2=0,\\ Ak\omega-Bk^3+\alpha Ak^2+\beta Bk+2\gamma u_1 B\mathrm{k}+3\delta u_1^2 Bk=0,\\ Bk\omega-Ak^3+\alpha Bk^2+\beta Ak+2\gamma u_1 Ak+3\delta u_1^2 Ak=0.\end{gathered}\tag{3.19}$$

In order to obtain nontrivial solutions of (1.1), we need to require that k, ω are all nonzero constants. Solving (3.19), one gets the following solutions.

Case 1. One has

$$\begin{gathered}A=B,\qquad C=\text{arbitrary constant},\\ \omega=k^2-\alpha k-\beta-2\gamma u_1-3\delta u_1^2,\quad\text{or }\omega=(3+\alpha\lambda)k^2+\left(\gamma\lambda+3\delta\lambda u_1-\alpha u_1\right)k,\end{gathered}\tag{3.20}$$

where k is a root of second-order algebraic equation $(2+\alpha\lambda)k^2+(\gamma\lambda+3\delta\lambda u_1+\alpha-\alpha u_1)k+\beta+2\gamma u_1+3\delta u_1^2=0$, $u_1=(-\gamma\pm\sqrt{\Delta})/2\delta$, $\lambda=(\alpha\pm\sqrt{\alpha^2+8\delta})/2\delta$, $\Delta=\gamma^2-4\beta\delta$.

Case 2. One has

$$\begin{gathered}A=-B,\quad C=\text{arbitrary constant},\quad \omega=\beta+2\gamma u_1+3\delta u_1^2-k^2-\alpha k,\\ \text{or }\omega=\left(\gamma\lambda+3\delta\lambda u_1-\alpha u_1\right)k-(3+\alpha\lambda)k^2,\end{gathered}\tag{3.21}$$

where k is a root of second-order algebraic equation $(2+\alpha\lambda)k^2+(\alpha u_1-\gamma\lambda-3\delta\lambda u_1-\alpha)k+\beta+2\gamma u_1+3\delta u_1^2=0$, $u_1=(-\gamma\pm\sqrt{\Delta})/2\delta$, $\lambda=(\alpha\pm\sqrt{\alpha^2+8\delta})/2\delta$, $\Delta=\gamma^2-4\beta\delta$.

By Case 1, we obtain the exact solutions of the (1.1) from (2.17), (3.1)

$$u(x,t)=\frac{-\gamma\pm\sqrt{\Delta}}{2\delta}+\lambda k\frac{\exp(kx+\omega t+\xi_0)}{\exp(kx+\omega t+\xi_0)+C}, \tag{3.22}$$

where $\lambda=(\alpha\pm\sqrt{\alpha^2+8\delta})/2\delta$, $\Delta=\gamma^2-4\beta\delta$, $\omega=k^2-\alpha k-\beta-2\gamma u_1-3\delta u_1^2$, k is a root of second-order algebraic equation $(2+\alpha\lambda)k^2+(\gamma\lambda+3\delta\lambda u_1+\alpha-\alpha u_1)k+\beta+2\gamma u_1+3\delta u_1^2=0$, $u_1=(-\gamma\pm\sqrt{\Delta})/2\delta$, ξ_0, $C\neq 0$ are arbitrary constants.

According to the result of Case 2 and (2.17), (3.1), one obtain the other exact solutions

$$u(x,t)=\frac{-\gamma\pm\sqrt{\Delta}}{2\delta}+\lambda k\frac{\exp(kx+\omega t+\xi_0)}{C\exp(kx+\omega t+\xi_0)-1}, \tag{3.23}$$

where $\lambda=(\alpha\pm\sqrt{\alpha^2+8\delta})/2\delta$, $\Delta=\gamma^2-4\beta\delta$, $\omega=\beta+2\gamma u_1+3\delta u_1^2-k^2-\alpha k$, k is a root of second-order algebraic equation $(2+\alpha\lambda)k^2+(\alpha u_1-\gamma\lambda-3\delta\lambda u_1-\alpha)k+\beta+2\gamma u_1+3\delta u_1^2=0$, $u_1=(-\gamma\pm\sqrt{\Delta})/2\delta$, ξ_0, $C\neq 0$ are arbitrary constants.

According to formulas (3.5), (3.6), we can get multiple new soliton solutions of kink type and multiple new soliton-like solutions of singular type from (3.22) and (3.23).

Analogously, we assume that (2.18) has solutions of the form (3.7); substituting (3.7) into (2.18), one gets a set of algebraic equations

$$\begin{aligned}
&A\omega+\left(\alpha u_1-\gamma\lambda-3\delta\lambda u_1\right)Ak+(3+\alpha\lambda)Bk^2=0,\\
&-B\omega-\left(\alpha u_1-\gamma\lambda-3\delta\lambda u_1\right)Bk+(3+\alpha\lambda)Ak^2=0,\\
&-Ak\omega-Bk^3-\alpha Ak^2-\beta Bk-2\gamma u_1Bk-3\delta u_1^2Bk=0,\\
&-Bk\omega+Ak^3-\alpha Bk^2+\beta Ak+2\gamma u_1Ak+3\delta u_1^2Ak=0.
\end{aligned} \tag{3.24}$$

In order to obtain a nontrivial solution of (1.1), we also need to assume that k,ω are all nonzero constants. Solving (3.24), we obtain the following.

Case 1. One has

$$\begin{gathered}
A=Bi, \qquad C=\text{arbitrary constant},\\
\omega=ik^2-\alpha k+i\left(\beta+2\gamma u_1+3\delta u_1^2\right), \quad \text{or } \omega=(3+\alpha\lambda)ik^2+\left(\gamma\lambda+3\delta\lambda u_1-\alpha u_1\right)k,
\end{gathered} \tag{3.25}$$

where k is a root of second-order algebraic equation $(2+\alpha\lambda)ik^2+(\gamma\lambda+3\delta\lambda u_1+\alpha-\alpha u_1)k-i(\beta+2\gamma u_1+3\delta u_1^2)=0$, $u_1=(-\gamma\pm\sqrt{\Delta})/2\delta$, $\lambda=(\alpha\pm\sqrt{\alpha^2+8\delta})/2\delta$, $\Delta=\gamma^2-4\beta\delta$.

Case 2. One has

$$A = -Bi, \qquad C = \text{arbitrary constant},$$
$$\omega = -ik^2 - \alpha k - i\left(\beta + 2\gamma u_1 + 3\delta u_1^2\right), \quad \text{or} \quad \omega = -(3+\alpha\lambda)ik^2 + \left(\gamma\lambda + 3\delta\lambda u_1 - \alpha u_1\right)k, \tag{3.26}$$

where k is a root of second-order algebraic equation $(2+\alpha\lambda)ik^2 - (\gamma\lambda + 3\delta\lambda u_1 + \alpha - \alpha u_1)k - i(\beta + 2\gamma u_1 + 3\delta u_1^2) = 0$, $u_1 = (-\gamma \pm \sqrt{\Delta})/2\delta$, $\lambda = (\alpha \pm \sqrt{\alpha^2 + 8\delta})/2\delta$, $\Delta = \gamma^2 - 4\beta\delta$.

Collecting (2.17), (3.7), (3.25), and (3.26), we obtain the following explicit exact periodic traveling wave solutions

$$u(x,t) = \frac{-\gamma \pm \sqrt{\Delta}}{2\delta} + i\lambda k \frac{\exp(i\xi)}{\exp(i\xi) + C}, \tag{3.27}$$

where $\lambda = (\alpha \pm \sqrt{\alpha^2 + 8\delta})/2\delta$, $\Delta = \gamma^2 - 4\beta\delta$, $\xi = kx + \omega t + \xi_0$, $\omega = ik^2 - \alpha k + i(\beta + 2\gamma u_1 + 3\delta u_1^2)$, k is a root of second-order algebraic equation $(2+\alpha\lambda)ik^2 + (\gamma\lambda + 3\delta\lambda u_1 + \alpha - \alpha u_1)k - i(\beta + 2\gamma u_1 + 3\delta u_1^2) = 0$, $u_1 = (-\gamma \pm \sqrt{\Delta})/2\delta$, ξ_0, $C \neq 0$ are arbitrary constants,

$$u(x,t) = \frac{-\gamma \pm \sqrt{\Delta}}{2\delta} - i\lambda k \frac{C\exp(i\xi)}{\exp(i\xi) + 1}, \tag{3.28}$$

where $\lambda = (\alpha \pm \sqrt{\alpha^2 + 8\delta})/2\delta$, $\Delta = \gamma^2 - 4\beta\delta$, $\xi = kx + \omega t + \xi_0$, $\omega = -ik^2 - \alpha k - i(\beta + 2\gamma u_1 + 3\delta u_1^2)$, k is a root of second order algebraic equation $(2+\alpha\lambda)ik^2 - (\gamma\lambda + 3\delta\lambda u_1 + \alpha - \alpha u_1)k - i(\beta + 2\gamma u_1 + 3\delta u_1^2) = 0$, $u_1 = (-\gamma \pm \sqrt{\Delta})/2\delta$, ξ_0, $C \neq 0$ are arbitrary constants.

By using of formulas (3.11) and (3.12), we can obtain multiple new periodic wave solutions in form $\tan\xi$ and $\cot\xi$.

Choosing the solutions (3.3) ((3.4), (3.13), (3.14), (3.18), (3.22), (3.23), (3.27) and (3.28), resp.) as a new "seed" solution $u_1(x,t)$ and solving the linear PDEs (2.11), (2.12), one gets a quasisolution $\phi(x,t)$. Then substituting the quasisolution $\phi(x,t)$ and $u_1(x,t)$ chosen above into (2.14), we can obtain more and more new exact particular solutions of (1.1). Taking $C = 1$ in solutions (3.3), (3.4), (3.22), and (3.23), we can obtain shock wave solutions and singular traveling wave solutions of (1.1). Putting $C = 1$ in solutions (3.9), (3.10), (3.27), and (3.28), we can obtain periodic wave solutions in form $\tan\xi$ and $\cot\xi$.

4. Conclusion

It is worthwhile pointing out that the exact solutions obtained in this paper have more general form than some known solutions in previous studies. In addition to rederiving all known solutions in a systematic way, several entirely new exact solutions can also be obtained. Specially, choosing $\alpha = 0$ in the all solutions above, one can obtain abundant explicit and exact solutions to the Kolmogorov-Petrovskii-Piskunov equation [16]. Setting $\alpha = 0$, $\gamma = 0$, $\beta = -\delta$ in the all solutions above, one can get abundant explicit and exact solutions to the Chaffee-Infante reaction diffusion equation [17]. We can also obtain abundant explicit and exact solutions to the Burgers-Huxley equation [18] by taking $\alpha \neq 0$, $\beta = \delta\eta$, $\gamma = -(1+\eta)\delta$, η arbitrary in the all solutions above. Go a step further, taking $\alpha = 0$, $\beta = \eta$, $\gamma = -(1+\eta)$, η arbitrary in the all solutions above, we also obtain abundant explicit and exact solutions to the FitzHugh-Nagumo equation [19]. We can obtain abundant explicit exact solutions to the

Newell-Whitehead equation when taking $\alpha = 0$, $\beta = -1$, $\gamma = 0$, $\delta = 1$ in the all solutions above [17]. Putting $\alpha = 0$, $\beta = 1 - (3\eta/2)$, $\gamma = (5\eta/2) - 2$, $\delta = 1 - \eta$ in the all solutions above, we can obtain abundant explicit and exact solutions to an isothermal autocatalytic system [20].

Acknowledgments

This work is supported by the National Science Foundation of China (10771041, 40890150, 40890153), the Scientific Program (2008B080701042) of Guangdong Province, China. The authors would like to thank Professor Wang Mingliang for his helpful suggestions.

References

[1] M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, vol. 149, Cambridge University Press, Cambridge, UK, 1991.

[2] B. Grammaticos, A. Ramani, and J. Hietarinta, "A search for integrable bilinear equations: the Painlevé approach," *Journal of Mathematical Physics*, vol. 31, no. 11, pp. 2572–2578, 1990.

[3] G. B. Whitham, "Comments on periodic waves and solitons," *IMA Journal of Applied Mathematics*, vol. 32, no. 1–3, pp. 353–366, 1984.

[4] G. B. Whitham, "On shocks and solitary waves," *Scripps Institution of Oceanography Reference Series*, pp. 91–124, 1991.

[5] J. Weiss, M. Tabor, and G. Carnevale, "The Painlevé property for partial differential equations," *Journal of Mathematical Physics*, vol. 24, no. 3, pp. 522–526, 1983.

[6] J. D. Gibbon, A. C. Newell, M. Tabor, and Y. B. Zeng, "Lax pairs, Bäcklund transformations and special solutions for ordinary differential equations," *Nonlinearity*, vol. 1, no. 3, pp. 481–490, 1988.

[7] J. Weiss, "Bäcklund transformation and the Painlevé property," in *Partially Integrable Evolution Equations in Physics*, R. Conte and N. Boccara, Eds., Kluwer Academic Publishers, Dordrecht, The Netherlands, 1990.

[8] J. Hietarinta, "Hirota's bilinear method and partial integrability," in *Partially Integrable Evolution Equations in Physics*, R. Conte and N. Boccara, Eds., Kluwer Academic Publishers, Dordrecht, The Netherlands, 1990.

[9] N. G. Berloff and L. N. Howard, "Solitary and periodic solutions of nonlinear nonintegrable equations," *Studies in Applied Mathematics*, vol. 99, no. 1, pp. 1–24, 1997.

[10] M. Wang, Y. Zhou, and Z. Li, "Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics," *Physics Letters A*, vol. 216, no. 1–5, pp. 67–75, 1996.

[11] M. L. Wang, "Exact solutions for a compound KdV-Burgers equation," *Physics Letters A*, vol. 213, no. 5-6, pp. 279–287, 1996.

[12] M. L. Wang, Y. B. Zhou, and H. Q. Zhang, "A nonlinear transformation of the shallow water wave equations and its application," *Advances in Mathematics*, vol. 28, no. 1, pp. 72–75, 1999.

[13] E. G. Fan and H. Q. Zhang, "A new approach to Bäcklund transformations of nonlinear evolution equations," *Applied Mathematics and Mechanics*, vol. 19, no. 7, pp. 645–650, 1998.

[14] Y. D. Shang, "Bäcklund transformation, Lax pairs and explicit exact solutions for the shallow water waves equation," *Applied Mathematics and Computation*, vol. 187, no. 2, pp. 1286–1297, 2007.

[15] Y. D. Shang, J. Qin, Y. Huang, and W. Yuan, "Abundant exact and explicit solitary wave and periodic wave solutions to the Sharma-Tasso-Olver equation," *Applied Mathematics and Computation*, vol. 202, no. 2, pp. 532–538, 2008.

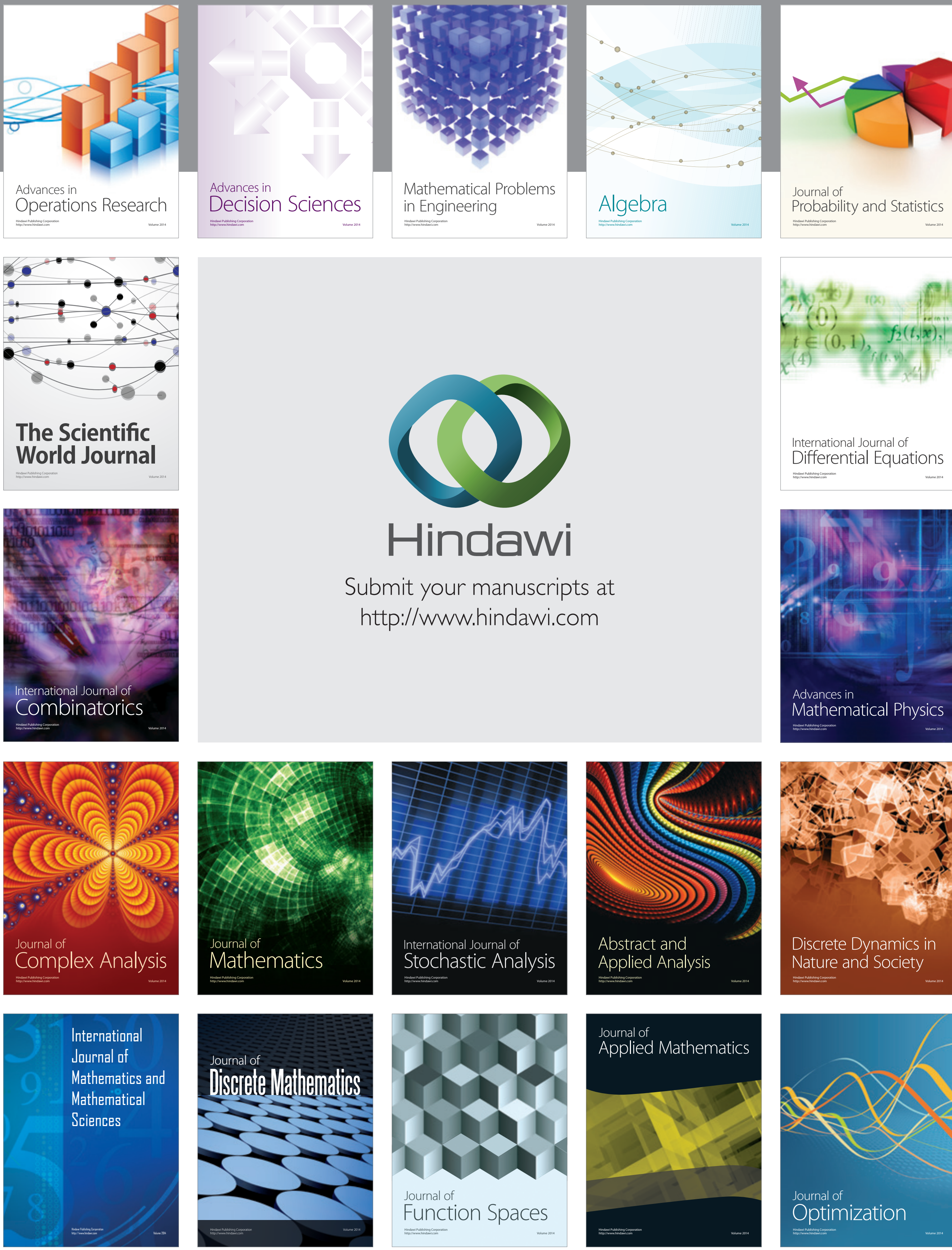
[16] C. Roman and P. Oleksii, "New conditional symmetries and exact solutions of nonlinear reaction-diffusion-convection equations," *Journal of Physics A*, vol. 40, no. 33, pp. 10049–10070, 2007.

[17] J. F. Zhang, "Exact and explicit solitary wave solutions to some nonlinear equations," *International Journal of Theoretical Physics*, vol. 35, no. 8, pp. 1793–1798, 1996.

[18] A. M. Wazwaz, "Travelling wave solutions of generalized forms of Burgers, Burgers-KdV and Burgers-Huxley equations," *Applied Mathematics and Computation*, vol. 169, no. 1, pp. 639–656, 2005.

[19] R. Cherniha, "New Q symmetries and exact solutions of some reaction-diffusion-convection equations arising in mathematical biology," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 2, pp. 783–799, 2007.

[20] J. H. Merkin, R. A. Satnoianu, and S. K. Scott, "Travelling waves in a differential flow reactor with simple autocatalytic kinetics," *Journal of Engineering Mathematics*, vol. 33, no. 2, pp. 157–174, 1998.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