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Research Article

On the Convergence of Absolute Summability for Functions of Bounded Variation in Two Variables

Ying Mei[1] and Dansheng Yu[2]

[1] *Department of Mathematics, Lishui University, Lishui, Zhejiang 323000, China*
[2] *Department of Mathematics, Hangzhou Normal University, Hangzhou, Zhejiang 310036, China*

Correspondence should be addressed to Dansheng Yu, danshengyu@yahoo.com.cn

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By adopting some new ideas, we obtain the estimates of an absolute convergence for the functions of the bounded variation in two variables. Our results generalize the related results of Humphreys and Bojanic (1999) and Wang and Yu (2003) from one dimension to two dimensions and can be applied to several summability methods.

1. Introduction

Let f be 2π-periodic functions integrable on $[-\pi,\pi]$. Denote its Fourier series by

$$S(f,x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx), \tag{1.1}$$

where the Fourier coefficients a_k and b_k are defined as follows:

$$a_k := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt, \qquad b_k := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,dt. \tag{1.2}$$

Denoted by $S_n(f,x)$ the nth partial sums of the Fourier series (1.1), that is,

$$S_n(f,x) := \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx). \tag{1.3}$$

When f is of bounded variation, Bojanić [1] obtained the following result on the rate of the convergence of approximation by $S_n(f,x)$.

Theorem A. *If a periodic function f is of bounded variation on the interval $[-\pi,\pi]$, then the following estimate holds for every x and $n = 1,2,\ldots$:*

$$\left|S_n(f,x) - f(x)\right| \leq \frac{3}{n}\sum_{k=1}^{n} V_0^{\pi/k}(\varphi_x), \tag{1.4}$$

where $\varphi_x(t) := f(x+t) + f(x-t) - f(x+0) - f(x-0)$ and $V(\varphi_x,[0,t])$ is the total variation of φ_x on $[0,t]$.

There are a lot of interesting generalizations that have been achieved by many authors (see [1–9]). Among them, Jenei [4] and Móricz [5] generalized Theorem A to the double Fourier series of functions of bounded variation; Humphreys and Bojanic [3], Wang and Yu [7] investigated the absolute convergence of Cesàro means of Fourier series.

Let $C_\alpha := (c_{nk}^{\alpha})$ be the Cesàro matrix of order α, that is,

$$c_{nk}^{\alpha} := \frac{A_{n-k}^{\alpha-1}}{A_n^{\alpha}}, \quad k = 0,1,\ldots,n, \tag{1.5}$$

where

$$A_n^{\alpha} := \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)}, \quad n = 0,1,\ldots. \tag{1.6}$$

Then, the so-called Cesàro means of order α of (1.1) is defined by (denote by $\sigma_n^{\alpha}(f,x)$):

$$\sigma_n^{\alpha}(f,x) = \frac{1}{A_n^{\alpha}}\sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_k(f,x). \tag{1.7}$$

Humphreys and Bojanic [3] investigated the rate of the absolute convergence of Cesàro means of the series (1.1). Their result can be read as follows.

Theorem B. *Let $x \in [0,\pi]$ and f be 2π-periodic functions of bounded variation on $[-\pi,\pi]$. Then for $\alpha > 0$ and $n \geq 2$, one has*

$$R_n^{\alpha}(f,x) \leq \frac{4\alpha}{n\pi}\sum_{k=1}^{n} V_0^{\pi/k}(\varphi_x), \tag{1.8}$$

where

$$\begin{gathered} R_n^{\alpha}(f,x) := \sum_{k=n+1}^{\infty}\left|\sigma_k^{\alpha}(f,x) - \sigma_{k-1}^{\alpha}(f,x)\right|, \\ \varphi_x(t) := f(x+t) + f(x-t) - f(x+0) - f(x-0) \end{gathered} \tag{1.9}$$

and $\operatorname{var}_a^b(f)$ be the total variation of f on $[a,b]$.

However, Wang and Yu [7] showed that Theorem A is not correct when $0 < \alpha < 1$. In fact, they proved the following.

Theorem C. *Suppose* $0 < \alpha < 1$, $x \in [0,\pi]$ *and* f *are* 2π*-periodic functions of bounded variation on* $[-\pi,\pi]$. *Then for* $n \geq 2$, *one has*

$$R_n^\alpha(f,x) \leq \frac{100}{\alpha^2 n^\alpha}\sum_{k=1}^{n} k^{\alpha-1} V_0^{\pi/k}(\varphi_x), \tag{1.10}$$

and there exists a 2π*-periodic function* f^* *of bounded variation on* $[-\pi,\pi]$ *and a point* $x \in [0,\pi]$ *such that*

$$R_n^\alpha(f^*,x) > \frac{1}{20000\alpha n^\alpha}\sum_{k=1}^{n} k^{\alpha-1} V_0^{\pi/k}(\varphi_x) \quad (n \geq 8). \tag{1.11}$$

Motivated by Theorems B and C, and the results of Jenei [4] and Móricz [5] on the double Fourier series, we will investigate the absolute convergence of a kind of very general summability of the double Fourier series. Our new results not only generalize Theorems B and C to the double case, but also can be applied to many other classical summability methods. We will present our main results on Section 2. Proofs will be given in Section 3. In Section 4, we will apply our results to some classical summability methods.

2. The Main Results

Let $f(x,y)$ be a function periodic in each variable with period 2π and integrable on the two-dimensional torus $\mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$ in Lebesgue's sense, in symbol, $f \in L(\mathbb{T}^2)$. The double Fourier series of a complex-valued function $f \in L(\mathbb{T}^2)$ is defined by

$$f(x,y) \sim \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \widehat{f}(k,l)e^{i(kx+ly)}, \tag{2.1}$$

where the $\widehat{f}(k,l)$ are the Fourier coefficients of f:

$$\widehat{f}(k,l) := \frac{1}{4\pi^2}\int_{\mathbb{T}^2} f(u,v)e^{-i(ku+lv)}du\,dv, \quad (k,l) \in \mathbb{Z}^2. \tag{2.2}$$

Let $R := [a_1,b_1] \times [a_2,b_2]$ be a bounded and closed rectangle on the plane. A function $f(x,y)$ defined on R is said to be of bounded variation over R in the sense of Hardy-Krause, in symbol, $f(x,y) \in BV_H(R)$, if

(i) the total variation $V(f,R)$ of $f(x,y)$ over R is finite, that is,

$$V(f,R) := \sup \sum_{j=1}^{m}\sum_{k=1}^{n} \left| f(x_j,y_k) - f(x_{j-1},y_k) - f(x_j,y_{k-1}) + f(x_j,y_k) \right| < \infty, \tag{2.3}$$

where the supremum is extended for all finite partitions

$$a_1 = x_0 < x_1 < \cdots < x_m = b_1, \qquad a_2 = y_0 < y_1 < \cdots < y_n = b_2 \tag{2.4}$$

of the intervals $[a_1, b_1]$ and $[a_2, b_2]$;

(ii) the marginal functions $f(\cdot, a_2)$ and $f(a_1, \cdot)$ are of bounded variation over the intervals $[a_1, b_1]$ and $[a_2, b_2]$, respectively.

For any $f(x, y) \in BV_H(\mathbb{T})$, define

$$\begin{gathered} f(x,y) := \frac{1}{4}\left(f(x+0,y+0) + f(x-0,y+0) + f(x+0,y-0) + f(x-0,y-0)\right), \\ \varphi_{xy}(u,v) := f(x+u,y+v) + f(x-u,y+v) + f(x+u,y-v) + f(x-u,y-v) - 4f(x,y). \end{gathered} \tag{2.5}$$

For convenience, write

$$V_{00}^{st}(\varphi_{xy}) := V(\varphi_{xy}, [0,s] \times [0,t]). \tag{2.6}$$

For any double sequence $\{a_{mn}\}$, define

$$\begin{gathered} \Delta_{11} a_{mn} := a_{mn} - a_{m-1,n} - a_{m,n-1} + a_{m-1,n-1}, \\ \Delta_{10} a_{mn} := a_{mn} - a_{m-1,n}, \\ \Delta_{01} a_{mn} := a_{mn} - a_{m,n-1}. \end{gathered} \tag{2.7}$$

For any fourfold sequence $\{a_{mnjk}\}$, write

$$\begin{gathered} \Delta_{11} a_{mnjk} := a_{mnjk} - a_{m,n,j+1,k} - a_{m,n,j,k+1} + a_{m,n,j+1,k+1}, \\ \Delta_{01} a_{mnjk} := a_{mnjk} - a_{m,n,j,k+1}, \;\; \Delta_{10} a_{mnjk} := a_{m,n,j,k} - a_{m,n,j+1,k}. \end{gathered} \tag{2.8}$$

A doubly infinite matrix $T := (t_{mnjk})$ is said to be doubly triangular if $t_{mnjk} = 0$ for $j > m$ or $k > n$. The mnth term of the T-transform of the double Fourier series (2.1) is defined by

$$T_{mn}(x,y) := \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} t_{mn\mu\nu} S_{\mu\nu}(x,y), \tag{2.9}$$

where $S_{\mu\nu}(x, y)$ is the $\mu\nu$th partial sum of (2.1), that is,

$$S_{\mu\nu}(x,y) := \sum_{|k|=0}^{\mu} \sum_{|l|=0}^{\nu} \widehat{f}(k,l) e^{i(kx+ly)} := \sum_{|k|=0}^{\mu} \sum_{|l|=0}^{\nu} A_{kl}(x,y). \tag{2.10}$$

Write

$$\bar{t}_{mnjk} := \sum_{\mu=j}^{m}\sum_{\nu=k}^{n} t_{mnjk}, \quad 0 \le j \le m,\ 0 \le k \le n,$$
$$\bar{t}_{mnjk} = 0, \quad \text{if } j > m \text{ or } k > n, \tag{2.11}$$
$$\widehat{t}^{(1,1)}_{mnjk} := \bar{t}_{mnij} - \bar{t}_{m-1,n,i,j} - \bar{t}_{m,n-1,i,j} + \bar{t}_{m-1,n-1,i,j}.$$

Set

$$R^{(1,1)}_{mn}(f;x,y) := \sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty} \left|\Delta_{11}T_{jk}(x,y)\right|. \tag{2.12}$$

Our main results are the following Theorems 2.1 and 2.2.

Theorem 2.1. *Let $f(x,y)$ be a periodic function and $f(x,y) \in BV_H(\mathbb{T}^2)$. Assume that T is a lower doubly triangular matrix satisfying*

$$\widehat{t}^{(1,1)}_{mnk0} = \widehat{t}^{(1,1)}_{mn0l} = 0, \quad k,l = 0,1,\ldots, \tag{2.13}$$

$$\sum_{k=1}^{m}\sum_{l=1}^{n}\left|\widehat{t}^{(1,1)}_{mnkl}\right| = O(1), \tag{2.14}$$

and there exist constants α, β $(0 < \alpha, \beta \le 1)$ such that

$$\left|\sum_{k=1}^{m}\sum_{l=1}^{n}\frac{\widehat{t}^{(1,1)}_{mnkl}}{kl}\sin kx \sin ly\right| = \begin{cases} O\left(l^{-1-\beta}xy^{-\beta}\right), & (x,y) \in \left[0,\dfrac{\pi}{m}\right] \times \left[\dfrac{\pi}{n},\pi\right], \\ O\left(k^{-1-\alpha}x^{-\alpha}y\right), & (x,y) \in \left[\dfrac{\pi}{m},\pi\right] \times \left[0,\dfrac{\pi}{n}\right], \\ O\left(k^{-1-\alpha}l^{-1-\beta}x^{-\alpha}y^{-\beta}\right), & (x,y) \in \left[\dfrac{\pi}{m},\pi\right] \times \left[\dfrac{\pi}{n},\pi\right]. \end{cases} \tag{2.15}$$

Then for $n \ge 2$, $(x,y) \in \mathbb{T}^2$, one has

$$R^{(1,1)}_{mn}(f;x,y) \le C\frac{1}{m^{\alpha}n^{\beta}}\sum_{k=1}^{m}\sum_{l=1}^{n} k^{\alpha-1}l^{\beta-1}V_{00}^{(\pi/k)(\pi/l)}(\varphi_{xy}). \tag{2.16}$$

Theorem 2.2. *Let $f(x,y)$ and T satisfy all the conditions of Theorem 2.1, except* (2.15) *is replaced by*

$$\sum_{k=1}^{m}\sum_{l=1}^{n}\left|\Delta_{k^*l^*}\left(\frac{\widehat{t}^{(1,1)}_{mnkl}}{k^{\sigma(k^*)}l^{\sigma(l^*)}}\right)\right| = O\left(m^{\tau(k^*)}n^{\tau(l^*)}\right), \tag{2.17}$$

where $k^*, l^* = 0$ *or* 1, $k^* + l^* \geq 1$, $\tau(0) = \sigma(0) = 0$, $\tau(1) = \sigma(1) + 1 = 2$. *Then for* $n \geq 2$, $(x, y) \in \mathbb{T}^2$, *one has*

$$R_{mn}^{(1,1)}(f; x, y) \leq C\frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n} V_{00}^{(\pi/k)(\pi/l)}(\varphi_{xy}). \tag{2.18}$$

3. Proofs of Results

Lemma 3.1 (see [10]). *If* $g(x, y)$ *is continuous on a rectangle* $R := [a_1, b_1] \times [a_2, b_2]$ *and* $f(x, y) \in BV_H(R)$, *then*

(i) $g(x, y)$ *is integrable with respect to* $f(x, y)$ *over* R *in the sense of Riemann-Stieltjes integral;*

(ii) $f(x, y)$ *is integrable with respect to* $g(x, y)$ *over* R *in the sense of Riemann-Stieltjes integral and*

$$\begin{aligned}\int_R f(x,y)d_x d_y g(x,y) = &\int_R g(x,y)d_x d_y f(x,y)\\ &+ \int_{a_1}^{b_1} f(x,b_2)d_x g(x,b_2) - \int_{a_1}^{b_1} f(x,a_2)d_x g(x,a_2)\\ &+ \int_{a_2}^{b_2} f(b_1,y)d_y g(b_1,y) - \int_{a_2}^{b_2} f(a_1,y)d_y g(a_1,y)\\ &- f(b_1,b_2)g(b_1,b_2) + f(a_1,b_2)g(a_1,b_2)\\ &+ f(a_2,b_1)g(a_2,b_1) - f(a_1,a_2)g(a_1,a_2).\end{aligned} \tag{3.1}$$

Lemma 3.2 (see [10]). *If* $f(x, y) \in BV_H(R)$ *and* $g(x, y)$ *is absolutely continuous on* R, *that is,*

$$g(x,y) = \int_{a_1}^{x}\int_{a_2}^{y} h(u,v)du\,dv + C, \quad (x,y) \in R, \tag{3.2}$$

for some function $h(x, y) \in L(R)$ *and constant* C, *then*

$$\int_R f(x,y)d_x d_y g(x,y) = \int_R f(x,y)h(x,y)dx\,dy. \tag{3.3}$$

Proof of Theorem 2.1. By the definition of $T_{mn}(x, y)$ (see (2.9)), we have

$$\begin{aligned}T_{jk}(x,y) &= \sum_{i_1=0}^{j}\sum_{i_2=0}^{k} t_{jki_1i_2}\sum_{|\mu|=0}^{i_1}\sum_{|\nu|=0}^{i_2} A_{\mu\nu}(x,y)\\ &= \sum_{|\mu|=0}^{j}\sum_{|\nu|=0}^{k} A_{\mu\nu}(x,y)\bar{t}_{jk|\mu||\nu|}.\end{aligned} \tag{3.4}$$

Therefore,

$$\begin{aligned}
\Delta_{11}T_{jk}(x,y) &= \sum_{|\mu|=0}^{j}\sum_{|\nu|=0}^{k}A_{\mu\nu}(x,y)\bar{t}_{jk|\mu||\nu|} - \sum_{|\mu|=0}^{j-1}\sum_{|\nu|=0}^{k}A_{\mu\nu}(x,y)\bar{t}_{j-1,k,|\mu|,|\nu|} \\
&\quad - \sum_{|\mu|=0}^{j}\sum_{|\nu|=0}^{k-1}A_{\mu\nu}(x,y)\bar{t}_{j,k-1,|\mu|,|\nu|} + \sum_{|\mu|=0}^{j-1}\sum_{|\nu|=0}^{k-1}A_{\mu\nu}(x,y)\bar{t}_{j-1,k-1,|\mu|,|\nu|} \\
&= \sum_{|\mu|=0}^{j}\sum_{|\nu|=0}^{k}A_{\mu\nu}(x,y)\hat{t}^{(1,1)}_{jk|\mu||\nu|} \quad (\text{by } (2.11)) \\
&= \sum_{\mu=1}^{j}\sum_{\nu=1}^{k}\left(A_{\mu\nu}(x,y)+A_{-\mu,\nu}(x,y)+A_{\mu,-\nu}(x,y)+A_{-\mu,-\nu}(x,y)\right)\hat{t}^{(1,1)}_{jk\mu\nu} \quad (\text{by } (2.13)).
\end{aligned} \tag{3.5}$$

Noting that

$$\begin{aligned}
&A_{\mu\nu}(x,y)+A_{-\mu,\nu}(x,y)+A_{\mu,-\nu}(x,y)+A_{-\mu,-\nu}(x,y) \\
&\quad = \hat{f}(\mu,\nu)e^{i(\mu x+\nu y)}+\hat{f}(-\mu,\nu)e^{i(-\mu x+\nu y)}+\hat{f}(\mu,-\nu)e^{i(\mu x-\nu y)}+\hat{f}(-\mu,-\nu)e^{i(-\mu x-\nu y)} \\
&\quad = \frac{1}{4\pi^2}\int_{\mathbb{T}^2} f(s,t)\cos\mu(s-x)\cos\nu(t-y)\,ds\,dt,
\end{aligned} \tag{3.6}$$

we have

$$\begin{aligned}
\Delta_{11}T_{jk}(x,y) &= \frac{1}{4\pi^2}\sum_{\mu=1}^{j}\sum_{\nu=1}^{k}\hat{t}^{(1,1)}_{jk\mu\nu}\int_{\mathbb{T}^2} f(s,t)\cos\mu(s-x)\cos\nu(t-y)\,ds\,dt \\
&= \frac{1}{4\pi^2}\sum_{\mu=1}^{j}\sum_{\nu=1}^{k}\hat{t}^{(1,1)}_{jk\mu\nu}\iint_0^{\pi}\varphi_{xy}(s,t)\cos\mu s\cos\nu t\,ds\,dt \\
&= \frac{1}{4\pi^2}\sum_{\mu=1}^{j}\sum_{\nu=1}^{k}\frac{\hat{t}^{(1,1)}_{jk\mu\nu}}{\mu\nu}\iint_0^{\pi}\varphi_{xy}(s,t)d_s d_t \sin\mu s\sin\nu t,
\end{aligned} \tag{3.7}$$

where in the last equality, Lemma 3.2 is applied.

By (3.7) and Lemma 3.1, we have

$$\begin{aligned}
R^{(1,1)}_{mn}(f;x,y) &= \frac{1}{4\pi^2}\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}\left|\sum_{\mu=1}^{j}\sum_{\nu=1}^{k}\frac{\hat{t}^{(1,1)}_{jk\mu\nu}}{\mu\nu}\iint_0^{\pi}\varphi_{xy}(s,t)d_s d_t \sin\mu s\sin\nu t\right| \\
&= \frac{1}{4\pi^2}\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}\left|\sum_{\mu=1}^{j}\sum_{\nu=1}^{k}\frac{\hat{t}^{(1,1)}_{jk\mu\nu}}{\mu\nu}\iint_0^{\pi}\sin\mu s\sin\nu t\,d_s d_t\varphi_{xy}(s,t)\right|
\end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{4\pi^2}\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}\iint_0^{\pi}\left|\sum_{\mu=1}^{j}\sum_{\nu=1}^{k}\frac{\widehat{t}_{jk\mu\nu}^{(1,1)}}{\mu\nu}\sin\mu s\sin\nu t\right|d_sd_tV_{00}^{st}(\varphi_{xy})\\
&= \frac{1}{4\pi^2}\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}\left(\int_0^{\pi/j}\int_0^{\pi/k}+\int_0^{\pi/j}\int_{\pi/k}^{\pi}+\int_{\pi/j}^{\pi}\int_0^{\pi/k}+\int_{\pi/j}^{\pi}\int_{\pi/k}^{\pi}\right)\\
&\quad\times\left|\sum_{\mu=1}^{j}\sum_{\nu=1}^{k}\frac{\widehat{t}_{jk\mu\nu}^{(1,1)}}{\mu\nu}\sin\mu s\sin\nu t\right|d_sd_tV_{00}^{st}(\varphi_{xy})\\
&=: I_1+I_2+I_3+I_4.
\end{aligned} \tag{3.8}$$

By (2.14), we have (Denote by $\chi_{(a_1,b_1)\times(a_2,b_2)}(x,y)$ and $\chi_{(a_1,b_1)}(x)$ the characteristic functions of $(a_1,b_1)\times(a_2,b_2)$ and (a_1,b_1), respectively.)

$$\begin{aligned}
I_1 &= \frac{1}{4\pi^2}\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}\int_0^{\pi/j}\int_0^{\pi/k}\left|\sum_{\mu=1}^{j}\sum_{\nu=1}^{k}\frac{\widehat{t}_{jk\mu\nu}}{\mu\nu}\sin\mu s\sin\nu t\right|d_sd_tV_{00}^{st}(\varphi_{xy})\\
&\le \frac{1}{4\pi^2}\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}\int_0^{\pi/j}\int_0^{\pi/k}\sum_{\mu=1}^{j}\sum_{\nu=1}^{k}\left|\widehat{t}_{jk\mu\nu}\right|st\,d_sd_tV_{00}^{st}(\varphi_{xy})\\
&\le C\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}\int_0^{\pi/j}\int_0^{\pi/k}st\,d_sd_tV_{00}^{st}(\varphi_{xy})\\
&\le C\int_0^{\pi/(m+1)}\int_0^{\pi/(n+1)}\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}st\chi_{(0,\pi/j)\times(0,\pi/k)}(s,t)d_sd_tV_{00}^{st}(\varphi_{xy})\\
&\le C\int_0^{\pi/(m+1)}\int_0^{\pi/(n+1)}d_sd_tV_{00}^{st}(\varphi_{xy})\\
&\le CV_{00}^{(\pi/(m+1))(\pi/(n+1))}(\varphi_{xy})\\
&\le \frac{C}{m^{\alpha}n^{\beta}}\sum_{j=1}^{m}\sum_{k=1}^{n}j^{\alpha-1}k^{\beta-1}V_{00}^{(\pi/j)(\pi/k)}(\varphi_{xy}).
\end{aligned} \tag{3.9}$$

For I_2, by the first inequality of (2.15),

$$\begin{aligned}
I_2 &= \frac{1}{4\pi^2}\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}\int_0^{\pi/j}\int_{\pi/k}^{\pi}\left|\sum_{\mu=1}^{j}\sum_{\nu=1}^{k}\frac{\widehat{t}_{jk\mu\nu}}{\mu\nu}\sin\mu s\sin\nu t\right|d_sd_tV_{00}^{st}(\varphi_{xy})\\
&\le C\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}k^{-1-\beta}\int_0^{\pi/j}\int_{\pi/k}^{\pi}st^{-\beta}d_sd_tV_{00}^{st}(\varphi_{xy})
\end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}k^{-1-\beta}\left(\int_0^{\pi/j}\int_{\pi/k}^{\pi/(n+1)}+\int_0^{\pi/j}\int_{\pi/(n+1)}^{\pi}\right)st^{-\beta}d_sd_tV_{00}^{st}(\varphi_{xy})\\
&\le C\int_0^{\pi/(m+1)}\int_0^{\pi/(n+1)}\sum_{j=m+1}^{\infty}s\chi_{(0,\pi/j)}(s)\sum_{k>(\pi/t)}^{\infty}k^{-1-\beta}t^{-\beta}d_sd_tV_{00}^{st}(\varphi_{xy})\\
&\quad+C\int_0^{\pi/(m+1)}\int_{\pi/(n+1)}^{\pi}\sum_{j=m+1}^{\infty}s\chi_{(0,\pi/j)}(s)\sum_{k=n+1}^{\infty}k^{-1-\beta}t^{-\beta}d_sd_tV_{00}^{st}(\varphi_{xy})\\
&\le C\int_0^{\pi/(m+1)}\int_0^{\pi/(n+1)}d_sd_tV_{00}^{st}(\varphi_{xy})+Cn^{-\beta}\int_0^{\pi/(m+1)}\int_{\pi/(n+1)}^{\pi}t^{-\beta}d_sd_tV_{00}^{st}(\varphi_{xy})\\
&\le CV_{00}^{(\pi/(m+1))(\pi/(n+1))}(\varphi_{xy})+Cn^{-\beta}\sum_{k=1}^{n}\int_0^{\pi/(m+1)}\int_{\pi/(k+1)}^{\pi/k}t^{-\beta}d_sd_tV_{00}^{st}(\varphi_{xy})\\
&\le CV_{00}^{(\pi/(m+1))(\pi/(n+1))}(\varphi_{xy})+Cn^{-\beta}\sum_{k=1}^{n}k^{\beta-1}V_{00}^{(\pi/(m+1))(\pi/k)}(\varphi_{xy})\\
&\le\frac{C}{m^{\alpha}n^{\beta}}\sum_{j=1}^{m}\sum_{k=1}^{n}j^{\alpha-1}k^{\beta-1}V_{00}^{(\pi/j)(\pi/k)}(\varphi_{xy}).
\end{aligned}
\tag{3.10}
$$

Similarly, we also have

$$
I_3\le\frac{C}{m^{\alpha}n^{\beta}}\sum_{j=1}^{m}\sum_{k=1}^{n}j^{\alpha-1}k^{\beta-1}V_{00}^{(\pi/j)(\pi/k)}(\varphi_{xy}).
\tag{3.11}
$$

By (2.15), we deduce that

$$
\begin{aligned}
I_4&\le C\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}\int_{\pi/j}^{\pi}\int_{\pi/k}^{\pi}j^{-1-\alpha}k^{-1-\beta}s^{-\alpha}t^{-\beta}d_sd_tV_{00}^{st}(\varphi_{xy})\\
&\le C\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}\left(\int_{\pi/j}^{\pi/(m+1)}\int_{\pi/k}^{\pi/(n+1)}+\int_{\pi/(m+1)}^{\pi}\int_{\pi/k}^{\pi/(n+1)}+\int_{\pi/j}^{\pi/(m+1)}\int_{\pi/(n+1)}^{\pi}+\int_{\pi/(m+1)}^{\pi}\int_{\pi/(n+1)}^{\pi}\right)\\
&\quad\times j^{-1-\alpha}k^{-1-\beta}s^{-\alpha}t^{-\beta}d_sd_tV_{00}^{st}(\varphi_{xy})\\
&\le C\int_0^{\pi/(m+1)}\int_0^{\pi/(n+1)}\left(\sum_{j>(\pi/s)}^{\infty}j^{-1-\alpha}s^{-\alpha}\right)\left(\sum_{k>(\pi/t)}^{\infty}k^{-1-\beta}t^{-\beta}\right)d_sd_tV_{00}^{st}(\varphi_{xy})\\
&\quad+C\int_0^{\pi/(m+1)}\int_{\pi/(n+1)}^{\pi}\left(\sum_{j>(\pi/s)}^{\infty}j^{-1-\alpha}s^{-\alpha}\right)\left(\sum_{k=n+1}^{\infty}k^{-1-\beta}t^{-\beta}\right)d_sd_tV_{00}^{st}(\varphi_{xy})
\end{aligned}
$$

$$
\begin{aligned}
&+C\int_{\pi/(m+1)}^{\pi}\int_{0}^{\pi/(n+1)}\left(\sum_{j=m+1}^{\infty}j^{-1-\alpha}s^{-\alpha}\right)\left(\sum_{k>(\pi/t)}^{\infty}k^{-1-\beta}t^{-\beta}\right)d_s d_t V_{00}^{st}(\varphi_{xy})\\
&+C\int_{\pi/(m+1)}^{\pi}\int_{\pi/(n+1)}^{\pi}\left(\sum_{j=m+1}^{\infty}j^{-1-\alpha}s^{-\alpha}\right)\left(\sum_{k=n+1}^{\infty}k^{-1-\beta}t^{-\beta}\right)d_s d_t V_{00}^{st}(\varphi_{xy})\\
&\leq CV_{00}^{(\pi/(m+1))(\pi/(n+1))}(\varphi_{xy})+Cn^{-\beta}\sum_{k=1}^{n}k^{\beta-1}V_{00}^{(\pi/(m+1))(\pi/k)}(\varphi_{xy})\\
&+Cm^{-\alpha}\sum_{j=1}^{m}j^{\alpha-1}V_{00}^{(\pi/j)(\pi/(n+1))}(\varphi_{xy})\\
&+\frac{C}{m^{\alpha}n^{\beta}}\sum_{j=1}^{m}\sum_{k=1}^{n}j^{\alpha-1}k^{\beta-1}V_{00}^{(\pi/j)(\pi/k)}(\varphi_{xy})\\
&\leq C\frac{C}{m^{\alpha}n^{\beta}}\sum_{j=1}^{m}\sum_{k=1}^{n}j^{\alpha-1}k^{\beta-1}V_{00}^{(\pi/(m+1))(\pi/(n+1))}(\varphi_{xy}).
\end{aligned}
\tag{3.12}
$$

We get (2.16) by combining (3.8)–(3.12). □

Proof of Theorem 2.2. Set

$$
\widetilde{D}_i(t)=\sum_{j=1}^{i}\sin it=\frac{\cos(1/2)t-\cos(i+(1/2))t}{2\sin(1/2)t},\quad i=1,2,\ldots. \tag{3.13}
$$

By Abel's transformation, we have

$$
\begin{aligned}
\sum_{k=1}^{m}\sum_{l=1}^{n}\frac{\widehat{t}_{mnkl}^{(1,1)}}{kl}\sin ks\sin lt&=\sum_{k=1}^{m}\frac{\sin ks}{k}\sum_{l=1}^{n}\Delta_{01}\frac{\widehat{t}_{mnkl}^{(1,1)}}{l}\widetilde{D}_l(t)\\
&=\sum_{l=1}^{n}\frac{\sin lt}{l}\sum_{k=1}^{m}\Delta_{10}\frac{\widehat{t}_{mnkl}^{(1,1)}}{k}\widetilde{D}_k(s)\\
&=\sum_{k=1}^{m}\sum_{l=1}^{n}\Delta_{11}\frac{\widehat{t}_{mnkl}^{(1,1)}}{kl}\widetilde{D}_k(s)\widetilde{D}_l(t).
\end{aligned}
\tag{3.14}
$$

Therefore, by the following well-known inequalities: $|\widetilde{D}_i(t)| = O(t^{-1})$, $|\sin t| \leq |t|$, and condition (2.17), we see that (2.15) holds with $\alpha=\beta=1$, and thus we get Theorem 2.2 from Theorem 2.1. □

4. Applications

4.1. Cesàro's Means

Let $T = (C,\gamma,\delta) = C_\gamma \times C_\delta$ be the double Cesàro matrix of order (γ,δ) with $\gamma,\delta > -1$, that is, T has entries

$$t_{mnjk} = C^{\gamma}_{mj} \times C^{\delta}_{nk} = \frac{A^{\gamma-1}_{m-j}A^{\delta-1}_{n-k}}{A^{\gamma}_{m}A^{\delta}_{n}}. \tag{4.1}$$

Then for $0 \le j \le m-1$, $0 \le k \le n-1$,

$$\begin{aligned}
\widehat{t}^{(1,1)}_{mnjk} &= \Delta_{11}\bar{t}_{m-1,n-1,j,k} \\
&= \left(\sum_{u=j}^{m-1}\frac{A^{\gamma-1}_{m-1-u}}{A^{\gamma}_{m-1}} - \sum_{u=j}^{m}\frac{A^{\gamma-1}_{m-u}}{A^{\gamma}_{m}}\right)\left(\sum_{v=k}^{n-1}\frac{A^{\delta-1}_{n-1-v}}{A^{\delta}_{n-1}} - \sum_{v=k}^{n}\frac{A^{\delta-1}_{n-v}}{A^{\delta}_{n}}\right) \\
&= \left(\frac{A^{\gamma}_{m-j}}{A^{\gamma}_{m}} - \frac{A^{\gamma}_{m-1-j}}{A^{\gamma}_{m-1}}\right)\left(\frac{A^{\delta}_{n-k}}{A^{\delta}_{n}} - \frac{A^{\delta}_{n-1-k}}{A^{\delta}_{n-1}}\right) \\
&= \frac{jkA^{\gamma-1}_{m-j}A^{\delta-1}_{n-k}}{mnA^{\gamma}_{m}A^{\delta}_{n}},
\end{aligned} \tag{4.2}$$

$$\widehat{t}^{(1,1)}_{mnmn} = t_{mnmn} = \frac{1}{A^{\alpha}_{m}A^{\beta}_{n}}.$$

In particular,

$$\widehat{t}^{(1,1)}_{mn0k} = \widehat{t}^{(1,1)}_{mnj0} = 0, \quad j,k = 0,1,\ldots. \tag{4.3}$$

In other words, T satisfies condition (2.13) of Theorem 2.1.

By the well-known inequality (e.g., see [11])

$$A^{\gamma}_{n} = \frac{n^{\gamma}}{\Gamma(\gamma+1)}\left(1 + O\left(\frac{1}{n}\right)\right), \tag{4.4}$$

we deduce that

$$\begin{aligned}
\sum_{j=1}^{m}\sum_{k=1}^{n}\left|\widehat{t}^{(1,1)}_{mnjk}\right| &= \left(\sum_{j=1}^{m}\frac{jA^{\gamma-1}_{m}}{mA^{\gamma}_{m}} + \frac{1}{A^{\gamma}_{m}}\right)\left(\sum_{k=1}^{n}\frac{kA^{\delta-1}_{n-k}}{nA^{\delta}_{n}} + \frac{1}{A^{\delta}_{n}}\right) \\
&= \left(O\left(m^{-2}\right)\sum_{j=1}^{m/2} j + O\left(m^{-\gamma}\right)\sum_{j=m/2}^{m}A^{\gamma-1}_{m-j} + \frac{1}{A^{\gamma}_{m}}\right)
\end{aligned}$$

$$\times \left(O\left(n^{-2}\right) \sum_{k=1}^{n/2} k + O\left(n^{-\delta}\right) \sum_{k=n/2}^{n} A_{n-k}^{\delta-1} + \frac{1}{A_n^{\delta}} \right)$$
$$= O(1), \quad \gamma, \delta > 0, \tag{4.5}$$

which means condition (2.14) of Theorem 2.1.

For $0 < \gamma < 1$, we have

$$\sum_{k=1}^{n} \frac{A_{n-k}^{\gamma-1}}{nA_n^{\gamma}} \sin kx = \operatorname{Im}\left(\frac{1}{nA_n^{\gamma}} \sum_{k=1}^{n} A_{n-k}^{\gamma-1} e^{ikx} \right) = \operatorname{Im}\left(\frac{e^{inx}}{nA_n^{\gamma}} \sum_{k=0}^{n-1} A_k^{\gamma-1} e^{-ikx} \right)$$
$$=: \operatorname{Im}(J_1 + J_2), \tag{4.6}$$

where

$$J_1 := \frac{e^{inx}}{nA_n^{\gamma}} \left(1 - e^{-ix}\right)^{-\gamma}, \qquad J_2 := \frac{e^{inx}}{nA_n^{\gamma}} \sum_{k=n}^{\infty} A_k^{\gamma-1} e^{-ikx}. \tag{4.7}$$

By (4.4), we observe that

$$|J_1| = O\left(n^{-1-\gamma} x^{-\gamma}\right). \tag{4.8}$$

Since $A_k^{\gamma-1}$ decreases monotonically to 0, J_2 converges for $0 < x \le \pi$, and

$$|J_2| = O\left(\frac{A_n^{\gamma-1}}{nA_n^{\gamma}} \left| 1 - e^{-ix} \right|^{-1} \right) = O\left(n^{-2} x^{-1}\right) = O\left(n^{-1-\gamma} x^{-\gamma}\right) \tag{4.9}$$

for $t \in [(\pi/n), \pi]$.

By (4.4)–(4.9), we obtain that

$$\sum_{k=1}^{n} \frac{A_{n-k}^{\gamma-1}}{nA_n^{\gamma}} \sin kx = O\left(n^{-1-\gamma} x^{-\gamma}\right), \quad x \in \left[\frac{\pi}{n}, \pi\right], \ 0 < \gamma \le 1. \tag{4.10}$$

On the other hand, we have

$$\sum_{k=1}^{n} \frac{A_{n-k}^{\gamma-1}}{nA_n^{\gamma}} \sin kx = O(x) \sum_{k=1}^{n} \frac{kA_{n-k}^{\gamma-1}}{nA_n^{\gamma}} = O(x), \quad x \in \left[0, \frac{\pi}{n}\right], \ \gamma > 0. \tag{4.11}$$

For $\gamma \geq 1$, we have that

$$\sum_{k=1}^{n-1}\left|\Delta\left(\frac{A_{n-k}^{\gamma-1}}{nA_n^{\gamma}}\right)\right| = \frac{1}{nA_n^{\gamma}}\sum_{k=1}^{n-1}\left(A_{n-k}^{\gamma-1} - A_{n-k-1}^{\gamma-1}\right) + \frac{1}{nA_n^{\gamma}} = O\left(n^{-2}\right). \tag{4.12}$$

Now, by (4.10) and (4.11), we deduce that

$$\begin{aligned}\left|\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{\widehat{t}_{mnjk}^{(1,1)}}{jk}\sin jx\sin ky\right| &\leq \left|\sum_{j=1}^{m}\frac{A_{m-j}^{\gamma-1}}{mA_m^{\gamma}}\sin jx\right|\left|\sum_{k=1}^{n}\frac{A_{n-k}^{\delta-1}}{nA_n^{\delta}}\sin ky\right| \\ &= \begin{cases} O(k^{-1-\delta}xy^{-\delta}), & (x,y) \in \left[0,\dfrac{\pi}{m}\right]\times\left[\dfrac{\pi}{n},\pi\right], \\ O(j^{-1-\gamma}x^{-\gamma}y), & (x,y) \in \left[\dfrac{\pi}{m},\pi\right]\times\left[0,\dfrac{\pi}{n}\right], \\ O(j^{-1-\gamma}k^{-1-\delta}x^{-\gamma}y^{-\delta}), & (x,y) \in \left[\dfrac{\pi}{m},\pi\right]\times\left[\dfrac{\pi}{n},\pi\right] \end{cases}\end{aligned} \tag{4.13}$$

for $0 < \gamma, \delta \leq 1$.

When $\gamma > 1$, $0 < \delta \leq 1$, by (4.10) and (4.11) again,

$$\left|\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{\widehat{t}_{mnjk}^{(1,1)}}{jk}\sin jx\sin ky\right| \leq \left|\sum_{j=1}^{m}\frac{A_{m-j}^{\gamma-1}}{mA_m^{\gamma}}\sin jx\right|\left|\sum_{k=1}^{n}\frac{A_{n-k}^{\delta-1}}{nA_n^{\delta}}\sin ky\right| = O\left(k^{-1-\delta}xy^{-\delta}\right), \tag{4.14}$$

for $(x,y) \in [0,(\pi/m)]\times[(\pi/n),\pi]$. By (4.10)–(4.12), we have

$$\begin{aligned}\left|\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{\widehat{t}_{mnjk}^{(1,1)}}{jk}\sin jx\sin ky\right| &\leq \left|\sum_{j=1}^{m}\Delta\left(\frac{A_{m-j}^{\gamma-1}}{mA_m^{\gamma}}\right)D_j(x)\right|\left|\sum_{k=1}^{n}\frac{A_{n-k}^{\delta-1}}{nA_n^{\delta}}\sin ky\right| \\ &= \begin{cases} O(j^{-2}x^{-1}y), & (x,y) \in \left[\dfrac{\pi}{m},\pi\right]\times\left[0,\dfrac{\pi}{n}\right], \\ O(j^{-2}k^{-1-\delta}x^{-1}y^{-\delta}), & (x,y) \in \left[\dfrac{\pi}{m},\pi\right]\times\left[\dfrac{\pi}{n},\pi\right]. \end{cases}\end{aligned} \tag{4.15}$$

Similarly, we have

$$\left|\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{\widehat{t}_{mnjk}^{(1,1)}}{jk}\sin jx\sin ky\right| = \begin{cases} O(k^{-2}xy^{-1}), & (x,y) \in \left[0,\dfrac{\pi}{m}\right]\times\left[\dfrac{\pi}{n},\pi\right], \\ O(j^{-1-\gamma}x^{-\gamma}y), & (x,y) \in \left[\dfrac{\pi}{m},\pi\right]\times\left[0,\dfrac{\pi}{n}\right], \\ O(j^{-1-\gamma}k^{-2}x^{-\gamma}y^{-1}), & (x,y) \in \left[\dfrac{\pi}{m},\pi\right]\times\left[\dfrac{\pi}{n},\pi\right], \end{cases} \tag{4.16}$$

for $\delta > 1, 0 < \gamma \le 1$, and

$$\left|\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{\widehat{t}_{mnjk}^{(1,1)}}{jk}\sin jx\sin ky\right| = \begin{cases} O(k^{-2}xy^{-1}), & (x,y)\in\left[0,\dfrac{\pi}{m}\right]\times\left[\dfrac{\pi}{n},\pi\right],\\ O(j^{-2}x^{-1}y), & (x,y)\in\left[\dfrac{\pi}{m},\pi\right]\times\left[0,\dfrac{\pi}{n}\right],\\ O(j^{-2}k^{-2}x^{-1}y^{-1}), & (x,y)\in\left[\dfrac{\pi}{m},\pi\right]\times\left[\dfrac{\pi}{n},\pi\right],\end{cases} \tag{4.17}$$

for $\gamma, \delta > 1$.

Setting

$$\widetilde{\gamma} := \begin{cases}\gamma, & 0<\gamma\le 1,\\ 1, & \gamma>1,\end{cases} \qquad \widetilde{\delta} := \begin{cases}\delta, & 0<\delta\le 1,\\ 1, & \delta>1,\end{cases} \tag{4.18}$$

then we have

$$\left|\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{\widehat{t}_{mnjk}^{(1,1)}}{jk}\sin jx\sin ky\right| = \begin{cases} O\left(k^{-1-\widetilde{\delta}}xy^{-1}\right), & (x,y)\in\left[0,\dfrac{\pi}{m}\right]\times\left[\dfrac{\pi}{n},\pi\right],\\ O(j^{-1-\widetilde{\gamma}}x^{-1}y), & (x,y)\in\left[\dfrac{\pi}{m},\pi\right]\times\left[0,\dfrac{\pi}{n}\right],\\ O\left(j^{-1-\widetilde{\gamma}}k^{-1-\widetilde{\delta}}x^{-1}y^{-1}\right), & (x,y)\in\left[\dfrac{\pi}{m},\pi\right]\times\left[\dfrac{\pi}{n},\pi\right],\end{cases} \tag{4.19}$$

by (4.13)–(4.17).

From (4.19), we have the following corollary of Theorem 2.1, which generalizes Theorem B from one dimension to two dimensions.

Theorem 4.1. *Let $f(x,y)$ be a periodic function and $f(x,y)\in BV_H(\mathbb{T}^2)$ and $T=(C,\gamma,\delta)$ be the double Cesàro matrix of order (γ,δ) with $\gamma,\delta>0$. Then for $n\ge 2$, $(x,y)\in\mathbb{T}^2$, one has*

$$R_{mn}^{(1,1)}(f;x,y)\le C\frac{1}{m^{\widetilde{\gamma}}n^{\widetilde{\delta}}}\sum_{k=1}^{m}\sum_{l=1}^{n}k^{\widetilde{\gamma}-1}l^{\widetilde{\delta}-1}V_{00}^{(\pi/k)(\pi/l)}(\varphi_{xy}). \tag{4.20}$$

4.2. Bernstein-Rogosinski's Means

The so-called Bernstein-Rogosinski means of Fourier series (1.1) are defined by

$$B_n(f,x) := \frac{1}{2}\left(S_n\left(f,x+\frac{1}{2n+1}\pi\right)+S_n\left(f,x-\frac{1}{2n+1}\pi\right)\right). \tag{4.21}$$

Now, we introduce the following Bernstein-Rogosinski means of double Fourier series (2.1):

$$B_{mn}(f,x) := \frac{1}{4}\left(S_{mn}\left(f;x+\frac{\pi}{2m+1},y+\frac{\pi}{2n+1}\right)+S_{mn}\left(f;x+\frac{\pi}{2m+1},y-\frac{\pi}{2n+1}\right)\right.$$
$$\left.+S_{mn}\left(f;x-\frac{\pi}{2m+1},y+\frac{\pi}{2n+1}\right)+S_{mn}\left(f;x-\frac{\pi}{2m+1},y-\frac{\pi}{2n+1}\right)\right). \tag{4.22}$$

Direct calculations yield that

$$\begin{aligned}
B_{mn}(f,x) &= \sum_{j=0}^{m}\sum_{k=0}^{n}\cos\frac{j\pi}{2m+1}\cos\frac{k\pi}{2n+1}\Delta_{11}S_{jk}(x,y)\\
&= \sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\left(\cos\frac{j}{2m+1}\pi-\cos\frac{j+1}{2m+1}\pi\right)\left(\cos\frac{k}{2n+1}\pi-\cos\frac{k+1}{2n+1}\pi\right)S_{jk}(x,y)\\
&\quad+\sum_{j=0}^{m-1}\left(\cos\frac{j}{2m+1}\pi-\cos\frac{j+1}{2m+1}\pi\right)\cos\frac{n\pi}{2n+1}S_{jn}(x,y)\\
&\quad+\sum_{k=0}^{n-1}\left(\cos\frac{k}{2n+1}\pi-\cos\frac{k+1}{2n+1}\pi\right)\cos\frac{m\pi}{2m+1}S_{mk}(x,y)\\
&\quad+\cos\frac{m\pi}{2m+1}\cos\frac{n\pi}{2n+1}S_{mn}(x,y),
\end{aligned} \tag{4.23}$$

Thus, $B_{mn}(f,x)$ can be regarded as a T-transformation of Fourier series (1.1) with T of entries defined as

$$\begin{aligned}
t_{mnjk} &= \left(\cos\frac{j}{2m+1}\pi-\cos\frac{j+1}{2m+1}\pi\right)\left(\cos\frac{k}{2n+1}\pi-\cos\frac{k+1}{2n+1}\pi\right), \quad 0\le j<m,\ 0\le k<n,\\
t_{mnjn} &= \left(\cos\frac{j}{2m+1}\pi-\cos\frac{j+1}{2n+1}\pi\right)\cos\frac{n\pi}{2n+1}, \quad 0\le j<m,\\
t_{mnmk} &= \left(\cos\frac{k}{2n+1}\pi-\cos\frac{k+1}{2n+1}\pi\right)\cos\frac{m\pi}{2m+1}, \quad 0\le k<n,\\
t_{mnmn} &= \cos\frac{m\pi}{2m+1}\cos\frac{n\pi}{2n+1}.
\end{aligned} \tag{4.24}$$

By direct calculations, we have

$$\hat{t}^{(1,1)}_{mnjk} = \left(\cos\frac{j}{2m-1}\pi-\cos\frac{j}{2m+1}\pi\right)\left(\cos\frac{k}{2n-1}\pi-\cos\frac{k}{2n+1}\pi\right), \quad 0\le j<m,\ 0\le k<n, \tag{4.25}$$

$$\widehat{t}^{(1,1)}_{mnjn} = \left(\cos\frac{j}{2m-1}\pi - \cos\frac{j}{2m+1}\pi\right)\cos\frac{n\pi}{2n+1}, \quad 0 \le j < m, \tag{4.26}$$

$$\widehat{t}^{(1,1)}_{mnmk} = \left(\cos\frac{k}{2n-1}\pi - \cos\frac{k}{2n+1}\pi\right)\cos\frac{m\pi}{2m+1}, \quad 0 \le k < n, \tag{4.27}$$

$$\widehat{t}^{(1,1)}_{mnmn} = \cos\frac{m\pi}{2m+1}\cos\frac{n\pi}{2n+1}. \tag{4.28}$$

Therefore, (4.26) and (4.27) imply (2.13), while from (4.25)–(4.28), we get

$$\begin{aligned}
\sum_{j=1}^{m}\sum_{k=1}^{n}\left|\widehat{t}^{(1,1)}_{mnjk}\right| &\le \sum_{j=1}^{m-1}\sum_{k=1}^{n-1}\left|\cos\frac{j}{2m-1}\pi - \cos\frac{j}{2m+1}\pi\right|\left|\cos\frac{k}{2n-1}\pi - \cos\frac{k}{2n+1}\pi\right| \\
&\quad + \sum_{j=1}^{m-1}\left|\cos\frac{j}{2m-1}\pi - \cos\frac{j}{2m+1}\pi\right|\cos\frac{n\pi}{2n+1} \\
&\quad + \sum_{k=1}^{n-1}\left|\cos\frac{k}{2n-1}\pi - \cos\frac{k}{2n+1}\pi\right|\cos\frac{m\pi}{2m+1} + \cos\frac{m\pi}{2m+1}\cos\frac{n\pi}{2n+1} \\
&\le 4\sum_{j=1}^{m-1}\sum_{k=1}^{n-1}\sin\frac{2mj}{4m^2-1}\pi\sin\frac{j}{4m^2-1}\pi\sin\frac{2nk}{4n^2-1}\pi\sin\frac{k}{4n^2-1}\pi \\
&\quad + 2\sum_{j=1}^{m-1}\sin\frac{2mj}{4m^2-1}\pi\sin\frac{j}{4m^2-1}\pi + 2\sum_{k=1}^{n-1}\sin\frac{2nk}{4n^2-1}\pi\sin\frac{k}{4n^2-1}\pi + 1 \\
&\le 4\pi^4\sum_{j=1}^{m-1}\sum_{k=1}^{n-1}\frac{2mj^2}{(4m^2-1)^2}\frac{2nk^2}{(4n^2-1)^2} + 2\pi^2\sum_{j=1}^{m-1}\frac{2mj^2}{(4m^2-1)^2} + 2\pi^2\sum_{k=1}^{n-1}\frac{2nk^2}{(4n^2-1)^2} + 1 \\
&= O(1),
\end{aligned} \tag{4.29}$$

which implies (2.14).

Finally, we verify that T satisfies condition (2.17); thus T satisfies all the conditions of Theorem 2.2. By direct calculations, we deduce that

$$\begin{aligned}
\sum_{j=1}^{m}\sum_{k=1}^{n}\left|\Delta_{01}\frac{\widehat{t}^{(1,1)}_{mnjk}}{k}\right| &= \left(\sum_{j=1}^{m}\left|\cos\frac{j}{2m-1}\pi - \cos\frac{j}{2m+1}\pi\right| + \cos\frac{m\pi}{2m+1}\right) \\
&\quad \times \left(\sum_{k=1}^{n-1}\left|\frac{\cos(k/(2n-1))\pi - \cos(k/(2n+1))\pi}{k}\right.\right. \\
&\qquad \left.\left. - \frac{\cos((k+1)/(2n-1))\pi - \cos((k+1)/(2n+1))\pi}{k+1}\right| + \frac{1}{n}\cos\frac{n\pi}{2n+1}\right)
\end{aligned}$$

$$
\begin{aligned}
&= O\left(\sum_{k=1}^{n-1}\left|\frac{\cos(k/(2n-1))\pi-\cos(k/(2n+1))\pi}{k}\right.\right.\\
&\qquad\left.\left.-\frac{\cos((k+1)/(2n-1))\pi-\cos((k+1)/(2n+1))\pi}{k+1}\right|+n^{-2}\right)\\
&= O(1)\sum_{k=1}^{n-1}\frac{1}{k}\left(\left(\cos\frac{k}{2n-1}\pi-\cos\frac{k}{2n+1}\pi\right)-\left(\cos\frac{k+1}{2n-1}\pi-\cos\frac{k+1}{2n+1}\pi\right)\right)\\
&\quad+O(1)\sum_{k=1}^{n-1}\left(\frac{1}{k}-\frac{1}{k+1}\right)\left|\cos\frac{k+1}{2n-1}\pi-\cos\frac{k+1}{2n+1}\pi\right|\\
&= \sum_{k=1}^{n-1}\frac{1}{k}\left|\left(\sin\frac{2nk\pi}{4n^2-1}-\sin\frac{2n(k+1)\pi}{4n^2-1}\right)\sin\frac{k\pi}{4n^2-1}\right.\\
&\qquad\left.+\left(\sin\frac{k\pi}{4n^2-1}-\sin\frac{(k+1)\pi}{4n^2-1}\right)\sin\frac{2n(k+1)\pi}{4n^2-1}\right|\\
&\quad+\sum_{k=1}^{n-1}\frac{1}{k(k+1)}\sin\frac{2n(k+1)\pi}{4n^2-1}\sin\frac{(k+1)\pi}{4n^2-1}\\
&= O(1)\sum_{k=1}^{n-1}\frac{1}{k}\sin\frac{n\pi}{4n^2-1}\sin\frac{k\pi}{4n^2-1}+O(1)\sum_{k=1}^{n-1}\frac{1}{k^2}\sin\frac{2n(k+1)\pi}{4n^2-1}\sin\frac{(k+1)\pi}{4n^2-1}\\
&= O(1)\sum_{k=1}^{n-1}n^{-3}=O\left(n^{-2}\right).
\end{aligned}
\tag{4.30}
$$

Analogously,

$$
\begin{aligned}
\sum_{j=1}^{m}\sum_{k=1}^{n}\left|\Delta_{10}\frac{\widehat{t}^{(1,1)}_{mnjk}}{j}\right| &= O\left(m^{-2}\right),\\
\sum_{j=1}^{m}\sum_{k=1}^{n}\left|\Delta_{11}\frac{\widehat{t}^{(1,1)}_{mnjk}}{j}\right| &= O\left(m^{-2}n^{-2}\right).
\end{aligned}
\tag{4.31}
$$

Now, (4.30) and (4.31) imply (2.17).

4.3. Riesz's Means

Let $\{p_n\}$ and $\{q_n\}$ be positive sequences such that $P_n = p_0+p_1+\cdots+p_n \to \infty$, $Q_n = q_0+q_1+\cdots+q_n \to \infty$, and let $T := (t_{mnjk})$ be a lower triangular matrix with entries $t_{mnjk} = p_jq_k/P_mQn$, $j = 0,1,\dots m$, $k = 0,1,\dots,n$; $m,n = 1,2,\dots$. Then the T-transformation of Fourier series (2.1) is known as Riesz's mean.

Proposition 4.2. *Let $\{p_n\}$ and $\{q_n\}$ be positive nondecreasing sequences such that (one says $A_n \sim B_n$, if there are two positive constants C_1 and C_2 such that $C_1 \le (A_n/B_n) \le C_2$.)*

$$np_n \sim P_n, \qquad nq_n \sim Q_n, \tag{4.32}$$

$$\sum_{k=1}^{n} \frac{p_k}{k} = O(p_n), \qquad \sum_{k=1}^{n} \frac{q_k}{k} = O(q_n). \tag{4.33}$$

Then T satisfies all the conditions of Theorem 2.2.

Proof. Direct calculations yield that (set $P_{-1} = Q_{-1} = 0$)

$$\hat{t}^{(1,1)}_{mnjk} = \left(\sum_{i=j}^{m} \frac{p_i}{P_m} - \sum_{i=j}^{m-1} \frac{p_i}{P_{m-1}}\right)\left(\sum_{i=k}^{n} \frac{q_i}{Q_n} - \sum_{i=k}^{n-1} \frac{q_i}{Q_{n-1}}\right) = \frac{p_m q_n P_{j-1} Q_{k-1}}{P_m P_{m-1} Q_n Q_{n-1}}, \quad 0 \le j < m,\ 0 \le k < n, \tag{4.34}$$

$$\begin{aligned} \hat{t}^{(1,1)}_{mnmk} &= \frac{p_m}{P_m}\left(\sum_{i=k}^{n} \frac{q_i}{Q_n} - \sum_{i=k}^{n-1} \frac{q_i}{Q_{n-1}}\right) = \frac{p_m q_n Q_{k-1}}{P_m Q_n Q_{n-1}}, \quad 0 \le k < n, \\ \hat{t}^{(1,1)}_{mnjn} &= \left(\sum_{i=j}^{m} \frac{p_i}{P_m} - \sum_{i=j}^{m-1} \frac{p_i}{P_{m-1}}\right)\frac{q_n}{Q_n} = \frac{p_m q_n P_{j-1}}{P_m P_{m-1} Q_n}, \quad 0 \le j < m, \end{aligned} \tag{4.35}$$

$$\hat{t}^{(1,1)}_{mnmn} = \frac{p_m q_n}{P_m Q_n}. \tag{4.36}$$

It follows from (4.35) that T satisfies condition (2.13).

Since $\{p_n\}$ and $\{q_n\}$ are nondecreasing, by (4.32), (4.34)–(4.36), we have

$$\sum_{j=1}^{m} \sum_{k=1}^{n} \left|\hat{t}^{(1,1)}_{mnjk}\right| = \frac{p_m q_n}{P_m P_{m-1} Q_n Q_{n-1}} \sum_{j=1}^{m} \sum_{k=1}^{n} P_{j-1} Q_{k-1} = O\left(\frac{mnp_m q_n}{P_m Q_n}\right) = O(1). \tag{4.37}$$

Hence, T satisfies condition (2.14).

Since $\{p_n\}$ and $\{q_n\}$ are nondecreasing, by (4.32)–(4.36), we have

$$\begin{aligned} \sum_{j=1}^{m} \sum_{k=1}^{n} \left|\Delta_{01}\left(\frac{\hat{t}^{(1,1)}_{mnjk}}{k}\right)\right| &= \left(\frac{p_m}{P_m P_{m-1}} \sum_{j=1}^{m-1} P_{j-1} + \frac{p_m}{P_m}\right) \\ &\quad \times \left(\frac{q_n}{Q_n Q_{n-1}} \sum_{k=1}^{n-1} \left|\frac{Q_{k-1}}{k} - \frac{Q_k}{k+1}\right| + \frac{q_n}{nP_n}\right) \end{aligned}$$

$$
\begin{aligned}
&= O(1)\left(\frac{q_n}{Q_nQ_{n-1}}\left(\sum_{k=1}^{n-1}\frac{Q_{k-1}}{k(k+1)}+\sum_{k=2}^{n-1}\frac{q_k}{k+1}\right)+O\left(n^{-2}\right)\right)\\
&= O(1)\left(\frac{1}{nQ_{n-1}}\sum_{k=1}^{n-1}\frac{q_k}{k}+n^{-2}\right)\\
&= O\left(\frac{q_{n-1}}{nQ_{n-1}}\right)\\
&= O\left(n^{-2}\right).
\end{aligned}
\tag{4.38}
$$

Similarly,

$$
\begin{gathered}
\sum_{j=1}^{m}\sum_{k=1}^{n}\left|\Delta_{10}\frac{\widehat{t}^{(1,1)}_{mnjk}}{j}\right| = O\left(m^{-2}\right),\\
\sum_{j=1}^{m}\sum_{k=1}^{n}\left|\Delta_{11}\frac{\widehat{t}^{(1,1)}_{mnjk}}{j}\right| = O\left(m^{-2}n^{-2}\right).
\end{gathered}
\tag{4.39}
$$

Now, by (4.38) and (4.39), we show that T also satisfies condition (2.17). □

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