

Research Article

Moment Stability of the Critical Case of PWM Feedback Systems with Stochastic Perturbations

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This paper further studies the moment stability of pulse-width-modulated (PWM) feedback system which is subjected to multiplicative and additive random disturbance modeled by the derivative of Wiener process. Different from the existing investigation, we focus on its critical case. The linear plant considered herein is assumed to be critically stable; that is, the plant has one and only one pole at the origin, and the rest of the poles are left half of complex plane. We establish several globally asymptotically stability criteria for such PWM feedback systems and then propose an algorithm to calculate the stability bound effectively. Furthermore, we present two numerical examples to show the effectiveness of the theoretical results.

1. Introduction

Pulse-width modulation has extensively been used in attitude control systems, adaptive control systems, signal processing, power control systems, modeling of neuron behavior, and the like (e.g., see [1–5]). In many areas, especially engineering applications, it is always operated in all kinds of accidental or continued disturbances. It is of prime importance whether or not keeping the scheduled operation or work of the state. Therefore, there has been a growing research interest on the stability analysis for PWM feedback systems, and a set of stability results have been established by a variety of methods [6–20]. By the method of positive kernels of integral operator, Halanař in [10] proposed two versions of direct Lyapunov and Lagrange stability results for PWM feedback system with Hurwitz stable plant. In [12, 13], the authors studied the stability properties of nonlinear PWM feedback systems.

There are, however, only a few results concerning the qualitative properties of PWM feedback system subjected to random disturbance. Gupta and Jury [14] developed a method

of determining the mean square value of the output of a PWM system with Gaussian random input. Sun et al. [15] presented a set of theoretical results for mean square exponential stability, asymptotical mean square stability, and the p th moment exponential stability by using spectrum technique. Besides, in [16], the authors gave some definitions on the p th moment stable in mean and established several p th moment globally stability criteria in mean. Recently, the authors in [17, 18] investigated the dynamical systems subjected to noise disturbance by linear matrix inequalities (LMIs) and control Lyapunov methods. In [20], the authors established new Lyapunov and Lagrange stability results for pulse-width-modulated (PWM) feedback systems subjected to random disturbance. The linear plant considered therein was assumed to be *Hurwitz stable*. An optimization procedure was also presented in [20], which is expected to improve the analysis performance significantly. And then they have shown that when the parameters of PWM are within a certain computable range, the PWM feedback system is stable.

To the best of the authors' knowledge, there are a few (if any) results for the stability analysis of the critical case of PWM systems with stochastic perturbations. In the present paper, we try to make the contribution on this issue. It is noted that the linear plant considered herein is *critically stable*; that is, the linear plant has one and only one pole at the origin, and the rest of the poles are left half of complex plane. Obviously, such systems are somewhat complex in comparison with most of the systems in literature. We will establish several Lyapunov and Lagrange criteria for the p th moment uniform asymptotical stability in mean and then present an algorithm to compute the upper bound for the parameters of PWM. We will characterize the relationship between the parameters of PWM and the coefficients of state vectors of the feedback systems. It will be shown that when the random disturbance is sufficiently small, such PWM feedback system is the p th moment globally asymptotically stable in mean provided that the upper bounds of parameters of pulse-width modulator are selected properly. We also demonstrate the effectiveness of our results by means of two numerical examples.

2. Notations and Some Definitions

We let (Ω, F, P) denote the underlying probability space for all the systems that will be considered, where Ω is the sample space, F is the σ -algebra of subsets of the sample space, and P is the probability measure. An R^n -valued random variable x with domain X is a measurable function from Ω to $X \subset R^n$. A family $\{x(t), t \in I\}$ of R^n -valued random variables with domain X defined on a probability space (Ω, F, P) is called a stochastic process with index set I and state space (X, F^n) .

Definition 2.1. Let (X, d) be a metric space, $X \subset R^n$, $A \subset X$, and let $T \subset R^+$. For any fixed $a \in A$ (a is called the initial state), $t_0 \in T$, a stochastic process $\{x(t, \omega, a, t_0), t \in T_{a, t_0}\}$ with domain X is called a *stochastic motion* if $x(t_0, \omega, a, t_0) = a$ for all $\omega \in \Omega$, where $T_{a, t_0} = [t_0, t_1) \cap T$, $t_1 > t_0$, and t_1 is finite or infinite.

Definition 2.2. Let S be a family of stochastic motions with domain X given by

$$S \subset \{x(\cdot, \cdot, a, t_0) : x(t_0, \omega, a, t_0) = a, \omega \in \Omega, a \in A, t_0 \in T\}. \quad (2.1)$$

We call the four-tuple $\{T, X, A, S\}$ a *stochastic dynamical system*.

Definition 2.3. Let $\{T, X, A, S\}$ be a stochastic dynamical system. A set $M \subset A$ is said to be *invariant* with respect to system S (or short, (S, M) is invariant) if $a \in M$ implies that

$$P\{\omega : x(t, \omega, a, t_0) \in M \forall t \in T_{a, t_0}\} = 1, \quad \forall t_0 \in T, \text{ and all } x(\cdot, \cdot, a, t_0) \in S. \quad (2.2)$$

Definition 2.4. $x_0 \in A$ is called an *equilibrium point* of a stochastic dynamical system $\{T, X, A, S\}$ if the set $\{x_0\}$ is invariant with respect to S .

Definition 2.5. S is said to be the *p*th moment uniformly bounded if, for every $\eta > 0$ and for every $t_0 \in T$, there exists a $\beta = \beta(\eta) > 0$ such that if $d(a, x_0) < \eta$, then for all $x(\cdot, \cdot, a, t_0) \in S$, $Ed(x(t, \omega, a, t_0), x_0)^p < \beta$ for all $t \in T_{a, t_0}$, where t_0 is a fixed point in X . S is said to be the *p*th moment uniformly ultimately bounded in mean if there exists $B > 0$ and if for every $\delta > 0$ and for every $t_0 \in T$ there exists a $\tau = \tau(\delta)$ such that for all $x(\cdot, \cdot, a, t_0) \in S$, $Ed(x(t, \omega, a, t_0), x_0)^p < \beta$, for all $t > t_0 + \tau$, whenever $d(a, x_0) < \delta$, where x_0 is a fixed point in X .

Definition 2.6. Let $\{T, X, A, S\}$ be a stochastic dynamical system, and let d be the metric on X . A set $M \subset A$ is said to be the *p*th moment stable in mean (i.e., (S, M) is said to be the *p*th moment stable in mean) if, for any $\varepsilon > 0$, $t_0 \in T$, there exists $\delta = \delta(t_0, \varepsilon)$ such that $Ed(x(t, \omega, a, t_0), M)^p < \varepsilon$ for any process $x(\cdot, \cdot, a, t_0) \in S$, whenever $d(a, M) < \delta$, where E denotes the expectation of a random process. If δ is independent of t_0 , (S, M) is said to be the *p*th moment uniformly stable in mean. (S, M) is said to be the *p*th moment uniformly asymptotically stable in mean if it is uniformly stable in the *p*th mean and if, for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ and a $\tau = \tau(\varepsilon)$ such that for any process $x(\cdot, \cdot, a, t_0) \in S$, whenever $d(a, M) < \delta$ implies that $Ed(x(t, \omega, a, t_0), M)^p < \varepsilon$ for all $t > t_0 + \tau$.

We can similarly define the *p*th moment asymptotic stability in mean, the *p*th moment asymptotic stability in mean in the large, the *p*th moment uniform asymptotic stability in mean in the large, the *p*th moment exponential stability in mean, and the *p*th moment exponential stability in mean in the large. When $p = 2$, we speak of various stability and bound concepts in the mean square.

3. Stability Analysis of PWM Feedback System: The Critical Case

The PWM feedback system to be considered in this paper is shown in Figure 1.

The pulse-width modulator is described by

$$u(t) = m(e(t)) = \begin{cases} M \operatorname{sgn}(e(kT)) & t \in [kT, kT + T_k), \\ 0 & t \in [kT + T_k, (k + 1)T), \end{cases} \quad (3.1)$$

where $e(t) = r(t) - y(t)$ with $r(t)$ being the external input and $y(t)$ the system output, and, for $k = 0, 1, 2, \dots$, the pulse-width T_k and the sign function sign are given, respectively, by

$$T_k = \begin{cases} \beta |e(kT)|, & |e(kT)| \leq \frac{T}{\beta}; \\ T, & |e(kT)| > \frac{T}{\beta}, \end{cases} \quad (3.2)$$

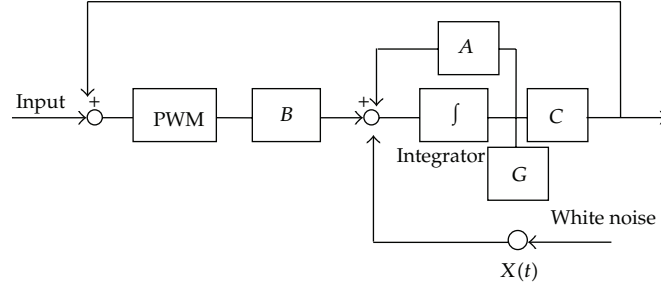


Figure 1: Block diagram of PWM feedback systems subjected to multiplicative disturbances [20].

$$\text{sgn}(\sigma) = \begin{cases} 1 & \sigma > 0 \\ 0 & \sigma = 0 \\ -1 & \sigma < 0. \end{cases} \quad (3.3)$$

The sampling period T , the amplitude of the pulse M , and β are all assumed to be constants. And throughout this paper, we always assume that $r(t) \equiv 0$. Under these assumptions, the PWM feedback system with the output function can be described by

$$\begin{aligned} dx(t) &= Ax(t)dt + Bu(t)dt + Gx(t)dW_t, \\ y(t) &= Cx(t), \end{aligned} \quad (3.4)$$

where $x \in R^n$, $y \in R$, $u \in R$ is output of the pulse-width modulator, A , B , C , and G are matrices of appropriate dimensions, and $W(t)$ is a scalar wiener process.

Furthermore, we assume that the linear plant is critically stable; that is, the matrix A is stable with one and only one eigenvalue equal to zero. Without loss of generality, we assume that

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ b \end{pmatrix}, \quad C = [C_1 \ 1], \quad G = \begin{pmatrix} G_1 & 0 \\ 0 & g \end{pmatrix}, \quad (3.5)$$

where A_1 is Hurwitz stable, $x_1 \in R^{n-1}$ and $x_2 \in R$, and system (3.4) can be rewritten as

$$\begin{aligned} dx_1(t) &= A_1x_1(t)dt + B_1u(t)dt + G_1x_1(t)dW_t, \\ dx_2(t) &= bu(t)dt + gx_2(t)dW_t, \\ y(t) &= C_1x_1(t) + x_2(t). \end{aligned} \quad (3.6)$$

Note that $(x_1^T, x_2) = (0^T, 0)$ is an equilibrium point of PWM feedback system (3.6).

Lemma 3.1 (The Itô isometry). *Assume that $W(t)$ is a scalar wiener process, and If $\phi(t, \omega)$ is bounded and elementary, then*

$$E \left[\left(\int_S^T \phi(t, \omega) dW_t \right)^2 \right] = E \left[\int_S^T \phi(t, \omega)^2 dt \right]. \tag{3.7}$$

Proof. Put $\Delta W_j = W_{t_{j+1}} - W_{t_j}$, then

$$E[e_i e_j \Delta W_i \Delta W_j] = \begin{cases} 0 & \text{if } i \neq j \\ E[e_j^2] (t_{j+1} - t_j) & \text{if } i = j. \end{cases} \tag{3.8}$$

Using that $e_i e_j \Delta W_i$ and ΔW_j are independent if $i < j$. Thus

$$\begin{aligned} E \left[\left(\int_S^T \phi(t, \omega) dW_t \right)^2 \right] &= \sum_{i,j} E[e_i e_j \Delta W_i \Delta W_j] = \sum_j e_j^2 \cdot (t_i - t_j) \\ &= E \left[\int_S^T \phi(t, \omega)^2 dt \right]. \end{aligned} \tag{3.9}$$

□

Theorem 3.2. *Assume that the matrix A_1 in (3.6) is Hurwitz stable, $b > 0$, and $C_1 \neq 0$. Then the equilibrium $x_e = 0$ of PWM feedback system (3.6) is the p th moment uniformly asymptotically stable in mean in the large whenever there exists a positive definite matrix $P = \begin{pmatrix} P_1 & 0 \\ 0 & \xi \end{pmatrix}$ such that $M\beta < \inf_{T_k \in [0, T]} (2b\xi / (G_2(T_k) + b^2\xi + \max(0, \theta(T_k))))$, and there exists a $\delta > 0$ (an upper bounded of δ will be given in the proof) such that $\|G\| < \delta$, where*

$$\begin{aligned} \theta(T_k) &= b^2 \xi^2 \lambda_M(C_1^T C_1) - b \xi \lambda_m(U(T_k) C_1 + (U(T_k) C_1)^T) + \lambda_M(U(T_k) U^T(T_k)), \\ U(T_k) &= (P_1 - I) \left(\frac{I - e^{-A_1 T_k}}{T_k} A_1^{-1} B \right), \\ G_2(T_k) &= \left(\frac{I - e^{-A_1 T_k}}{T_k} A_1^{-1} B \right)^T (P_1 - I) \left(\frac{I - e^{-A_1 T_k}}{T_k} A_1^{-1} B \right). \end{aligned} \tag{3.10}$$

Proof. Since $(E\|X\|^p)^{1/p}$ is nondecreasing in p , the equilibrium $x_e = 0$ is stable (resp., asymptotically stable, etc.) and implies that it is the q th moment stable (resp., asymptotically stable, etc.) in mean for all $q < p$. Therefore, it suffices to prove the theorem for even integers.

Integrating (3.6), we have

$$\begin{aligned} x_1(t) &= e^{A_1(t-kT)} x_1(kT) + \int_{kT}^t e^{A_1(t-s)} B_1 u(s) ds + \int_{kT}^t e^{A_1(t-s)} G_1 x_1(s) dW_s, \\ x_2(t) &= x_2(kT) + \int_{kT}^t b u(s) ds + \int_{kT}^t g x_2(s) dW_s. \end{aligned} \tag{3.11}$$

Therefore, when $t = kT + T$,

$$\begin{aligned} x_1(kT + T) &= e^{A_1 T} \left(x_1(kT) - \int_0^{T_k} e^{-A_1 s} B_1 M \operatorname{sgn}(C_1 x_1(kT) + x_2(kT)) ds \right) \\ &\quad + \int_{kT}^{kT+T} e^{A_1(kT+T-s)} G_1 x_1(s) dW_s \\ &= e^{A_1 T} \left[(I - M\beta W(\tau_k) C_1) x_1(kT) - M\beta W_1(\tau_k) x_2(kT) \right] \end{aligned} \quad (3.12)$$

$$\begin{aligned} &\quad + \int_{kT}^{kT+T} e^{A_1(kT+T-s)} G_1 x_1(s) dW_s, \\ x_2(kT + T) &= x_2(kT) - \int_0^{T_k} b M \operatorname{sgn}(C_1 x_1(kT) + x_2(kT)) ds + \int_{kT}^{kT+T} g x_2(s) dW_s \\ &= -M\beta b \frac{T_k}{\tau_k} C_1 x_1(kT) + \left(1 - M\beta b \frac{T_k}{\tau_k} \right) x_2(kT) + \int_{kT}^{kT+T} g x_2(s) dW_s \end{aligned} \quad (3.13)$$

with

$$\begin{aligned} \tau_k &= \beta |C_1 x_1(kT) + x_2(kT)| \\ &= \begin{cases} T_k, & T_k < T, \\ \geq T, & T_k = T, \end{cases} \\ W(\tau_k) &= \begin{cases} 0 & \tau_k = 0, \\ \frac{I - e^{-A_1 T_k}}{T_k} A_1^{-1} B & \tau_k < T, \\ \frac{I - e^{-A_1 T}}{\tau_k} A_1^{-1} B = \frac{T}{\tau_k} W(T) & \tau_k \geq T. \end{cases} \end{aligned} \quad (3.14)$$

To simplify our notations, we let

$$\begin{aligned} H &= [I - M\beta W(\tau_k) \quad -M\beta W(\tau_k)], \\ Z(kT) &= \int_{kT}^{kT+T} e^{A_1(kT+T-s)} G_1 x_1(s) dW_s, \end{aligned}$$

$$\begin{aligned} \tilde{Z}(kT) &= \int_{kT}^{kT+T} g x_2(s) dW_s, \\ \rho(\tau_k) &= \begin{cases} 0, & \tau_k = 0, \\ \frac{T_k}{\tau_k}, & \tau_k \in (0, \infty), \end{cases} \\ L(\tau_k) &= \begin{bmatrix} -M\beta b \frac{T_k}{\tau_k} C_1 & 1 - M\beta b \frac{T_k}{\tau_k} \end{bmatrix} \\ &= [-M\beta b \rho(\tau_k) C_1 \quad 1 - M\beta b \rho(\tau_k)]. \end{aligned} \tag{3.15}$$

Then (3.12) and (3.13) are reduced to

$$x_1(kT + T) = e^{A_1 T} H x(kT) + Z(kT), \tag{3.16}$$

$$x_2(kT + T) = L(\tau_k) x(kT) + \tilde{Z}(kT). \tag{3.17}$$

Since A_1 is Hurwitz stable, then $e^{A_1 T}$ is *Schur stable*. Therefore, there exists a positive definite matrix $P_1 = P_1^T$ such that

$$\left(e^{A_1 T} \right)^T P_1 e^{A_1 T} - P_1 = -I. \tag{3.18}$$

Choosing the quadratic Lyapunov function

$$V : R^n \longrightarrow R^+, \quad V(x) = x^T P x, \tag{3.19}$$

where

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & \xi \end{pmatrix}, \quad P_1 > 0, \quad \xi > 0. \tag{3.20}$$

Then,

$$\begin{aligned} &E[V(x(kT + T))] - E[V(x(kT))] \\ &= E\left[x^T(kT + T) P x(kT + T)\right] - E\left[x^T(kT) P x(kT)\right] \\ &= E\left[x_1^T(kT + T) P_1 x_1(kT + T)\right] + E\left[x_2^T(kT + T) \xi x_2(kT + T)\right] \\ &\quad - E\left[x^T(kT) P x(kT)\right] \\ &= E\left[\left(e^{A_1 T} H x(kT) + Z(kT)\right)^T P_1 \left(e^{A_1 T} H x(kT) + Z(kT)\right)\right] \end{aligned}$$

$$\begin{aligned}
& + E \left[\left(L(\tau_k)x(kT) + \tilde{Z}(kT) \right)^T \xi \left(L(\tau_k)x(kT) + \tilde{Z}(kT) \right) \right] \\
& - E \left[x^T(kT)Px(kT) \right] \\
& = E \left[x^T(kT) \left(H^T(P_1 - I)H + \xi L^T(\tau_k)L(\tau_k) - P \right) x(kT) \right] \\
& + 2E \left[Z^T(kT)P_1e^{A_1T}Hx_1(kT) \right] + 2E \left[\tilde{Z}^T(kT)\xi x_2(kT) \right] \\
& + E \left[Z^T(kT)P_1Z(kT) \right] + E \left[\tilde{Z}^T(kT)\xi\tilde{Z}(kT) \right] \\
& = E \left[x^T(kT) \left(H^T(P_1 - I)H + \xi L^T(\tau_k)L(\tau_k) - P \right) x(kT) \right] \\
& + 2E \left[h^T(kT)\Phi x(kT) \right] + E \left[h^T(kT)Ph(kT) \right],
\end{aligned} \tag{3.21}$$

where

$$\begin{aligned}
x(kT) &= \begin{pmatrix} x_1(kT) \\ x_2(kT) \end{pmatrix}, \quad h(kT) = \begin{pmatrix} Z(kT) \\ \tilde{Z}(kT) \end{pmatrix}, \\
\Phi &= \begin{pmatrix} P_1e^{A_1T}H \\ \xi L(\tau_k) \end{pmatrix} = \begin{pmatrix} P_1e^{A_1T}(I - M\beta W(\tau_k)C_1) & -M\beta P_1e^{A_1T}W(\tau_k) \\ -M\beta b\rho(\tau_k)\xi C_1 & \xi - M\beta b\rho(\tau_k)\xi \end{pmatrix} = \Phi_1 + \begin{pmatrix} 0 & 0 \\ 0 & \xi \end{pmatrix},
\end{aligned} \tag{3.22}$$

with

$$\Phi_1 = \begin{pmatrix} P_1e^{A_1T}(I - M\beta W_1(\tau_k)C_1) & -M\beta P_1e^{A_1T}W_1(\tau_k) \\ M\beta b\rho(\tau_k)\xi & -M\beta b\rho(\tau_k)\xi \end{pmatrix}. \tag{3.23}$$

Let

$$\begin{aligned}
Q &= H^T(P_1 - I)H + \xi L^T(\tau_k)L(\tau_k) - P \\
&= \begin{pmatrix} -I - M\beta \left(U(\tau_k)C_1 + (U(\tau_k)C_1)^T \right) + (\gamma + 2M\beta b\rho(\tau_k)\xi)C_1C_1^T & D_{12} \\ D_{12}^T & \gamma \end{pmatrix},
\end{aligned} \tag{3.24}$$

where

$$\begin{aligned}
\gamma &= M^2\beta^2 \left[G_2(\tau_k) + b^2\rho^2(\tau_k)\xi \right] - 2M\beta b\rho(\tau_k)\xi, \\
D_{12} &= -M\beta U(\tau_k) + [\gamma + M\beta b\rho(\tau_k)\xi]C_1^T,
\end{aligned} \tag{3.25}$$

with

$$\begin{aligned} U(\tau_k) &= (P_1 - I)W(\tau_k), \\ G_2(\tau_k) &= W^T(\tau_k)(P_1 - I)W(\tau_k). \end{aligned} \tag{3.26}$$

Then (3.21) is reduced to

$$\begin{aligned} &E[V(x(kT + T))] - E[V(x(kT))] \\ &= E\left[x^T(kT)Qx(kT)\right] + 2E\left[h^T(kT)\Phi_1x(kT)\right] \\ &\quad + E\left[h^T(kT)Ph(kT)\right] + 2\xi E\left[\tilde{Z}(kT)x_2(kT)\right]. \end{aligned} \tag{3.27}$$

By elementary transformation of matrix, we have

$$Q = \begin{pmatrix} I & D_{12} \\ 0 & 1 \end{pmatrix} D \begin{pmatrix} I & 0 \\ \frac{D_{12}^T}{\gamma} & 1 \end{pmatrix}, \tag{3.28}$$

where

$$D = -\begin{pmatrix} D_{11} & 0 \\ 0 & -\gamma \end{pmatrix} \tag{3.29}$$

with

$$\begin{aligned} D_{11} &= \frac{1}{\gamma} \left[\gamma I + M^2 \beta^2 \left(U(\tau_k)U^T(\tau_k) - M^2 \beta^2 b \rho(\tau_k) \xi \left(U(\tau_k)C_1 + (U(\tau_k)C_1)^T \right) \right) \right. \\ &\quad \left. + M^2 \beta^2 b^2 \rho^2(\tau_k) \xi^2 C_1^T C_1 \right]. \end{aligned} \tag{3.30}$$

Let $\lambda_m(\cdot)$ and $\lambda_M(\cdot)$ denote the minimum and maximum eigenvalues of a matrix, respectively. It is obvious that D is a negative definite matrix if D_{11} is positive definite matrix with $\gamma < 0$. Note that, with $\gamma < 0$, we have

$$\begin{aligned} \lambda_m(D_{11}) &\geq 1 + (M\beta)^2 \gamma^{-1} \left\{ \lambda_M \left[U(\tau_k)U^T(\tau_k) \right] \right. \\ &\quad \left. - b \rho(\tau_k) \xi \lambda_m \left[U(\tau_k)C_1 + (U(\tau_k)C_1)^T \right] + b^2 \rho^2(\tau_k) \xi^2 \lambda_M \left[C_1^T C_1 \right] \right\}. \end{aligned} \tag{3.31}$$

Therefore, the claim that D_{11} is positive definite is true if

$$\begin{aligned} & (M\beta)\left(G_2(\tau_k) + b^2\rho^2(\tau_k)\xi\right) - 2b\rho(\tau_k)\xi \\ & + (M\beta)\left\{\lambda_M\left[U(\tau_k)U^T(\tau_k)\right] - b\rho(\tau_k)\xi\lambda_m\left[U(\tau_k)C_1 + (U(\tau_k)C_1)^T\right] \right. \\ & \left. + b^2\rho^2(\tau_k)\xi^2\lambda_M\left[C_1^TC_1\right]\right\} \leq 0 \end{aligned} \quad (3.32)$$

with

$$M\beta\left(G_2(\tau_k) + b^2\rho^2(\tau_k)\xi\right) - 2b\rho(\tau_k)\xi < 0. \quad (3.33)$$

That is,

$$\begin{aligned} M\beta & < \inf_{\tau_k \in [0, \infty]} \frac{2b\rho(\tau_k)\xi}{G_2(\tau_k) + b^2\xi + \max(0, \theta(\tau_k))} \\ & = \inf_{T_k \in [0, T]} \frac{2b\xi}{G_2(T_k) + b^2\xi + \max(0, \theta(T_k))}, \end{aligned} \quad (3.34)$$

where

$$\begin{aligned} \theta(\tau_k) & = b^2\rho^2(\tau_k)\xi^2\lambda_M\left(C_1^TC_1\right) - b\rho(\tau_k)\xi\lambda_m\left(U(\tau_k)C_1 + (U(\tau_k)C_1)^T\right) \\ & \quad + \lambda_M\left(U(\tau_k)U^T(\tau_k)\right), \\ \theta(T_k) & = b^2\xi^2\lambda_M\left(C_1^TC_1\right) - b\xi\lambda_m\left(U(T_k)C_1 + (U(T_k)C_1)^T\right) \\ & \quad + \lambda_M\left(U(T_k)U^T(T_k)\right). \end{aligned} \quad (3.35)$$

Thus, whenever $M\beta$ satisfies (3.34) for some positive constant ξ , then D is negative definite matrix, and, therefore, Q is negative definite matrix. By elementary transform of matrix, one observes that

$$\Phi_1 = \begin{pmatrix} I & P_1 e^{A_1 T} & \frac{W(\tau_k)}{b\rho(\tau_k)\xi} \\ 0 & & 1 \end{pmatrix} \Phi_2 \begin{pmatrix} I & 0 \\ C_1 & 1 \end{pmatrix}, \quad (3.36)$$

where

$$\Phi_2 = \begin{pmatrix} P_1 e^{A_1 T} & 0 \\ 0 & -M\beta\rho(\tau_k)b\xi \end{pmatrix}. \quad (3.37)$$

So, we have $\|\Phi_1\| \leq \|\Phi_2\|$.

Now we establish an estimate for $h(kT)$. Let $\mu > 0$ be arbitrary. We will show that there exists $\delta > 0$ such that $E\|h(kT)\|^2 < \mu E\|x(kT)\|^2$ whenever $\|G\| < \delta$.

For $t \in [kT, kT + T]$, we have

$$x(t) = x(kT) + \int_{kT}^t (Ax(s) + Bu(s))ds + \int_{kT}^t Gx(s)dW_s. \quad (3.38)$$

Note that

$$E\left(\int_{kT}^t \|B\| |u(s)| ds\right)^2 \leq M^2 \beta^2 \|B\|^2 \|C\|^2 E\|x(kT)\|^2. \quad (3.39)$$

Then,

$$\begin{aligned} \frac{E\|x(t)\|^2}{4} &\leq E\|x(kT)\|^2 + E\left[\left\|\int_{kT}^t Ax(s)ds\right\|^2\right] \\ &\quad + E\left[\left(\int_{kT}^t \|B\| |u(s)| ds\right)^2\right] + E\left[\left(\int_{kT}^t Gx(s)dW_s\right)^T \left(\int_{kT}^t Gx(s)dW_s\right)\right]. \end{aligned} \quad (3.40)$$

By Lemma 3.1, one observes that

$$E\left[\left(\int_{kT}^t Gx(s)dW_s\right)^T \left(\int_{kT}^t Gx(s)dW_s\right)\right] = E\left[\int_{kT}^t x^T(s)G^T Gx(s)ds\right]. \quad (3.41)$$

Then (3.40) is reduced to

$$\frac{E\|x(t)\|^2}{4} \leq K_0 E\|x(kT)\|^2 + (\|A\|^2 + \|G\|^2) \int_{kT}^t E\|x(s)\|^2 ds, \quad (3.42)$$

where $K_0 = 1 + M^2 \beta^2 \|B\|^2 \|C\|^2$.

By Gronwall inequality, we have

$$\begin{aligned} E\|x(t)\|^2 &\leq 4K_0 e^{(\|A\|^2 + \|G\|^2)(t-kT)} E\|x(kT)\|^2 \\ &\leq 4K_0 e^{(\|A\|^2 + \|G\|^2)T} E\|x(kT)\|^2. \end{aligned} \quad (3.43)$$

Thus,

$$\begin{aligned}
E[\|h(kT)\|^2] &= E\left[\left(\int_{kT}^{kT+T} e^{A_1(kT+T-s)} G_1 x_1(s) dW_s\right)^T \left(\int_{kT}^{kT+T} e^{A_1(kT+T-s)} G_1 x_1(s) dW_s\right)\right] \\
&\quad + E\left[\left(\int_{kT}^{kT+T} g x_2(s) dW_s\right)^T \left(\int_{kT}^{kT+T} g x_2(s) dW_s\right)\right] \\
&= E\left[\int_{kT}^{kT+T} x_1^T(s) G_1^T e^{A_1^T(kT+T-s)} e^{A_1(kT+T-s)} G_1 x_1(s) ds\right] + E\left[\int_{kT}^{kT+T} g^2 x_2^2(s) ds\right] \\
&= E\left[\int_{kT}^{kT+T} x^T(s) \tilde{G}^T \tilde{G} x(s) ds\right] \\
&\leq K_1 \|G\|^2 \int_{kT}^{kT+T} E[\|x(s)\|^2] ds \\
&\leq 4K_0 K_1 \|G\|^2 e^{(\|A\|^2 + \|G\|^2)T} E[\|x(kT)\|^2],
\end{aligned} \tag{3.44}$$

where

$$\begin{aligned}
\tilde{G} &= \begin{pmatrix} e^{A_1(kT+T-s)} & 0 \\ 0 & g \end{pmatrix}, \\
K_1 &= \max_{s \in [0, T]} \|e^{A_1^T s} e^{A_1 s}\|, \quad K = 4K_0 K_1 e^{\|A\|^2 T}.
\end{aligned} \tag{3.45}$$

Thus, there exists a $\delta > 0$ such that $K\|G\|^2 e^{\|G\|^2 T} < \mu$ whenever $\|G\| < \delta$. Choosing $\mu > 0$, so that $c(\mu) = 1 - 2(\|\Phi_2\| + \xi)\sqrt{\mu} - \|P\|\mu > 0$. Then, we obtain that

$$E[V(x(kT+T))] - E[V(x(kT))] \leq -\frac{c(\mu)}{\lambda_M(P)} E[V(x(kT))]. \tag{3.46}$$

Therefore, $E[V(x(kT+T))] - E[V(x(kT))]$ is negative definite.

If we define $\tilde{M} = \{0\}$ and $\tilde{S} = \{\tilde{x} : \tilde{x}(t, \tilde{a}, t_0) = E[V(x(t, a, t_0))]\}$, then we have determined in an unambiguous way a deterministic dynamical system $\{R^+, R^+, R^+, S\}$, corresponding to the stochastic dynamical system $\{R^+, R^n, R^n, S_{(4)}\}$ ($S_{(4)}$ is the trivial solution set of system (3.6)). In this case M is an invariant set for $\{R^+, R^+, R^+, S\}$.

In fact, $\{EV(x(kT))\}$ is uniformly asymptotically stable (see, e.g., [21]). For $t \in [kT, kT+T]$, we have

$$E[V(x(t))] \leq \lambda_M(P) K e^{\|G\|^2 T} E[\|x(kT)\|^2] \leq \frac{\lambda_M(P)}{\lambda_m(P)} V(x(kT)). \tag{3.47}$$

This implies that $E[V(x(t))]$ in the interval $[kT, kT + T]$ is bounded by $rEV(x(kT))$, where $r > 0$ is constant. Therefore, $E[V(x(t))]$ converges to the origin simultaneously with $EV(x(kT))$. We conclude that the trivial solution of deterministic system \tilde{S} is asymptotically stable. Now recall that (\tilde{S}, \tilde{M}) is uniformly stable if, for every $\varepsilon > 0$, $t_0 \in \mathbb{R}^+$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $\tilde{x}(t, \tilde{a}, t_0) < \varepsilon$ for all $t \in \mathbb{R}_{\tilde{a}, t_0}^+ = \mathbb{R}_{a, t_0}^+$, and for all $\tilde{x}(\cdot, \tilde{a}, t_0) \in \tilde{S}$, whenever $\tilde{a} < \delta$.

Since $\tilde{x}(t, \tilde{a}, t_0) = EV(x(t))$, thus, (\tilde{S}, \tilde{M}) is uniformly stable means that, for every $\varepsilon > 0$, $t_0 \in \mathbb{R}^+$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $EV(x(t)) < \varepsilon$ for $t \in \mathbb{R}_{a, t_0}^+$ and for all $x(\cdot, a, t_0) \in S$, whenever $EV(a) < \delta$, since $EV(x(t))/\lambda_M(P) \leq E\|x(t)\|^2 \leq EV(x(t))/\lambda_m(P)$; hence, for every $\varepsilon > 0$, $t_0 \in \mathbb{R}^+$, for any process $x(\cdot, \cdot, a, t_0) \in S$, we have, for $t \in \mathbb{R}_{a, t_0}^+$, $E\|x(t)\|^2 \leq \varepsilon/\lambda_m(P)$, whenever $\|a\| < \delta_1$, where $\delta_1 = \delta/\lambda_M(P)$. This is precisely the definition of uniform stability in the mean square of the trivial solution $x_e = 0$ of $S_{(4)}$. Therefore, the uniform stability of (\tilde{S}, \tilde{M}) is equivalent to the uniform stability in the mean square of $(S_{(4)}, \{x_e\})$.

We can now conclude that the equilibrium $x_e = 0$ of PWM feedback system (3.6) is uniformly asymptotically stable in mean square.

For $p = 2q$, $q \geq 1$, we have

$$\begin{aligned}
 & E[V(x(kT + T))^q] - E[V(x(kT))^q] \\
 &= E\left[(V(x(kT + T)) - V(x(kT))) \times (V(x(kT + T))^{q-1} + \dots + V(x(kT))^{q-1})\right] \\
 &\leq E\left[\left(-\frac{1}{\lambda_M(P)}V(x(kT)) + 2Eh^T(kT)\Phi x(kT) + Eh^T(kT)Ph(kT)\right)\right. \\
 &\quad \left. \times (V(x(kT + T))^{q-1} + \dots + V(x(kT))^{q-1})\right] \tag{3.48} \\
 &\leq -\frac{1}{\lambda_M(P)}E[V(x(kT))^q] + E\left[\left(2h^T(kT)\Phi x(kT) + 2h^T(kT)\Phi h(kT)\right)\right. \\
 &\quad \left. \times (V(x(kT + T))^{q-1} + \dots + V(x(kT))^{q-1})\right].
 \end{aligned}$$

It can easily be verified that the second expectation in the above inequality can always be chosen to be less than $\mu EV(x(kT))^q$ for arbitrary μ , when $\|G\|$ is sufficiently small. Similarly as in [22], we can show that $EV(x(t))^q$ is bounded by $EV(x(kT))^q$ when $t \in [kT, kT + T]$, where $a > 0$ is a constant.

The rest of the proof of the p th moment uniform asymptotic stability of the trivial solution of (3.6) with $p = 2q$ proceeds similarly as the proof of uniform asymptotic stability in the mean square of the trivial solution of (3.6).

Therefore, we have shown that the trivial solution of system (3.6) is the p th moment uniformly asymptotically stable in mean for even integers, and, hence, we conclude the proof for all $p > 0$. \square

Remark 3.3. The upper bound of $M\beta$ is given by (3.34) that can easily be computed and optimized. A simple procedure is presented in [20] to accomplish that. We will employ this procedure in a specific example in Section 4.

Theorem 3.4. *Assume that A_1 is Hurwitz stable, $C_1 = 0$, and $b > 0$. Then the trivial solution of the PWM feedback system (3.6) is the p th moment uniformly asymptotically stable in mean whenever*

$M\beta < 2/b$, and there exists a $\delta > 0$ (an upper bounded of δ will be given in the proof of Theorem 3.2) such that $\|G\| < \delta$.

Proof. When $C_1 = 0$, we have

$$D_{11} = \frac{1}{\gamma} \left[\gamma I + M^2 \beta^2 U(\tau_k) U^T(\tau_k) \right]. \quad (3.49)$$

If $\lambda_m(D_{11}) > 0$, D_{11} is positive definite matrix. And if $-\gamma > 0$, then D is positive negative definite matrix. Therefore, whenever $\lambda_m(D_{11}) > 0$ and $\gamma < 0$, it can easily be verified that

$$\left(M\beta b^2 - 2b \right) \xi + \lambda_M \left(U(\tau_k) U^T(\tau_k) \right) + G_2(\tau_k) > 0. \quad (3.50)$$

If $M\beta < 2/b$, there exists ξ such that (3.50) and $\lambda < 0$ are true for all $M\beta$. Hence, whenever $M\beta < 2/b$ for some positive constant ξ , then Q is negative definite matrix.

The rest of the proof of the uniformly p th moment asymptotic stability in mean of the trivial solution of (3.6) proceeds similarly as that of Theorem 3.2. \square

Remark 3.5. From stability results of Theorem 3.4, we can obtain when upper bound of the parameters of PWM feedback system ($M\beta$) is sufficiently large, and if we choose that b is sufficiently small, then the trivial solution of the system (3.6) is the p th moment uniformly asymptotic stability in mean.

Next, we consider PWM feedback system with additive noise in the plant, described by equations of the form. We have

$$\begin{aligned} dx_1(t) &= A_1 x_1(t) dt + B_1 u(t) dt + \varepsilon_1 x_1(t) dW_t, \\ dx_2(t) &= bu(t) dt + \varepsilon_2 x_2(t) dW_t, \\ y(t) &= C_1 x_1(t) + x_2(t), \end{aligned} \quad (3.51)$$

where $\varepsilon_1 \in R^{n-1}$, $\varepsilon_2 \in R$ have positive components. We let $\varepsilon = (\varepsilon_1 \ \varepsilon_2) \in R^n$ denote the magnitude of the random noise. A block diagram of system (3.51) is shown in Figure 2.

Theorem 3.6. *Assume that A_1 is Hurwitz stable, and the solution processes of the PWM feedback system (3.6) are the p th moment uniformly ultimately bounded provided by $b > 0$, all $M\beta \in (0, \infty)$ (for random disturbance). When $b \leq 0$, the solution processes are unbounded.*

Proof. The proof of Theorem 3.6 is similar as that of [20]. \square

To end this section, we present an algorithm for computing stability bound.

Stability Bound Algorithm

An upper bound of $M\beta$ that satisfies (3.34) can be computed and optimized in the following manner.

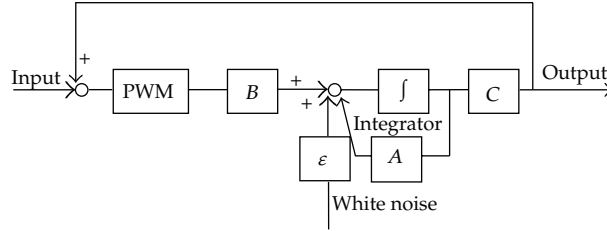


Figure 2: Block diagram of PWM feedback systems subjected to additive disturbances.

- (S1) Determine the matrix P_1 by solving $(e^{A_1 T})^T P_1 e^{A_1 T} - P_1 = -I$.
- (S2) Choose a precision level $\delta > 0$ and a correspondingly dense partition of the interval $[0, T]$, and say the set $\{t_0 = 0, t_1, \dots, t_N = T\}$, where $0 < t_{j+1} - t_j < \delta, j = 0, 1, 2, \dots, N - 1$.
- (S3) For each $i, j = 0, 1, 2, \dots, N - 1$, calculate

$$W(t_j) = \frac{I - e^{-A_1 t_j}}{t_j} A_1^{-1} B, \quad G_2(t_j) = W(t_j)^T (P_1 - I) W(t_j), \quad (3.52)$$

$$U(t_j) = (P_1 - I) W(t_j).$$

- (S4) Search for the largest $M\beta$ such that

$$M\beta < \inf_{t_j \in [0, \infty]} \frac{2b\rho(t_j)\xi}{G_2(t_j) + b^2\xi + \max(0, \theta(t_j))}, \quad (3.53)$$

where

$$\theta(t_j) = b^2\xi^2 \lambda_M(C_1^T C_1) - b\xi \lambda_m(U(t_j)C_1 + (U(t_j)C_1)^T) + \lambda_M(U(t_j)U^T(t_j)). \quad (3.54)$$

- (S5) Repeat steps 1–4, using finer partition of the interval $[0, T]$ until there is no further significant improvement for $M\beta_{\text{opt}}$.

4. Examples

Example 4.1. Consider the system (3.6) with second-order critical stable plant described by transfer function $T(s) = k/s(1 + (s/a)), a > 0$. The state space representation of this system is given by

$$A = \begin{pmatrix} -a & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} k \\ k \end{pmatrix}, \quad C = [1 \ 1]. \quad (4.1)$$

Hence $A_1 = -a$, $B_1 = k$, $C_1 = 1$, and P_1 , $W(T_k)$, $U(T_k)$ in Theorem 3.2 are calculated in this case as

$$\begin{aligned} P_1 &= \frac{1}{1 - e^{-2aT}}, & W(T_k) &= \frac{(e^{aT_k} - 1)k}{aT_k}, \\ U(T_k) &= \frac{k(e^{aT_k} - 1)}{aT_k(e^{2aT} - 1)}, & G_2(T_k) &= \frac{k^2(e^{aT_k} - 1)^2}{a^2T_k^2(e^{2aT} - 1)}, \\ \theta(T_k) &= k^2\xi^2 - \frac{2k^2(e^{aT_k} - 1)}{aT_k(e^{2aT} - 1)}\xi + \frac{k^2(e^{aT_k} - 1)^2}{a^2T_k^2(e^{2aT} - 1)^2} \\ &= k^2\left(\xi - \frac{e^{aT_k} - 1}{aT_k(e^{2aT} - 1)}\right)^2 \geq 0. \end{aligned} \tag{4.2}$$

Choosing $\xi = 1/aT(1 + e^{-aT})$, it can be seen that the stability bound is given by

$$M\beta < \frac{1}{(((e^{aT} - 1)/aT(e^{aT} + 1)) + (1/2))k}. \tag{4.3}$$

Assume that $M = 1$, and then the condition for uniformly asymptotic stability in p th mean is given by

$$\frac{1}{k\beta} > \frac{1}{aT} - \frac{2}{aT(e^{aT} + 1)} + \frac{1}{2}. \tag{4.4}$$

In Example 4.1, we obtain the stability results depicted in Figure 3 ($k\beta_{\text{opt}}$ versus aT).

Example 4.2. Consider PWM feedback system (3.6) with transfer function $T(s) = 1/s(1 + s)$. The state space representation of this system is given by

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad C = [1 \ 1]. \tag{4.5}$$

Letting $\delta = 0.01$, and assuming $T = 1$, we obtain the estimation of the upper bound of $M\beta$, that is, $M\beta \leq 1.0394$. For $M\beta \in (0, 1.0394)$, we compute μ_{\max} such that $c(\mu) = 1 - 2(\|R_2\| + \xi)\sqrt{\mu} - \|P\|\mu > 0$ is true for all $\mu < \mu_{\max}$. Next, we compute δ_{\max} such that $K\delta^2e^{\delta^2T} < \mu_{\max}$ is true for all $\delta < \delta_{\max}$, where $K = 4K_1(1 + M^2\beta^2\|B\|^2\|C\|^2)e^{\|A\|^2T}$ and $K_1 = \max_{s \in [0, T]} \|e^{A_1^T s} e^{A_1 s}\|$. In Figure 4, we depict the estimates of the upper bound δ_{\max} of $\|G\|$ versus $M\beta$.

We observe that δ_{\max} decreases as $M\beta$ increases. When the states are sufficiently far away from the origin so that $T_k = T$, and the output of pulse-width modulator is either $+M$ or $-M$. Therefore, as M increases (for fixed β), the maximum $\|G\|$ allowable to ensure the p th moment uniform asymptotic stability in mean will decrease; besides, if disturbance of the feedback system (3.6) is increase (less than δ_{\max}), the trivial solution of system (3.6) is the p th moment uniformly asymptotically stable in mean by decreasing the value of $M\beta$, as shown in Figure 4.

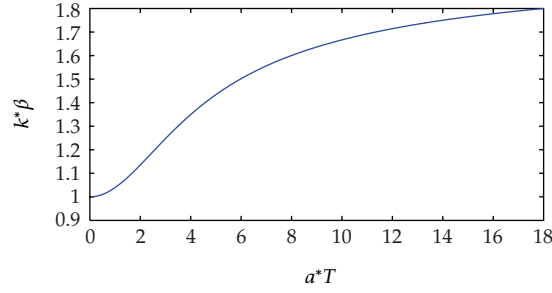


Figure 3: Estimates of upper bounds for $k\beta$ for the second-order PWM feedback system of Example 4.1.

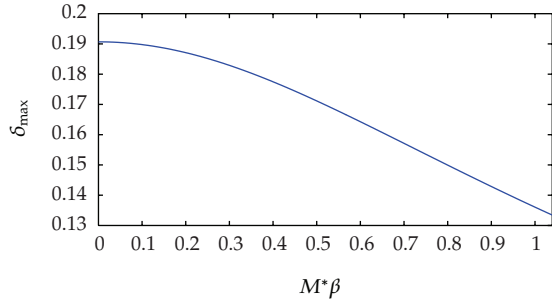


Figure 4: Upper bounds for $\|G\|$ when $M\beta \in (0, 1.0394)$.

In Figure 5, we plot the sample response of $x = (x_1 \ x_2)^T$ with $M = 0.2$, $T = \beta = 1$, $G = \begin{pmatrix} 0.02 & 0 \\ 0 & 0.006 \end{pmatrix}$ and $x(0) = (2 \ 0.5)^T$. We generated 100 sample responses of $x(t)$ and computed the average $\overline{\|x(t)\|}$ of $\|x(t)\|$. Figure 6 shows the average $\overline{\|x(t)\|}$ tending to zero as time t increases. However, $\overline{\|x(t)\|}$ does not diminish entirely to zero since it is an approximation to the mean $E\|x(t)\|$.

5. Conclusions

We studied the critical case of PWM feedback systems with random perturbations and establish several Lyapunov and Lagrange criteria for the p th moment uniform asymptotical stability in mean, and then we presented an algorithm to compute the upper bound for the parameters of PWM and finally gave two numerical examples to verify the effectiveness of theoretical results. We characterized the relationship between the parameters of PWM and the coefficients of state vectors of the feedback systems and showed that when the random disturbance is sufficiently small such that PWM feedback system is the p th moment globally asymptotically stable in mean provided that the upper bounds of parameters of pulse-width modulator are selected properly.

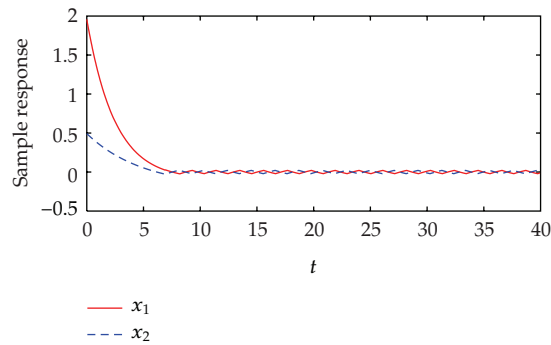


Figure 5: Sample response of PWM feedback system (3.6) in Example 4.2.

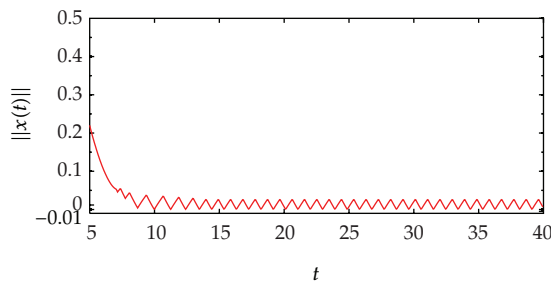


Figure 6: $\|x(t)\|$ of PWM feedback system (3.6) in Example 4.2.

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