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Research Article

Iterative Algorithms for General Multivalued Variational Inequalities

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We introduce and study some new classes of variational inequalities and the Wiener-Hopf equations. Using essentially the projection technique, we establish the equivalence between these problems. This equivalence is used to suggest and analyze some iterative methods for solving the general multivalued variational in equalities in conjunction with nonexpansive mappings. We prove a strong convergence result for finding the common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the general multivalued variational inequalities under some mild conditions. Several special cases are also discussed.

1. Introduction

Variational inequality problems were initially studied by Stampacchia in 1964. Variational inequalities have applications in diverse disciplines such as partial differential equations, optimal control, optimization, mathematical programming, mechanics, and finance, see [1–33] and the references therein. Variational inequalities have been extended and generalized in several directions using novel and innovative techniques. It is a common practice to study these variational inequalities in the setting of convexity. It has been observed that the optimality conditions of the differentiable convex functions can be characterized by the variational inequalities. In recent years, it has been shown that the minimum of the differentiable nonconvex functions can also be characterized by the variational inequalities. Motivated and inspired by these developments, Noor [19] has introduced a new type of variational inequality involving two nonlinear operators, which is called the general variational inequality. It is worth mentioning that this general variational inequality is remarkable different from the so-called general variational inequality which was introduced by Noor [16] in 1988. Noor [19] proved that the general variational inequalities are equivalent to nonlinear projection equations and the Wiener-Hopf equations by using the projection technique. Using this equivalent formulation, Noor [19] suggested and analyzed some iterative algorithms for solving the special general variational inequalities and further proved these algorithms have strong convergence. In this paper, we introduce and consider a new class of variational inequalities, which is called the general multivalued variational inequality. Using essentially the projection technique, we establish the equivalence between the multivalued variational inequalities and the multivalued Wiener-Hopf equations.

Related to the variational inequalities, we have the problem of finding the fixed points of the nonexpansive mappings, which is the subject of current interest in functional analysis. It is natural to consider a unified approach to these two different problems. Noor and Huang [21] considered the problem of finding the common element of the set of the solutions of variational inequalities and the set of the fixed points of the nonexpansive mappings. We use the Wiener-Hopf technique to suggest and analyze some iterative methods for finding the common element the common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the special general variational inequalities. We also consider the convergence criteria of the proposed algorithms under suitable conditions. Several special cases are also discussed.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : H \to 2^H$ be a multivalued mapping. Let $F, g : H \to H$ be two nonlinear operators. We consider the problem of finding $u \in C$ and $w \in A(u)$ such that

$$\langle F(u) + w, g(v) - u \rangle \geq 0, \quad \forall v \in H,\ g(v) \in C. \tag{2.1}$$

Inequality of type (2.1) is called the general multivalued variational inequality. We will denote the set of solutions of the special general variational inequality (2.1) by SGVI(F, A, g). The general multivalued variational inequality (2.1) can be written in the following equivalent form, that is, find $u \in C,\ w \in A(u)$ and $g(u) \in C$ such that

$$\langle \rho(F(u) + w) + u - g(u), g(v) - u \rangle \geq 0, \quad \forall v \in H,\ g(v) \in C. \tag{2.2}$$

This equivalent formulation is very important and plays a crucial role in the development of the iterative methods for solving the general multivalued variational inequalities.

We now discuss several special cases.

Special Cases

(A) If $w = 0$, then (2.1) reduces to: find $u \in C$ such that

$$\langle F(u), g(v) - u \rangle \geq 0, \quad \forall v \in H,\ g(v) \in C, \tag{2.3}$$

which is called the general variational inequality, introduced and studied by Noor [19]. It has been shown that the minimum of a class of differentiable functions can be characterized by the general variational inequality of type (2.3).

(B) If $g \equiv I$, the identity operator, then (2.1) reduces to find $u \in C$ and $w \in A(u)$ such that

$$\langle F(u) + w, v - u \rangle \geq 0, \quad \forall v \in C, \tag{2.4}$$

which is known as the mildly nonlinear multivalued variational inequality and has been studied extensively.

If F and A are single-valued nonlinear operators, then problem (2.1) is equivalent to finding $u \in C$ such that

$$\langle F(u) + A(u), v - u \rangle \geq 0, \quad \forall v \in C, \tag{2.5}$$

which is known as the mildly nonlinear variational inequality, the origin of which can be traced back to Noor [15].

(C) If $w = 0$ and $g \equiv I$, then (2.1) reduces to: find $u \in C$ such that

$$\langle F(u), v - u \rangle \geq 0, \quad \forall v \in C, \tag{2.6}$$

which is wellknown as the variational inequality, originally introduced and studied by Stampacchia [24] in 1964. It is clear from the above discussion that general multivalued variational inequality is quite general one. It has been shown that a wide class of problems arising in various discipline of mathematical and engineering sciences can be studied via the general multivalued variational inequalities (2.1) and its special cases.

In the sequel, we need the following well-known lemma.

Lemma 2.1. *For a given $z \in H$, $u \in C$ satisfies the inequality*

$$\langle u - z, v - u \rangle, \quad \forall v \in C, \tag{2.7}$$

if and only if

$$u = P_C z, \tag{2.8}$$

where P_C is the projection of H into the closed convex set C.

By using Lemma 2.1, one can prove that the general multivalued variational inequality (2.1) is equivalent to the following fixed point problem.

Lemma 2.2. *$u \in C$ is a solution of the special general variational inequality* (2.1) *if and only if $u \in C$ satisfies the relation*

$$u = P_C\left[g(u) - \rho(F(u) + w)\right], \tag{2.9}$$

where $\rho > 0$ is a constant.

Related to the general multivalued variational inequality (2.1), we consider the problem of solving the Wiener-Hopf equations. Let $F, g : H \to H$ be two nonlinear operators and $A : H \to 2^H$ be a multi-valued relaxed monotone operator. Let $Q_C = I - gP_C$, where I is the identity operator. We consider the problem of finding $y \in H : w \in A(y)$ such that

$$FP_C y + w + \rho^{-1} Q_C y = 0, \tag{2.10}$$

which is called the special general multivalued Wiener-Hopf equations. We use SGWH(F, A, g) to denote the set of solutions of the special general multivalued Wiener-Hopf equations. For different and suitable choice of the operators F, A, we can obtain various forms of the Wiener-Hopf equations, which have been studied by Noor [17], Shi [22], and others.

Using essentially the technique of Noor [17, 18] and applying Lemma 2.2, one can establish the equivalence between the Wiener-Hopf equations and the general multivalued variational inequalities (2.1). To convey an idea of the technique and for the sake of completeness, we include its proof.

Lemma 2.3. *If $u \in$ SGVI(F, A, g), then $y \in H$ and $w \in A(y)$ satisfy the general Wiener-Hopf equations* (2.10)*, where*

$$\begin{gathered} y = g(u) - \rho(F(u) + w), \\ u = P_C y, \end{gathered} \tag{2.11}$$

where $\rho > 0$ is a constant.

Proof. Let $u \in$ SGVI(F, A, g). Then, from Lemma 2.2, we have

$$u = P_C[g(u) - \rho(F(u) + w)]. \tag{2.12}$$

Let

$$y = g(u) - \rho(F(u) + w), \quad \forall w \in A(u). \tag{2.13}$$

Then, we have

$$u = P_C y. \tag{2.14}$$

Therefore, from (2.13), we obtain

$$y = gP_C y - \rho(FP_C y + w). \tag{2.15}$$

It follows that

$$FP_C y + w + \rho^{-1} Q_C y = 0, \quad \forall w \in AP_C y, \tag{2.16}$$

where $Q_C = I - gP_C$, which is exactly the general Wiener-Hopf equations (2.10). This completes the proof. □

Remark 2.4. Let $S : C \to C$ be a nonexpansive mapping. If $u \in F(S) \cap \mathrm{SGVI}(F, A, g)$, then one can easy to see

$$u = Su = SP_C[g(u) - \rho(F(u) + w)], \tag{2.17}$$

which is implies that

$$u = (1 - \alpha_n)u + \alpha_n SP_C[g(u) - \rho(F(u) + w)], \tag{2.18}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$.

Using Remark 2.4 and Lemma 2.3, we can suggest the following algorithm for finding the common element of the solutions set of the variational inequalities and the set of fixed points of a nonexpansive mapping.

Algorithm 2.5. For a given $x_0 \in H$ arbitrarily, let the sequence $\{x_n\}$ be generated by

$$\begin{gathered} y_{n+1} = (1 - \alpha_n)x_n + \alpha_n[g(x_n) - \rho(F(x_n) + w_n)], \\ x_n = SP_C y_n, \end{gathered} \tag{2.19}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\rho > 0$ is some constant.

Note that, if $S \equiv I$, then Algorithm 2.5 reduces to the following iterative method for solving the general variational inequalities.

Algorithm 2.6. For a given $x_0 \in H$ arbitrarily, let the sequence$\{x_n\}$ be generated by

$$\begin{gathered} y_{n+1} = (1 - \alpha_n)x_n + \alpha_n[g(x_n) - \rho(F(x_n) + w_n)], \\ x_n = P_C y_n, \end{gathered} \tag{2.20}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\rho > 0$ is some constant.

If $\{w_n\} = 0$, then Algorithm 2.5 reduces to the following iterative method for solving the general variational inequalities (2.3), which was considered by Noor [19].

Algorithm 2.7. For a given $x_0 \in H$ arbitrarily, let the sequence $\{x_n\}$ be generated by

$$\begin{gathered} y_{n+1} = (1 - \alpha_n)x_n + \alpha_n(g(x_n) - \rho F(x_n)), \\ x_n = SP_C y_n, \end{gathered} \tag{2.21}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\rho > 0$ is some constant.

If $g = S \equiv I$ and $\{w_n\} = 0$, then Algorithm 2.5 reduces to the following iterative method for solving the variational inequalities (2.6).

Algorithm 2.8. For a given $x_0 \in H$ arbitrarily, let the sequence $\{x_n\}$ be generated by

$$\begin{aligned} y_{n+1} &= (1-\alpha_n)x_n + \alpha_n\big(x_n - \rho F(x_n)\big), \\ x_n &= P_C y_n, \end{aligned} \tag{2.22}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\rho > 0$ is some constant.

We recall the well-known concepts. The multivalued mapping A is said to be γ-*Lipschitzian* if there exists a constant $\gamma > 0$ such that

$$\|w_1 - w_2\| \le \gamma\|u - v\|, \quad \forall w_1 \in A(u),\ w_2 \in A(v). \tag{2.23}$$

Recall that a mapping $S : C \to C$ is called *nonexpansive* if

$$\|Sx - Sy\| \le \|x - y\|, \quad \forall x, y \in H. \tag{2.24}$$

We will use $F(S)$ to denote the set of fixed points of S.

A mapping $F : C \to H$ is called α-*strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle F(x) - F(y), x - y\rangle \ge \alpha\|x - y\|^2, \quad \forall x, y \in H, \tag{2.25}$$

and β-*Lipschitz continuous* if there exists a constant $\beta > 0$ such that

$$\|F(x) - F(y)\| \le \beta\|x - y\|, \quad \forall x, y \in H. \tag{2.26}$$

3. Main Results

Now we state and prove our main result.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \to H$ be an α-strongly monotone and β-Lipschitz continuous mapping, $g : C \to H$ an σ-strongly monotone and δ-Lipschitz continuous mapping and $A : H \to 2^H$ be a γ-Lipschitz continuous mapping. Let $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap \mathrm{SGVI}(F, A, g) \ne \emptyset$. Assume that*

$$\begin{gathered} \left|\rho - \frac{\alpha - \gamma(1-k)}{\beta^2 - \gamma^2}\right| < \frac{\sqrt{(\alpha - \gamma(1-k))^2 - (\beta^2 - \gamma^2)k(2-k)}}{\beta^2 - \gamma^2}, \\ \alpha > \gamma(1-k) + \sqrt{(\beta^2 - \gamma^2)k(2-k)}, \quad \gamma\rho < 1 - k,\ k < 1, \end{gathered} \tag{3.1}$$

where

$$k = \sqrt{1 - 2\sigma + \delta^2}, \qquad \sum_{n=0}^{\infty} \alpha_n = \infty, \tag{3.2}$$

then the approximate solution $\{y_{n+1}\}$ *obtained from Algorithm 2.5 converges strongly to* $y \in \mathrm{SGWH}(F, A, g)$.

Proof. Let $x^* \in F(S) \cap \mathrm{SGVI}(F, A, g)$. Then, from Remark 2.4, we have

$$x^* = SP_C y, \qquad y = (1 - \alpha_n)x^* + \alpha_n\left[g(x^*) - \rho(F(x^*) + w)\right], \tag{3.3}$$

where $y \in H$ and $w \in A(y)$ satisfy the general Wiener-Hopf equations (2.10).

From (2.19) and (3.1), we have

$$\|x_{n+1} - x^*\| = \left\|SP_C y_n - SP_C y\right\| \le \left\|y_n - y\right\|. \tag{3.4}$$

From (2.19), we have

$$\begin{aligned} \left\|y_{n+1} - y\right\| &\le (1 - \alpha_n)\|x_n - x^*\| \\ &\quad + \alpha_n\left\|g(x_n) - \rho(F(x_n) + w_n) - \left[g(x^*) - \rho(F(x^*) + w)\right]\right\| \\ &\le (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\left\|x_n - x^* - (g(x_n) - g(x^*))\right\| \\ &\quad + \alpha_n\left\|x_n - x^* - \rho(F(x_n) - F(x^*))\right\| + \rho\alpha_n\|w_n - w\|. \end{aligned} \tag{3.5}$$

Since g is an σ-strongly monotone and δ-Lipschitz continuous mapping, we have

$$\begin{aligned} &\left\|x_n - x^* - (g(x_n) - g(x^*))\right\|^2 \\ &\quad = \|x_n - x^*\|^2 - 2\langle g(x_n) - g(x^*), x_n - x^*\rangle + \left\|g(x_n) - g(x^*)\right\|^2 \\ &\quad \le \|x_n - x^*\|^2 - 2\sigma\|x_n - x^*\|^2 + \delta^2\|x_n - x^*\|^2 \\ &\quad = \left(1 - 2\sigma + \delta^2\right)\|x_n - x^*\|^2 = k^2\|x_n - x^*\|^2, \end{aligned} \tag{3.6}$$

where $k = \sqrt{1 - 2\sigma + \delta^2}$.

At the same time, we note that F is an α-strongly monotone and β-Lipschitz continuous mapping, so we have

$$\begin{aligned}
&\left\|x_n - x^* - \rho(F(x_n) - F(x^*))\right\|^2 \\
&\quad = \|x_n - x^*\|^2 - 2\rho\langle F(x_n) - F(x^*), x_n - x^*\rangle + \rho^2\|F(x_n) - F(x^*)\|^2 \\
&\quad = \left(1 - 2\rho\alpha + \rho^2\beta^2\right)\|x_n - x^*\|^2.
\end{aligned} \tag{3.7}$$

From (3.5)–(3.7), we have

$$\begin{aligned}
\left\|y_{n+1} - y\right\| &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\left(k + \rho\gamma + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\right)\|x_n - x^*\| \\
&= (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta\|x_n - x^*\|,
\end{aligned} \tag{3.8}$$

where

$$\theta = k + \gamma\rho + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}. \tag{3.9}$$

Using (3.1), we see that $\theta < 1$. Substituting (3.4) into (3.8), we have

$$\begin{aligned}
\left\|y_{n+1} - y\right\| &\leq \left[(1-\alpha_n) + \left(k + \gamma\rho + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\right)\alpha_n\right]\left\|y_n - y\right\| \\
&= [1 - (1-\theta)\alpha_n]\|x_n - x^*\| \\
&\leq \prod_{i=0}^{n}[1 - (1-\theta)\alpha_i]\left\|y_0 - y\right\|.
\end{aligned} \tag{3.10}$$

Since $\sum_{n=0}^{\infty}\alpha_n$ diverges and $1 - \theta > 0$, we have $\prod_{i=0}^{n}[1 - (1-\theta)\alpha_i] = 0$. Consequently, the sequence $\{y_n\}$ converges strongly to y in H, the required result. □

4. Conclusion

One of the most difficult and important problems in variational inequalities is the development of an efficient numerical methods. One of the technique is called the projection method and its variant forms. Projection method represent an important tool for finding the approximate solution of various types of variational inequalities. The projection type methods were developed in 1970s. The main idea in this techniques is to establish the equivalence between the variational inequalities and the fixed point problem using the concept of projection. These methods have been extended and modified in various ways. Shi [22] considered the problem of solving a system of nonlinear projections, which are called the Wiener-Hopf equations. It has been shown by Shi [22] that the Wiener-Hopf equations are equivalent to the variational inequalities. It turns out that this alternative formulation is more general and flexible. It has been shown that the Wiener-Hopf equations provide us a simple, natural, elegant, and convenient device to develop some efficient numerical methods for solving variational and complementarity problems. In this paper, we introduce and study some new classes of variational inequalities and Wiener-Hopf equations. Using essentially the projection technique, we establish the equivalence between these problems. This equivalence is used to suggest and analyze some iterative methods for solving the general multivalued variational inequalities in conjunction with nonexpansive mappings. We prove a strong convergence result for finding the common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the general multivalued variational inequalities under some mild conditions. Several special cases are also discussed. The ideas and techniques of this paper may be a starting point for a wide range of novel and innovative applications in various fields.

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