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Research Article

Singularities of Focal Surfaces of Null Cartan Curves in Minkowski 3-Space

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Singularities of the focal surfaces and the binormal indicatrix associated with a null Cartan curve will be investigated in Minkowski 3-space. The relationships will be revealed between singularities of the above two subjects and differential geometric invariants of null Cartan curves; these invariants are deeply related to the order of contact of null Cartan curves with tangential planar bundle of lightcone. Finally, we give an example to illustrate our findings.

1. Introduction

If we imagine a regular curve in $\mathbb{R}^3_1$, denote γ, and then imagine the set of all principal normal lines intersecting this curve. Unless γ is a line, these lines will all meet some locus, in fact, it is the locus of centres of curvatures of γ, which we call the *focal sets*. It is obvious that the focal set would be a point or a new curve depending on given curves $\gamma(s)$. Focal sets are useful to the study of certain optical phenomena (namely, scattering, in fact a rainbow is caused by caustics), expressing some geometrical results within fluid mechanics as well as describing many medical anomalies [1–4], and so it is important to study the geometric properties related to the *focal curve* (i.e., the locus of focal set is a curve) of a curve.

It is well known that there exist spacelike curve, timelike curve, and null curve in Minkowski spacetime. For nonnull curve in Minkowski space, many of the classical results from Riemannian geometry have Lorentz counterparts. In fact, spacelike curves or timelike curves can be studied by approaches similar to those taken in positive definite Riemannian geometry. Nonnull curves in Minkowski space, regarding singularity, have been studied

extensively by, among others, the second author and by Izumiya et al. [5–10]. The importance of the study of null curves and its presence in the physical theories are clear [4, 11–18]. Nersessian and Ramos [19] also show us that there exists a geometrical particle model based entirely on the geometry of the null curves, in Minkowskian 4-dimensional spacetime which under quantization yields the wave equations corresponding to massive spinning particles of arbitrary spin. They have also studied the simplest geometrical particle model which is associated with null curves in Minkowski 3-space [20]. However, null curves have many properties which are very different from spacelike and timelike curves [11, 21, 22]. In other words, null curve theory has many results which have no Riemannian analogues. In geometry of null curves difficulties arise because the arc length vanishes, so that it is impossible to normalize the tangent vector in the usual way. Bonner introduces the Cartan frame as the most useful one and he uses this frame to study the behaviors of a null curve [23]. Thus, one can use these fundamental results as the basic tools in researching the geometry of null curves. However, to the best of the authors' knowledge, the singularities of surfaces and curves as they relate to null Cartan curves (see Section 2) have not been considered in the literature, aside from our studies in de Sitter 3-space [24, 25]. Thus, the current study hopes to serve such a need; in this paper, we study *the focal surfaces* and *the binormal indicatrix* associated with a null Cartan curve in Minkowski 3-space from the standpoint of singularity theory.

A singularity is a point (or a function) at which a function (or surface resp.) blows up. It is a point at which a function is at a maximum/minimum or a surface is no longer smooth and regular. Much of the time, these singularities affect a surface not only at a certain point but around it also, and for this reason, we have focused our attention on germs in a local neighbourhood around a fixed point. To allow a useful study of these singularities, we consider *volume like distance functions* (denoted by $D : I \times \mathbb{R}^3_1 \to R, D(s, \mathbf{v}) = \langle \gamma(s) - \mathbf{v}, B(s) \rangle$) locally around the point $(s_0, \mathbf{v}_0)$ These functions are the unfoldings of these singularities in the local neighbourhood of $(s_0, \mathbf{v}_0)$, and depend only on the germ that they are unfolding. In this paper, we create these functions by varying a fixed point $\mathbf{v}$ in the volumelike distance function $D(s, \mathbf{v}) = \langle \gamma(s) - \mathbf{v}, B(s) \rangle$, to get a family of functions. We show that these singularities are versally unfolded by the family of volumelike distance functions. If the singularity of D_v is A_k-type ($k = 1, 2, 3$) and the corresponding 3-parameter unfolding is versal, then applying Bruce's theory (cf. [26]), we know that discriminant set of the 3-parameter unfolding is locally diffeomorphic to cuspidal edge or swallowtail; thus, we finished the classification of singularities of the focal surface (because the discriminant set of the unfolding is precisely the focal surface of a null Cartan curve). Moreover, we see the A_k-singularity ($k = 1, 2, 3$) of D_v are closely related to the new geometric invariant $\sigma(s)$. The singular point of the focal surface corresponds to the point of the null Cartan curve which has degenerated contact with a *tangential planer bundle* of a lightcone. As a consequence, the new Lorentzian invariant $\sigma(s)$ describes the contact between the tangential planer bundle of a lightcone and null Cartan curve $\gamma(s)$. It is important that the properties of volumelike distance function (or null Cartan curve $\gamma(s)$) needed to be generic [27]. Once we proved that the properties of the volumelike distance function were generic, we could deduce that all singularities were stable under small perturbations for our family of volumelike distance function. If transversality is satisfied for the volume like distance function of a null Cartan curve $\gamma(s)$, then the properties of the volumelike distance function are generic. By considering transversality, we prove that these properties given by us are generic. On the other hand, the *binormal indicatrix* of a null Cartan curve is a curve that lies in lightcone, by defining the *lightcone height function* and adopting the method similar to those taken in the study of focal surface, we can classify the singularities

of the binormal indicatrix the types of these singularities have a direct relation with the other Lorentzian invariant $k(s)$.

A brief description of the organization of this paper is as follows. The main results in this paper are stated in Theorems 2.1 and 6.3. In Section 3, we give volumelike distance functions and light-like height functions of a null Cartan curve, by which we can obtain several geometric invariants of the null Cartan curve. The geometric meaning of Theorem 2.1 is described in Section 4. We give the proof of Theorem 2.1 in Section 5. In Section 7, we give an example to illustrate the results of Theorem 2.1.

2. Preliminaries

Let $\mathbb{R}^3_1$ denote the 3-dimensional Minkowski space, that is to say, the manifold $\mathbb{R}^3$ with a flat Lorentz metric $\langle , \rangle$ of signature $(-,+,+)$, for any vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ in $\mathbb{R}^3$, we set $\langle \mathbf{x}, \mathbf{y} \rangle = -x_1y_1 + x_2y_2 + x_3y_3$. We also define a vector

$$\mathbf{x} \wedge \mathbf{y} = \begin{vmatrix} -\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}, \tag{2.1}$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is the canonical basis of $\mathbb{R}^3_1$. We say that a vector $\mathbf{x} \in \mathbb{R}^3_1 \setminus \{\mathbf{0}\}$ is *spacelike, null,* or *timelike* if $\langle \mathbf{x}, \mathbf{x} \rangle$ is positive, zero, or negative, respectively. The *norm* of a vector $\mathbf{x} \in \mathbb{R}^3_1$ is defined by $\|\mathbf{x}\| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$. We call $\mathbf{x}$ a *unit vector* if $\|\mathbf{x}\| = 1$.

Let $\gamma : I \to \mathbb{R}^3_1$ be a smooth regular curve in $\mathbb{R}^3_1$ (i.e., $\dot{\gamma}(t) \neq 0$ for any $t \in I$), parametrized by an open interval I. For any $t \in I$, the curve γ is called a *spacelike curve*, a *null (lightlike) curve,* or a *timelike curve* if all its velocity vectors satisfy $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle > 0$, $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle = 0$ or $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle < 0$, respectively. We call γ a *non-null curve* if γ is a timelike curve or a spacelike curve.

Let $\gamma : I \to \mathbb{R}^3_1$ be a null curve in $\mathbb{R}^3_1$ (i.e., $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle = 0$ for any $t \in I$). Now suppose that γ is framed by a null frame. A null frame $F = \{\xi = d\gamma/dt, N, B\}$ at a point of $\mathbb{R}^3_1$ is a positively oriented 3-tuple of vectors satisfying

$$\begin{aligned} \langle \xi, \xi \rangle = \langle B, B \rangle = 0, \quad \langle \xi, B \rangle = 1, \\ \langle \xi, N \rangle = \langle B, N \rangle = 0, \quad \langle N, N \rangle = 1. \end{aligned} \tag{2.2}$$

The Frenet formula of γ with respect to F is given by

$$\begin{aligned} \frac{d\xi}{dt} &= -h\xi + k_1 N, \\ \frac{dB}{dt} &= hB + k_2 N, \\ \frac{dN}{dt} &= -k_2\xi - k_1 B. \end{aligned} \tag{2.3}$$

The functions h, k_1, and k_2 are called the *curvature functions* of γ (cf. [11]). Employing the usual terminology, the spacelike unit vector filed N will be called the *principal normal vector*

filed. The null vector filed B is called the *binormal vector filed*. Null frames of null curves are not uniquely determined. Therefore, the curve and a frame must be given together.

There always exists a parameter s of γ such that $h = 0$ in (2.3). This parameter is called a *distinguished parameter* of γ, which is uniquely determined for prescribed screen vector bundle (i.e., a complement in $\langle d\gamma/dt\rangle^{\perp}$ to $\langle d\gamma/dt\rangle$) up to affine transformation [11].

Let $\gamma(s)$ be a null curve with a distinguished parameter in $\mathbb{R}^3_1$ (i.e., $h = 0$ in (2.3)). Moreover, we assume that $\gamma'(s), \gamma''(s), \gamma'''(s)$ are linearly independent for all s. Then, we consider the basis $E = \{\gamma'(s), \gamma''(s), \gamma'''(s)\}$ such that $\langle \gamma''(s), \gamma''(s)\rangle = k_1(s) = 1$. We choose the $\xi = d\gamma/ds, N = \gamma''$, then there exists only one null frame $F = \{\xi, N, B\}$ for which $\gamma(s)$ is a framed null curve with Frenet equations [11]:

$$\begin{aligned} \frac{d\xi}{ds} &= N, \\ \frac{dB}{ds} &= k_2 N, \\ \frac{dN}{ds} &= -k_2\xi - B, \end{aligned} \tag{2.4}$$

where $\xi = d\gamma/ds$, $N = \gamma''$, $B = -\gamma''' - k_2\gamma'$, $k_2 = (1/2)\langle \gamma''', \gamma'''\rangle$. We call (2.4) the *Cartan Frenet equations* and $\gamma(s)$ their *null Cartan curve* [11]. We remark that the curvature function k_2 is an invariant under Lorentzian transformations.

In case γ is a null Cartan curve, labeling $k_2(s) = k(s)$, then the Frenet formula of $\gamma(s)$ with respect to $F = \{\xi, N, B\}$ becomes

$$\begin{aligned} \frac{d\xi}{ds} &= N(s), \\ \frac{dB}{ds} &= k(s)N(s), \\ \frac{dN}{ds} &= -k(s)\xi(s) - B(s). \end{aligned} \tag{2.5}$$

This frame satisfies

$$\xi(s) \wedge B(s) = N(s), \qquad N(s) \wedge \xi(s) = \xi(s), \qquad B(s) \wedge N(s) = B(s). \tag{2.6}$$

Now we define surface $\mathcal{FS} : I \times \mathbb{R} \to \mathbb{R}^3_1$ by

$$\mathcal{FS}(s, \mu) = \gamma(s) + \frac{1}{k(s)} N(s) + \mu B(s). \tag{2.7}$$

We call $\mathcal{FS}(s,\mu)$ the *Focal surface* of null Cartan curve γ. We define the 2-dimensional *future lightcone* vertex at $\mathbf{v}_0$ by

$$LC_+^*(\mathbf{v}_0) = \left\{\mathbf{x} \in \mathbb{R}_1^3 : \langle \mathbf{x} - \mathbf{v}_0, \mathbf{x} - \mathbf{v}_0 \rangle = 0,\ x_0 > 0\right\}. \tag{2.8}$$

When $\mathbf{v}_0$ is the null vector $\mathbf{0}$, we simply denote $LC_+^*(\mathbf{0})$ by LC_+^*.

Let $\gamma : I \to \mathbb{R}_1^3$ be a regular null Cartan curve. We define the *binormal normal indicatrix* of $\gamma(s)$ as the map $\mathcal{BO}_\gamma : I \to LC_+^*$ given by

$$\mathcal{BO}_\gamma(s) = B(s), \tag{2.9}$$

and *the focal curve* of $\gamma(s)$ as the map $\mathcal{F}_\gamma : I \to \mathbb{R}_1^3$ given by

$$\mathcal{F}_\gamma(s) = \gamma(s) + \frac{1}{k(s)} N(s). \tag{2.10}$$

Defining the set: for any $\mathbf{v}_0 \in \mathbb{R}_1^3$, $\mathrm{TPB}(\mathbf{v}_0) = \{\mathbf{u} \in \mathbb{R}_1^3 \mid \langle \mathbf{u} - \mathbf{v}_0, B(s) \rangle = 0\} \setminus \{\mathbf{v}_0\}$, we call it the *tangential planar bundle of lightcone through* $\mathbf{v}_0$. It is obvious that the lightcone $LC_+^*(\mathbf{v}_0)$ is the envelope of the tangential planar bundle.

We give a geometric invariant σ of a null Cartan curve in $\mathbb{R}_1^3$ by

$$\sigma(s) = k^3(s) + 3k'^2(s) - k(s)k''(s), \tag{2.11}$$

which are related to the geometric meanings of the singularities of the focal surface.

We shall assume throughout the whole paper that all the maps and manifolds are C^∞ unless the contrary is explicitly stated.

Let $F : \mathbb{R}_1^3 \to \mathbb{R}$ be a submersion and $\gamma : I \to \mathbb{R}_1^3$ be a null Cartan curve. We say that γ and $F^{-1}(0)$ have *k-point contact* for $s = s_0$ if the function $g(s) = F \circ \gamma(t)$ satisfies $g(s_0) = g'(s_0) = \cdots = g^{k-1}(s_0) = 0, g^k(s_0) \neq 0$. We also say that γ and $F^{-1}(0)$ have *at least k-point contact* for $s = s_0$ if the function $g(s) = F \circ \gamma(s)$ satisfies $g(s_0) = g'(s_0) = \cdots = g^{k-1}(s_0) = 0$.

The main result in the paper is as follows.

Theorem 2.1. (**A**) *Let $\gamma : I \to \mathbb{R}_1^3$ be a regular null Cartan curve with $k(s) \neq 0$. For $\mathbf{v}_0 = \mathcal{FS}(s_0, \mu_0)$ and the tangential planar bundle $\mathcal{TPB}(\mathbf{v}_0) = \{\mathbf{u} \in \mathbb{R}_1^3 \mid \langle \mathbf{u} - \mathbf{v}_0, B(s) \rangle = 0\}$ of lightcone, one has the following.*

(1) *The null Crtan curve $\gamma(s)$ and $\mathcal{TPB}(\mathbf{v}_0)$ have at least 2-point contact for s_0.*

(2) *The null Crtan curve $\gamma(s)$ and $\mathcal{TPB}(\mathbf{v}_0)$ have 3-point contact for s_0 if and only if*

$$\mathbf{v}_0 = \gamma(s_0) + \frac{1}{k(s_0)} N(s_0) + \frac{k'(s_0)}{k^3(s_0)} B(s_0), \quad \sigma(s_0) \neq 0. \tag{2.12}$$

Under this condition, the germ of Image $\mathcal{FS}$ at $\mathcal{FS}(s_0, \mu_0)$ is locally diffeomorphic to the cuspidal edge $C \times \mathbb{R}$ (cf., Figure 1).

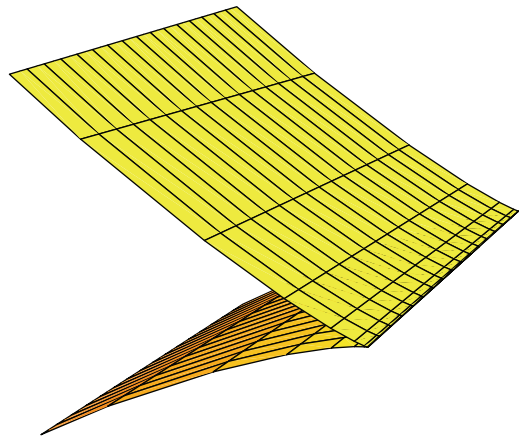

Figure 1: Cuspidal edge.

(3) *The null Crtan curve* $\gamma(s)$ *and* $\mathcal{TPB}(\mathbf{v}_0)$ *have 4-point contact for* s_0 *if and only if*

$$\mathbf{v}_0 = \gamma(s_0) + \frac{1}{k(s_0)}N(s_0) + \frac{k'(s_0)}{k^3(s_0)}B(s_0), \qquad \sigma(s_0) = 0,\ \sigma'(s_0) \neq 0. \tag{2.13}$$

Under this condition, the germ of Image $\mathcal{FS}$ *at* $\mathcal{FS}(s_0, \mu_0)$ *is locally diffeomorphic to the swallowtail SW (cf., Figure 2).*

(**B**) *Let* γ*:* $I \to \mathbb{R}^3_1$ *be a null Cartan curve. If* p *is a point of the binormal indicatrix of* γ *at* s_0*, then locally at* p*,*

(1) *the binormal indicatrix* $\mathcal{BO}_\gamma$ *is diffeomorphic to a line at* s_0 *if* $k(s_0) \neq 0$*,*

(2) *the binormal indicatrix* $\mathcal{BO}_\gamma$ *is diffeomorphic to the ordinary cusp* C *at* s_0 *if* $k(s_0) = 0$ *and* $k'(s_0) \neq 0$*.*

Here, $C \times \mathbb{R} = \{(x_1, x_2) \mid x_1^2 = x_2^3\} \times \mathbb{R}$ is the cuspidal edge and $SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\}$ is the swallowtail.

3. Volume-Like Distance Function and Lightcone Height Function of Null Cartan Curve

The purpose of this section is to obtain one geometric invariants of null Cartan curves by constructing a family of functions of the null Cartan curve.

Let $\gamma : I \to \mathbb{R}^3_1$ be a regular null Cartan curve with $k(s) \neq 0$. We define a three-parameter family of smooth functions

$$D : I \times \mathbb{R}^3_1 \longrightarrow \mathbb{R}, \tag{3.1}$$

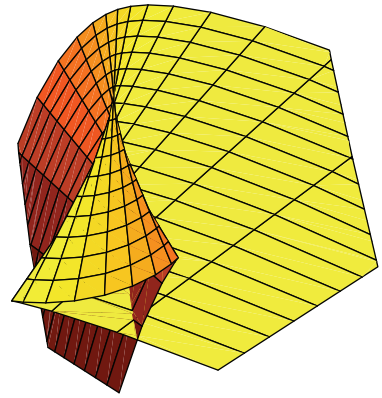

Figure 2: Swallowtail.

by $D(s,\mathbf{v}) = |B(s)N(s)\gamma(s) - \mathbf{v}| = \langle \gamma(s) - \mathbf{v}, B(s)\rangle$. Here, $|\mathbf{abc}|$ denotes the determinant of matrix $(\mathbf{abc})$. We call D the *volumelike distance function* of null Cartan curve γ. We denote that $d_v(s) = D(s,\mathbf{v})$ for any fixed vector $\mathbf{v}$ in $\mathbb{R}^3_1$. Using (2.5) and making a simple calculation, we can state the following facts.

Proposition 3.1. *Suppose $\gamma : I \to \mathbb{R}^3_1$ is a regular null Cartan curve with $k(s) \neq 0$ and $\mathbf{v} \in \mathbb{R}^3_1$. Then*

(1) *$d_v(s) = 0$ if and only if there exist real numbers λ, ω such that $\gamma(s) - \mathbf{v} = \mu B(s) + \omega N(s)$,*

(2) *$d_v(s) = d_v'(s) = 0$ if and only if $\mathbf{v} = \gamma(s) + \left(\dfrac{1}{k(s)}\right)N(s) - \mu B(s)$,*

(3) *$d_v(s) = d_v'(s) = d_v''(s) = 0$ if and only if $\mathbf{v} = \gamma(s) + \left(\dfrac{1}{k(s)}\right)N(s) + \left(\dfrac{k'(s)}{k^3(s)}\right)B(s)$,*

(4) *$d_v(s) = d_v'(s) = d_v''(s) = d_v'''(s) = 0$ if and only if $\mathbf{v} = \gamma(s) + (1/k(s))N(s) + (k'(s)/k^3(s))B(s)$ and $\sigma(s) = k^3(s) + 3k'^2(s) - k(s)k'(s) = 0$,*

(5) *$d_v(s) = d_v'(s) = d_v''(s) = d_v'''(s) = d_v^{(4)} = 0$ if and only if $\mathbf{v} = \gamma(s) + (1/k(s))N(s) + (k'(s)/k^3(s))B(s)$ and $\sigma(s) = \sigma'(s) = 3k^2(s)k'(s) + 5k(s)k''(s) - k^2(s)k'''(s) = 0$.*

Proof. (1) Let $\gamma(s) - \mathbf{v} = \lambda\xi(s) + \mu B(s) + \omega N(s)$. We have

$$\begin{aligned} d_v(s) &= \langle \gamma(s) - \mathbf{v}, B(s)\rangle \\ &= \langle \lambda\xi(s) + \omega B(s) + \mu N(s), B(s)\rangle \\ &= \lambda. \end{aligned} \tag{3.2}$$

The assertion (1) follows.

(2) By (1), we have $\gamma(s) - \mathbf{v} = \omega N(s) + \mu B(s)$. Using (3), we obtain

$$\begin{aligned} d_v'(s) &= \langle \xi(s), B(s)\rangle + \langle \gamma(s) - \mathbf{v}, B'(s)\rangle \\ &= 1 + \langle \omega N(s) + \mu B(s), k(s)N(s)\rangle \\ &= 1 + \omega k(s). \end{aligned} \tag{3.3}$$

It follows that $d_v(s) = d_v'(s) = 0$ if and only if $\mathbf{v} = \gamma(s) + (1/k(s))N(s) - \mu B(s)$.

(3) Under the assumption that $h_v(s) = d_v'(s) = 0$, we will compute $h_v'''(s)$

$$\begin{aligned} d_v''(s) &= \langle \xi(s), k(s)N(s)) \rangle + \left\langle \gamma(s) - \mathbf{v}, k'N(s) - k^2(s)\xi(s) - k(s)B(s) \right\rangle \\ &= \left\langle -\frac{1}{k(s)}N(s) + \mu B(s), k'N(s) - k^2(s)\xi(s) - k(s)B(s) \right\rangle \\ &= -k^2(s)\mu - \frac{k'(s)}{k(s)}. \end{aligned} \tag{3.4}$$

Hence, the assertion (3) holds.

(4) When $d_v(s) = d_v'(s) = d_v''(s) = 0$, the assertion (4) follows from the fact that

$$\begin{aligned} d_v'''(s) &= \left\langle \xi(s), k'(s)N(s) - k^2(s)\xi(s) - k(s)B(s) \right\rangle \\ &\quad + \left\langle \gamma(s) - \mathbf{v}, \left(k''(s) - 2k^2(s)\right)N(s) - 3k(s)k'(s)\xi(s) - 2k'(s)B(s) \right\rangle \\ &= -k(s) + \frac{3k'^2(s)}{k^2(s)} - \frac{k''(s)}{k(s)} + 2k(s) \\ &= k(s) + \frac{3k'^2(s)}{k^2(s)} - \frac{k''(s)}{k(s)}. \end{aligned} \tag{3.5}$$

(5) Under the condition that $d_v(s) = d_v'(s) = d_v''(s) = d_v'''(s) = 0$, this derivative is computed as follows:

$$\begin{aligned} d_v^{(4)}(s) &= -k'(s) + \left\langle \xi(s), \left(k''(s) - 2k^2(s)\right)N(s) - 3k(s)k'(s)\xi(s) - 2k'(s)B(s) \right\rangle \\ &\quad + \left\langle \gamma(s) - \mathbf{v}, \left(\left(k''(s) - 2k^2(s)\right)N(s) - 3k(s)k'(s)\xi(s) - 2k'(s)B(s)\right)' \right\rangle \\ &= -3k'(s) - \frac{k'(s)}{k^3(s)}\left(-4k(s)k''(s) + 2k^3(s) - 3k'^2(s)\right) - \frac{1}{k(s)}\left(k'''(s) - 9k(s)k'(s)\right), \end{aligned} \tag{3.6}$$

which implies that $d_v^{(4)}(s) = 0$ is equivalent to $4k^3(s)k'(s) + 4k(s)k'(s)k''(s) - k^2(s)k'''(s) + 3k'^3(s) = 0$. Moreover, in combination with $\sigma(s) = k^3(s) + 3k'^2(s) - k(s)k'(s) = 0$, it follows that $d_v(s) = d_v'(s) = d_v''(s) = d_v'''(s) = d_v^{(4)} = 0$ if and only if $\mathbf{v} = \gamma(s) + (1/k(s))N(s) + ((k'(s))/(k^3(s)))B(s)$ and $\sigma'(s) = 3k^2(s)k'(s) + 5k(s)k''(s) - k(s)k'''(s) = 0$. □

Let $\gamma : I \to \mathbb{R}_1^3$ be a regular null Cartan curve. We define a two-parameter family of functions

$$H : I \times LC_+^* \longrightarrow \mathbb{R}, \tag{3.7}$$

by $H(s,\mathbf{v}) = \langle \gamma(s), \mathbf{v}\rangle - s$. We call H the *lightcone height functions* of null Cartan curve $\gamma(s)$. We denote that $h_v(s) = H(s,\mathbf{v})$ for any fixed vector $\mathbf{v}$ in LC_+^*. We have the following proposition.

Proposition 3.2. *Suppose $\gamma : I \to \mathbb{R}_1^3$ is a regular null Cartan curve and $\mathbf{v} \in LC_+^*$. Then*

(1) *$h_v'(s) = 0$ if and only if there exist real numbers λ, ω such that $\mathbf{v} = \lambda\xi(s) + B(s) + \omega N(s)$ and $2\lambda + \omega^2 = 0$.*

(2) *$h_v'(s) = h_v''(s) = 0$ if and only if $\mathbf{v} = B(s)$.*

(3) *$h_v'(s) = h_v''(s) = h_v'''(s) = 0$ if and only if $\mathbf{v} = B(s)$ and $k(s) = 0$.*

(4) *$h_v'(s) = h_v''(s) = h_v'''(s) = h_v^{(4)}(s) = 0$ if and only if $\mathbf{v} = B(s)$ and $k(s) = k'(s) = 0$.*

Proof. Let $\mathbf{v} = \lambda\xi(s) + \mu B(s) + \omega N(s)$ in LC_+^*, where λ, ω, μ are real numbers.

(1) If

$$\begin{aligned} h_v'(s) &= \langle \xi(s), \mathbf{v}\rangle - 1 \\ &= \big\langle \xi(s), \lambda\xi(s) + \mu B(s) + \omega N(s)\big\rangle \\ &= \mu - 1 \\ &= 0, \end{aligned} \tag{3.8}$$

then $\mu = 1$. Moreover, in combination with $\mathbf{v}$ in LC_+^*, which means $2\lambda\mu + \omega^2 = 0$. It follows that $h_v'(s) = 0$ if and only if $\mathbf{v} = \lambda\xi(s) + B(s) + \omega N(s)$ and $2\lambda + \omega^2 = 0$.

(2) When $h_v'(s) = 0$, the second derivative

$$\begin{aligned} h_v''(s) &= \big\langle \xi'(s), \mathbf{v}\big\rangle \\ &= \langle N(s), \lambda\xi(s) + B(s) + \omega N(s)\rangle \\ &= \omega, \end{aligned} \tag{3.9}$$

then $h_v'(s) = h_v''(s) = 0$ if and only if $\mathbf{v} = B(s)$.

(3) When $h_v'(s) = h_v''(s) = 0$, the assertion (3) follows from the fact that

$$\begin{aligned} h_v'''(s) &= \big\langle N'(s), \mathbf{v}\big\rangle \\ &= \langle -k(s)\xi(s) - B(s), B(s)\rangle \\ &= -k(s). \end{aligned} \tag{3.10}$$

(4) Under the condition that $h_v'(s) = h_v''(s) = h_v'''(s) = 0$, this derivative is computed as follows:

$$\begin{aligned} h_v^{(4)}(s) &= \langle N''(s), \mathbf{v} \rangle \\ &= \langle -k'(s)\xi(s) - 2k(s)N(s), B(s) \rangle \\ &= -k'(s). \end{aligned} \tag{3.11}$$

The assertion (4) follows. □

4. Geometric Meanings of Invariant of a Null Cartan Curve

The purpose of this section is to study the geometric properties of the focal surface of a null Cartan curve in $\mathbb{R}_1^3$. Through these properties, one finds that the functions $\sigma(s) = k^3(s) + 3k'^2(s) - k(s)k''(s)$ have special meanings. These properties will be stated below.

Proposition 4.1. *Let $\gamma : I \to \mathbb{R}_1^3$ be a regular null Cartan curve with $k(s) \neq 0$. Then*

(1) *the singularities of $\mathcal{FS}$ are the set $\{(s,\mu) | \mu = k'(s)/k^3(s),\ s \in I\}$,*

(2) *if $\mathcal{FS}(s, k'(s)/k^3(s)) = \mathbf{v}_0$ is a constant vector, then $\mathcal{F}_\gamma(s)$ is in $TPB(\mathbf{v}_0)$ for any s in I and $\sigma(s) = k^3(s) + 3k'^2(s) - k(s)k''(s) \equiv 0$.*

Proof. (1) A straightforward computation shows that

$$\begin{aligned} \frac{\partial \mathcal{FS}}{\partial s} &= \left(\mu k(s) - \frac{k'(s)}{k^2(s)}\right) N(s) - \frac{1}{k(s)} B(s), \\ \frac{\partial \mathcal{FS}}{\partial \mu} &= B(s). \end{aligned} \tag{4.1}$$

The two equalities above imply that $\partial \mathcal{FS}/\partial s$ and $\partial \mathcal{FS}/\partial \mu$ are linearly dependent if and only if $\mu = k'(s)/k^3(s)$. This completes the proof of assertion (1). □

(2) For a smooth function $\mu : I \to \mathbb{R}$, define

$$f_\mu : I \longrightarrow \mathbb{R}_1^3, \qquad f_\mu(s) = \gamma(s) + \frac{1}{k(s)} N(s) + \mu(s) B(s). \tag{4.2}$$

$f_\mu(s) = \mathbf{v}_0$

$$\begin{aligned} \frac{df_\mu(s)}{ds} &= \left(\mu k(s) - \frac{k'(s)}{k^2(s)}\right) N(s) + \left(\mu'(s) - \frac{1}{k(s)}\right) B(s). \\ &= 0. \end{aligned} \tag{4.3}$$

$\mathcal{FS}(s,\mu)\mu(s) = k'(s)/k^3(s), \mu'(s) = -((k(s)k'(s) - 3k'^2(s))/k^4(s))\sigma(s) = k^3(s) + 3k'^2(s) - k(s)k''(s) = 0\gamma(s) + (1/k(s))N(s) + \mu B(s) = \mathbf{v}_0\mathcal{F}_\gamma$

$$\left\langle \mathcal{F}_\gamma(s) - \mathbf{v}_0, B(s) \right\rangle = \left\langle -\mu B(s), B(s) \right\rangle = 0. \tag{4.4}$$

5. Unfoldings of Functions of One Variable

In this section, we use some general results on the singularity theory for families of function germs [24].

Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \to \mathbb{R}$ be a function germ. We call F an *r-parameter unfolding* of f, where $f(s) = F_{x_0}(s, \mathbf{x}_0)$. We say that $f(s)$ has A_k-singularity at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leqslant p \leqslant k$ and $f^{(k+1)}(s_0) \neq 0$. We also say that $f(s)$ has $A_{\geqslant k}$-singularity at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leqslant p \leqslant k$. Let F be an unfolding of f and $f(s)$ has A_k-singularity ($k \geqslant 1$) at s_0. We denote the $(k-1)$-jet of the partial derivative $\partial F/\partial x_i$ at s_0 by $j^{(k-1)}((\partial F/\partial x_i)(s, \mathbf{x}_0))(s_0) = \sum_{j=1}^{k-1} \alpha_{ji}(s-s_0)^j$, for $i = 1, \ldots, r$. Then F is called a (p) *versal unfolding* if the $(k-1) \times r$ matrix of coefficients (α_{ji}) has rank $k-1 (k-1 \leqslant r)$. Under the same as the above, F is called a *versal unfolding* if the $k \times r$ matrix of coefficients $(\alpha_{0i}, \alpha_{ji})$ has rank $k (k \leqslant r)$, where $\alpha_{0i} = (\partial F/\partial x_i)(s_0, x_0)$.

We now introduce several important sets concerning the unfolding. The singular set of F is the set

$$\mathfrak{S}_F = \left\{ (s, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^r \mid \left(\frac{\partial F}{\partial s}\right)(s, \mathbf{x}) = 0 \right\}. \tag{5.1}$$

The *bifurcation set* $\mathfrak{B}_F$ of F is the critical value set of the restriction to $\mathfrak{S}_F$ of the canonical projection $\pi : \mathbb{R} \times \mathbb{R}^r \to \mathbb{R}^r$ and is given by

$$\mathfrak{B}_F = \left\{ \mathbf{x} \in \mathbb{R}^r \mid \text{there exists } s \text{ with } \frac{\partial F}{\partial s} = \frac{\partial^2 F}{\partial s^2} = 0 \text{ at } (s, \mathbf{x}) \right\}. \tag{5.2}$$

The *discriminant set* of F is the set

$$\mathfrak{D}_F = \left\{ \mathbf{x} \in \mathbb{R}^r \mid \text{there exists } s \text{ with } F = \frac{\partial F}{\partial s} = 0 \text{ at } (s, \mathbf{x}) \right\}. \tag{5.3}$$

Then we have the following well-known result [26].

Theorem 5.1. *Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \to \mathbb{R}$ be an r-parameter unfolding of $f(s)$ which has the A_k singularity at s_0.*

(1) *Suppose that F is a (p) versal unfolding.*

   (a) *If $k = 2$, then $(s_0, \mathbf{x}_0)$ is the fold point of $\pi|_{\mathfrak{S}_F}$ and $\mathfrak{B}_F$ is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$.*

   (b) *If $k = 3$, then $\mathfrak{B}_F$ is locally diffeomorphic to $C \times \mathbb{R}^{r-2}$.*

(2) *Suppose that F is a versal unfolding.*

(a) *If $k = 1$, then $\mathfrak{D}_F$ is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$.*

(b) *If $k = 2$, then $\mathfrak{D}_F$ is locally diffeomorphic to $C \times \mathbb{R}^{r-2}$.*

(c) *If $k = 3$, then $\mathfrak{D}_F$ is locally diffeomorphic to $SW \times \mathbb{R}^{r-3}$.*

Here, $SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2v,\ x_2 = 4u^3 + 2uv,\ x_3 = v\}$ is swallowtail and $C = \{(x_1, x_2) \mid x_1^2 = x_2^3\}$ is the ordinary cusp. We also say that a point $x_0 \in \mathbb{R}^r$ is a fold point of a map germ $f : (\mathbb{R}^r, \mathbf{x}_0) \to (\mathbb{R}^r, f(\mathbf{x}_0))$ if there exist diffeomorphism germs $\phi : (\mathbb{R}^r, \mathbf{x}_0) \to (\mathbb{R}^r, \mathbf{x}_0)$ and $\psi : (\mathbb{R}^r, f(\mathbf{x}_0)) \to (\mathbb{R}^r, f(\mathbf{x}_0)), \psi \circ \phi \to \psi^{-1} \circ f \circ \phi$ such that $\psi \circ \phi(x_1, \ldots, x_r) = (x_1, \ldots, x_{r-1}, x_r^2)$.

In the following propositions, the range of the index $i \in \{2, 3\}$ is used unless otherwise stated.

For the volumelike distance function D and the lightcone height function H, we can consider the following propositions.

Proposition 5.2. *Let $D : I \times \mathbb{R}_1^3 \to \mathbb{R}$ be the volumelike distance function on a null Cartan curve $\gamma(s)$ with $k(s) \neq 0$. If d_{v_0} has A_k-singularity at s_0 $(k = 1, 2, 3)$, then D is a versal unfolding of d_{v_0}.*

Proof. Let $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$, $\mathbf{v} = (v_1, v_2, v_3)$ in $\mathbb{R}_1^3$, and $B(s) = (B_1(s), B_2(s), B_3(s))$ in LC_+^*.

Under this notation the same as above proposition, we obtain

$$\begin{aligned}
D(s, v) &= \langle \xi(s) - \mathbf{v}, B(s) \rangle \\
&= -(\gamma_1(s) - v_1)B_1(s) + (\gamma_2(s) - v_2)B_2(s) + (\gamma_3(s) - v_3)B_3(s), \\
&\frac{\partial D}{\partial v_1}(s, \mathbf{v}) = B_1(s), \qquad \frac{\partial D}{\partial v_i}(s, \mathbf{v}) = -B_i(s), \\
&\frac{\partial}{\partial s}\frac{\partial D}{\partial v_1}(s, \mathbf{v}) = B_1'(s), \qquad \frac{\partial}{\partial s}\frac{\partial D}{\partial v_i}(s, \mathbf{v}) = -B_i'(s), \\
&\frac{\partial^2}{\partial s^2}\frac{\partial D}{\partial v_1}(s, \mathbf{v}) = B_1''(s), \qquad \frac{\partial^2}{\partial s^2}\frac{\partial D}{\partial v_i}(s, \mathbf{v}) = -B_i''(s).
\end{aligned} \tag{5.4}$$

Let $j^2((\partial D/\partial v_i)(s, \mathbf{v}_0))(s_0)$ be the 2-jet of $(\partial D/\partial v_i)(s, \mathbf{v}_0)$ at s_0 $(i = 1, 2, 3)$ and so

$$\begin{aligned}
&\frac{\partial D}{\partial v_i}(s_0, \mathbf{v}_0) + j^2\left(\frac{\partial D}{\partial v_i}(s, \mathbf{v}_0)\right)(s_0) \\
&\quad = \frac{\partial D}{\partial v_i}(s_0, \mathbf{v}_0) + \frac{\partial}{\partial s}\frac{\partial D}{\partial v_i}(s_0, \mathbf{v}_0)(s - s_0) + \frac{1}{2}\frac{\partial^2}{\partial s^2}\frac{\partial D}{\partial v_i}(s_0, \mathbf{v}_0)(s - s_0)^2 \\
&\quad = \alpha_{0,i} + \alpha_{1,i}(s - s_0) + \frac{1}{2}\alpha_{2,i}(s - s_0)^2.
\end{aligned} \tag{5.5}$$

We denote that

$$\mathcal{A} = \begin{pmatrix} \alpha_{0,1}\alpha_{0,2}\alpha_{0,3} \\ \alpha_{1,1}\alpha_{1,2}\alpha_{1,3} \\ \alpha_{2,1}\alpha_{2,2}\alpha_{2,3} \end{pmatrix} = \begin{pmatrix} B_1(s_0) - B_2(s_0) - B_3(s_0) \\ B_1'(s_0) - B_2'(s_0) - B_3'(s_0) \\ B_1''(s_0) - B_2''(s_0) - B_3''(s_0) \end{pmatrix}. \tag{5.6}$$

Thus,

$$\begin{aligned} \det \mathcal{A} &= \det\big(B(s_0)B'(s_0)B''(s_0)\big) \\ &= \Big\langle B(s_0) \wedge (k(s_0)N(s_0)), k'(s_0)N(s_0) \\ &\qquad -k^2(s_0)\xi(s_0) - k(s_0)B(s_0) \Big\rangle \\ &= -k^3(s_0)\langle B(s_0) \wedge N(s_0), \xi(s_0)\rangle \\ &= -k^3(s_0) \neq 0, \end{aligned} \tag{5.7}$$

which implies that the rank of $\mathcal{A}$ is 3, which finishes the proof. □

Proposition 5.3. *Let $H : I \times LC_+^* \to \mathbb{R}$ be the lightcone height function on a null Cartan curve $\gamma(s)$. If h_{v_0} has A_k-singularity $(k = 2, 3)$ at s_0, then H is a (p) versal unfolding of h_{v_0}.*

Proof. Consider a null Cartan curve $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$, and let $\mathbf{v} = (v_1, v_2, v_3)$ in LC_+^*, we have

$$\begin{aligned} H(s, \mathbf{v}) &= -\gamma_1(s)v_1 + \gamma_2(s)v_2 + \gamma_3(s)v_3 - s \\ &= \pm\gamma_1(s)\sqrt{v_2^2 + v_3^2} + \gamma_2(s)v_2 + \gamma_3(s)v_3 - s, \\ \frac{\partial H}{\partial v_i} &= \pm\frac{v_i}{\sqrt{v_2^2 + v_3^2}}\gamma_1(s) + \gamma_i(s), \\ \frac{\partial}{\partial s}\frac{\partial H}{\partial v_i} &= \pm\frac{v_i}{\sqrt{v_2^2 + v_3^2}}\gamma_1'(s) + \gamma_i'(s), \\ \frac{\partial^2}{\partial s^2}\frac{\partial H}{\partial v_i} &= \pm\frac{v_i}{\sqrt{v_2^2 + v_3^2}}\gamma_1''(s) + \gamma_i''(s). \end{aligned} \tag{5.8}$$

Let $j^2(\partial H/\partial v_i)(s, \mathbf{v}_0)(s_0)$ be the 2-jet of $(\partial H/\partial v_i)\,(s, \mathbf{v})$ $(i = 2, 3)$ at s_0, then we can show that

$$\begin{aligned} j^2\left(\frac{\partial H}{\partial v_i}(s, \mathbf{v}_0)\right)(s_0) &= \frac{\partial}{\partial s}\frac{\partial H}{\partial v_i}(s_0, \mathbf{v}_0)(s - s_0) + \frac{1}{2}\frac{\partial^2}{\partial s^2}\frac{\partial H}{\partial v_i}(s_0, \mathbf{v}_0)(s - s_0)^2 \\ &= \alpha_{1,i}(s - s_0) + \frac{1}{2}\alpha_{2,i}(s - s_0)^2. \end{aligned} \tag{5.9}$$

We denote that

$$\begin{aligned}\mathcal{P} &= \begin{pmatrix} \alpha_{1,2} & \alpha_{1,3} \\ \alpha_{2,2} & \alpha_{2,3} \end{pmatrix} \\ &= \begin{pmatrix} \pm\dfrac{v_{0,2}}{\sqrt{v_{0,2}^2+v_{0,3}^2}}\gamma_1'(s_0)+\gamma_2'(s_0) & \pm\dfrac{v_{0,2}}{\sqrt{v_{0,2}^2+v_{0,3}^2}}\gamma_1'(s_0)+\gamma_3'(s_0) \\ \pm\dfrac{v_{0,2}}{\sqrt{v_{0,2}^2+v_{0,3}^2}}\gamma_1''(s_0)+\gamma_2''(s_0) & \pm\dfrac{v_2}{\sqrt{v_{0,2}^2+v_{0,3}^2}}\gamma_1''(s_0)+\gamma_3''(s_0) \end{pmatrix}. \end{aligned} \tag{5.10}$$

We require rank $\mathcal{P} = 2$, which is verified from the fact that

$$\begin{aligned}\det\mathcal{P} &= \pm\frac{1}{\sqrt{v_2^2+v_3^2}}\det\left(\gamma'(s_0)\ \gamma''(s_0)\ \mathbf{v}_0\right) \\ &= \pm\frac{1}{\sqrt{v_2^2+v_3^2}}\langle\xi(s_0)\wedge N(s_0), B(s_0)\rangle \\ &= \pm\frac{1}{\sqrt{v_2^2+v_3^2}}\langle -\xi(s_0), B(s_0)\rangle \\ &= \pm\frac{1}{\sqrt{v_2^2+v_3^2}} \neq 0. \end{aligned} \tag{5.11}$$

This completes the proof. □

Proof of Theorem 2.1. Let $\gamma : I \to \mathbb{R}_1^3$ be a null Cartan curve with $k(s) \neq 0$. For $\mathbf{v}_0 = \mathcal{FS}(s_0, \mu_0)$, we define a function $\mathfrak{D} : \mathbb{R}_1^3 \to \mathbb{R}$ by $\mathfrak{D}(\mathbf{u}) = \langle \mathbf{u} - \mathbf{v}_0, B(s)\rangle$. Then we have $d_{\mathbf{v}_0}(s) = \mathfrak{D}(\gamma(s))$. Since $\mathcal{TPB}(\mathbf{v}_0) = \mathfrak{D}^{-1}(0)$ and 0 is a regular value of $\mathfrak{D}$, $d_{\mathbf{v}_0}(s)$ has the A_k-singularity at s_0 if and only if γ and $\mathcal{TPB}(\mathbf{v}_0)$ have $k+1$-point contact for s_0.

On the other hand, we can obtain from Proposition 3.1 that the discriminant set of $D = \mathfrak{D} \circ \gamma$ is

$$\mathfrak{D}_D = \left\{\gamma(s) + \frac{1}{k(s)}N(s) + \mu B(s) \mid s \in I,\ \mu \in R\right\}, \tag{5.12}$$

the assertion (**A**) of Theorem 2.1 follows from Propositions 3.1 and 5.2 and Theorem 5.1.

The bifurcation set $\mathfrak{B}_H$ of H is

$$\mathfrak{B}_H = \{B(s) \mid s \in I\}. \tag{5.13}$$

The assertion (**B**) of Theorem 2.1 follows from Propositions 3.2 and 5.3 and Theorem 5.1. □

6. Generic Properties of Null Cartan Curves

In this section, we consider generic properties of null Cartan curves in $\mathbb{R}^3_1$. The main tool is transversality theorem. Let $\mathrm{Emb}_{\mathrm{nu}}(I,\mathbb{R}^3_1)$ be the space of null embedding $\gamma : I \to \mathbb{R}^3_1$ equipped with Whitney C^∞-topology. We also consider the functions $\mathfrak{D} : \mathbb{R}^3_1 \times \mathbb{R}^3_1 \to \mathbb{R}$ defined by $\mathfrak{D}(\mathbf{u},\mathbf{v}) = \langle \mathbf{u}-\mathbf{v}, B(s)\rangle$. We claim that $\mathfrak{D}_v$ is a submersion for any $\mathbf{v}$ in $\mathbb{R}^3_1$, where $\mathfrak{D}_v(u) = \langle \mathbf{u}-\mathbf{v}, B(s)\rangle$. For any γ in $\mathrm{Emb}_{\mathrm{nu}}(I,\mathbb{R}^3_1)$, we have $D = \mathfrak{D}\circ(\gamma \times id_{\mathbb{R}^3_1})$. We also have the ℓ-jet extension

$$j^\ell_1 D : I \times \mathbb{R}^3_1 \longrightarrow J^\ell(I,\mathbb{R}), \tag{6.1}$$

defined by $j^\ell_1 D(s,v) = j^\ell d_v(s,v)$. We consider the trivialization $J^\ell(I,\mathbb{R}) \equiv I \times \mathbb{R} \times J^\ell(1,1)$. For any submanifold $\mathcal{O} \subset J^\ell(1,1)$, we denote that $\widetilde{\mathcal{O}} = I \times \{0\} \times \mathcal{O}$. It is evident that both $j^\ell_1 D$ is a submersions and $\widetilde{\mathcal{O}}$ is an immersed submanifold of $J^\ell(I,\mathbb{R})$. Then $J^\ell_1 D$ is transversal to $\widetilde{\mathcal{O}}$. We have the following proposition as a corollary of Lemma 6 in Wassermann [28].

Proposition 6.1. *Let $\mathcal{O}$ be submanifolds of $J^\ell(1,1)$. Then the set*

$$T_{\mathcal{O}} = \left\{ \gamma \in \mathrm{Emb}_{nu}\left(I,\mathbb{R}^3_1\right) \mid j^\ell_1 D \text{ is transversal to } \widetilde{\mathcal{O}}_1 \right\} \tag{6.2}$$

is residual subset of $\mathrm{Emb}_{nu}(I,\mathbb{R}^3_1)$. *If $\mathcal{O}$ is closed subset, then $T_{\mathcal{O}}$ is open.*

Let $f : (\mathbb{R},0) \to (\mathbb{R},0)$ be a function germ which has an A_k-singularity at 0. It is well known that there exists a diffeomorphism germ $\phi : (\mathbb{R},0) \to (\mathbb{R},0)$ such that $f\circ\phi(s) = \pm s^{k+1}$. This is the classification of A_k-singularities. For any $z = j^\ell f(0)$ in $J^\ell(1,1)$, we have the orbit $L^l(z)$ given by the action of the Lie group of ℓ-jet diffeomorphism germs. If f has an A_k-singularity, then the codimension of the orbit is k. There is another characterization of versal unfoldings as follows [27].

Proposition 6.2. *Let $F : (\mathbb{R}\times\mathbb{R}^r,0) \to (\mathbb{R},0)$ be an r-parameter unfolding of $f : (\mathbb{R},0) \to (\mathbb{R},0)$ which has an A_k-singularity at 0. Then F is a versal unfolding if and only if $j^\ell_1 F$ is transversal to the orbit $L^\ell(\widetilde{j^\ell f(0)})$ for $\ell \geqslant k+1$.*

Here, $j^\ell_1 F : (\mathbb{R}\times\mathbb{R}^r,0) \to J^\ell(\mathbb{R},\mathbb{R})$ is the ℓ-jet extension of F given by $j^\ell_1 F(s,\mathbf{x}) = j^\ell F_x(s)$.

The generic classification theorem is given as follows.

Theorem 6.3. *There exists an open and dense subset $T_{L^\ell_k} \subset \mathrm{Emb}_{nu}(I,\mathbb{R}^3_1)$ such that for any $\gamma \in T_{L^\ell_k}$, then the focal surface of γ is locally diffeomorphic to the cuspidal edge or the swallowtail at any singular point.*

Proof. For $\ell \geqslant 4$, we consider the decomposition of the jet space $J^\ell(1,1)$ into $L^\ell(1)$ orbits. We now define a semialgebraic set by

$$\Sigma^\ell = \left\{ z = j^\ell f(0) \in J^\ell(1,1) \mid f \ \text{ has an } A_{\geqslant 4}\text{-singularity} \right\}. \tag{6.3}$$

Then the codimension of Σ^ℓ is 4. Therefore, the codimension of $\widetilde{\Sigma}_0^\ell = I \times \{0\} \times \Sigma^\ell$ is 5. We have the orbit decomposition of $J^\ell(1,1) - \Sigma^\ell$ into

$$J^\ell(1,1) - \Sigma^\ell = L_0^\ell \cup L_1^\ell \cup L_2^\ell \cup L_3^\ell, \tag{6.4}$$

where L_k^ℓ is the orbit through an A_k-singularity. Thus, the codimension of $\widetilde{L_k^\ell}$ is $k+1$. We consider the ℓ-jet extension $j_1^\ell D$ of the volumelike distant function D. By Proposition 6.1, there exists an open and dense subset $T_{L_k^\ell} \subset \mathrm{Emb}_{\mathrm{nu}}(I, \mathbb{R}_1^3)$ such that $j_1^\ell D$ is transversal to $\widetilde{L_k^\ell}$ $(k = 0,1,2,3)$ and the orbit decomposition of $\widetilde{\Sigma^\ell}$. This means that $j_1^\ell D(I \times \mathbb{R}_1^3) \bigcap \widetilde{\Sigma^\ell} = \emptyset$ and D is a versal unfolding of g at any point $(s_0, \mathbf{v}_0)$. By Theorem 5.1, the discriminant set of D (i.e., the focal surface of γ) is locally diffeomorphic to cuspidal edge or swallowtail if the point is singular. □

7. Example

In this section, we give an example to illustrate the idea of Theorem 2.1.

Let $\gamma(s)$ be a null Cartan curve of $\mathbb{R}_1^3$ defined by

$$\gamma(s) = \left(s\sqrt{1 - \frac{1}{4}s^2} + 2\arcsin\frac{s}{2}, 2s - \frac{1}{6}s^3, -\frac{1}{6}\left(4 - s^2\right)^{3/2} \right), \tag{7.1}$$

with respect to a distinguished parameter s (Figure 3). The Cartan Frenet frame $F = \{\xi, N, B\}$ as follows:

$$\begin{aligned} \xi(s) &= \left(\sqrt{4 - s^2}, 2 - \frac{1}{2}s^2, \frac{1}{2}s\sqrt{4 - s^2} \right), \\ N(s) &= \left(-\frac{s}{\sqrt{4 - s^2}}, -s, \frac{2 - s^2}{\sqrt{4 - s^2}} \right), \\ B(s) &= \left(-\frac{2}{(4 - s^2)^{3/2}}, \frac{-1 + s^2}{-4 + s^2}, -\frac{s(-3 + s^2)}{(4 - s^2)^{3/2}} \right). \end{aligned} \tag{7.2}$$

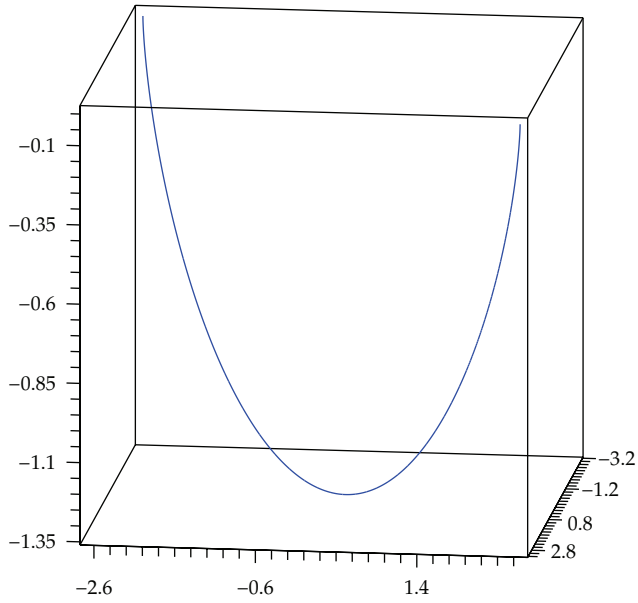

Figure 3: Null Cartan curve $\gamma(s)$.

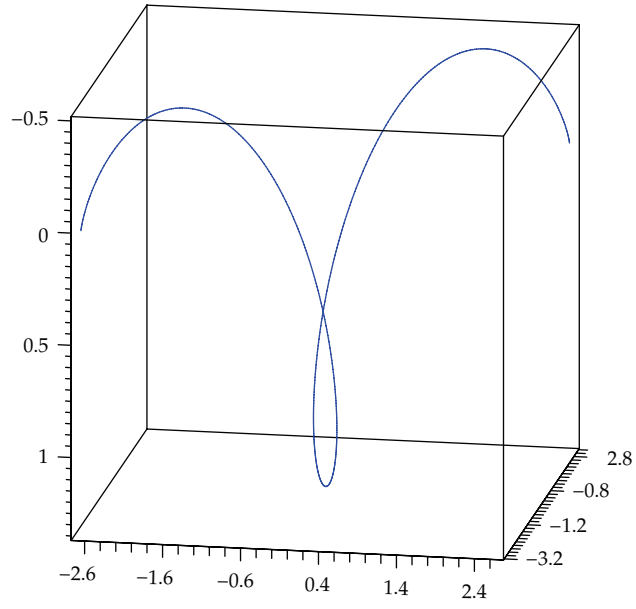

Figure 4: Focal curve of $\gamma(s)$.

Thus, using the Cartan Frenet equations (2.5), we obtain

$$k(s) = \frac{6}{\left(-4 + s^2\right)^2}. \tag{7.3}$$

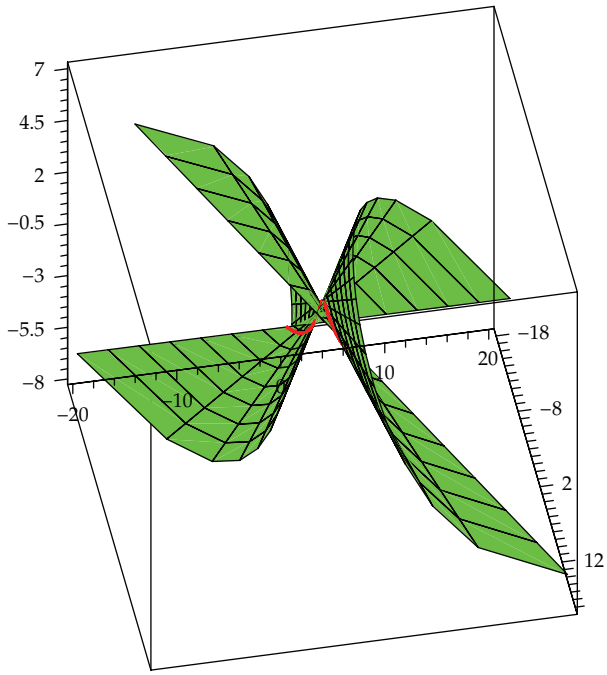

Figure 5: Focal surface and its singulaities.

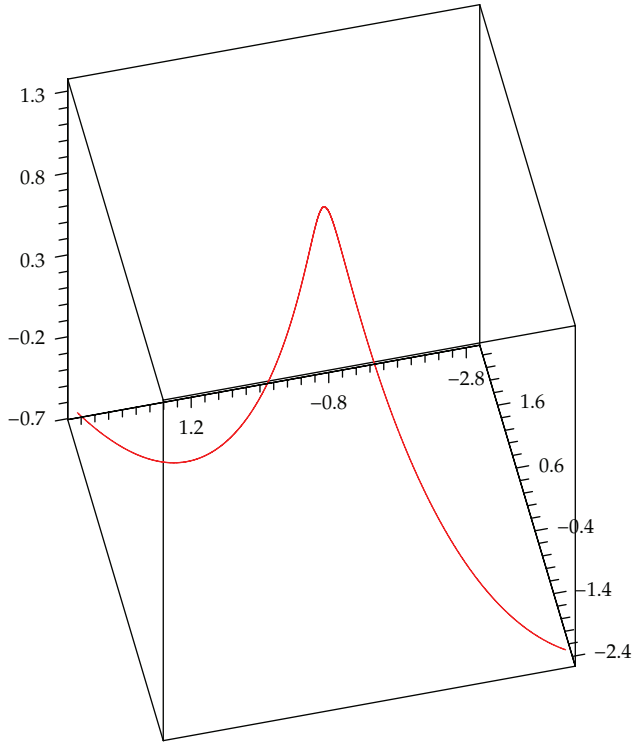

Figure 6: Singular locus of focal surface.

We give, respectively, the vector parametric equations of the focal curve $\mathcal{F}_\gamma$ (Figure 4), the focal surface $\mathcal{FS}(s,\mu) = \gamma(s) + (1/k(s))N(s) + \mu B(s)$ (Figure 5), and the singular locus $\mathcal{FS}(s, k'(s)/k^3(s))$ of the focal surface (Figure 6)

$$\begin{aligned}
&\frac{1}{6}\left\{ s\sqrt{4-s^2}\left(-1+s^2\right) + 12\arcsin\frac{s}{2}, -s\left(4-7s^2+s^4\right), -\left(-4+s^2\right)^{3/2}\left(-1+s^2\right)\right\},\\
&\left\{ \frac{s\left(-4+s^2\right)^2\left(-1+s^2\right)-12\mu}{6\left(4-s^2\right)^{3/2}} + 2\arcsin\frac{s}{2},\right.\\
&\left. -\frac{1}{6}s\left(4-7s^2+s^4\right) + \frac{-1+s^2}{-4+s^2}\mu, \frac{\left(-4+s^2\right)^3\left(-1+s^2\right)}{6\left(4-s^2\right)^{3/2}} - \frac{6s\left(-3+s^2\right)\mu}{6\left(4-s^2\right)^{3/2}}\right\}\\
&\frac{1}{18}\left\{ \frac{-76s+47s^3-7s^5+36\sqrt{4-s^4}\arcsin s/2}{\sqrt{4-s^2}},\right.\\
&\left. 20s-27s^3+15s^5-2s^7, \left(4-s^2\right)^{3/2}\left(3+3s^2-2s^4\right)\right\}.
\end{aligned} \tag{7.4}$$

We can calculate the geometric invariant

$$\begin{aligned}
\sigma(s) = k^3(s) + 3k'^2(s) - k(s)k''(s) &= \frac{72\left(-5+14s^2\right)}{\left(-4+s^2\right)^4},\\
\sigma'(s) &= -\frac{288s\left(13+35s^2\right)}{\left(-4+s^2\right)^7}.
\end{aligned} \tag{7.5}$$

We see that $\sigma(s) = 0$ gives two real roots $s = \pm\sqrt{5/14}$ and $\sigma'(s) = 0$ gives one real root $s = 0$, and two complex roots $s = \pm i\sqrt{13/15}$. Hence, we have $\mathcal{FS}$ is locally diffeomorphic to the cuspidal edge at any singularity $\mathcal{FS}(s,\mu)$, where $s \neq \pm\sqrt{5/14}, \mu = k'(s)/k^3(s) = -(1/9)s(-4+s^2)^3$. Moreover, $\mathcal{FS}$ is locally diffeomorphic to the swallowtail at $\mathcal{FS}(s,\mu)$, where $(s,\mu) = (\sqrt{5/14}, (14739/2744)\sqrt{5/14})$ or $(s,\mu) = (-\sqrt{5/14}, -(14739/2744)\sqrt{5/14})$.

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