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Research Article

Fixed Points, Maximal Elements and Equilibria of Generalized Games in Abstract Convex Spaces

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We firstly prove some new fixed point theorems for set-valued mappings in noncompact abstract convex space. Next, two existence theorems of maximal elements for class of $\mathcal{A}_{C,\theta}$ mapping and $\mathcal{A}_{C,\theta}$-majorized mapping are obtained. As in applications, we establish new equilibria existence theorems for qualitative games and generalized games. Our theorems improve and generalize the most known results in recent literature.

1. Introduction

Since Borglin and Keiding [1] proved a new existence theorem for a compact generalized games (=abstract economy) with KF-majorized preference correspondences. Following their ideas, many authors studied the existence of equilibria for generalized games, for example; see Ding [2], Shen [3], Chowdhury et al. [4], Briec and Horvath [5], Ding and Wang [6], Kim et al. [7], Kim and Tan [8], Lin et al. [9], Lin and Liu [10], Du and Deng [11], and so forth. In the setting, convexity assumptions play a crucial role. Since Horvath [12, 13] introduced H-space by replacing convex hulls by contract subsets, many authors have put forward abstract convex spaces without linear structure, for example: G-convex space [14] and FC-space [15]. As a result, many authors established existence theorems of maximal elements and equilibria of generalized games with majorized correspondences in H-space, G-convex space, and FC-space, respectively. For example, see Hou [16], Ding [17], Ding and Xia [18], Yang and Deng [19], Ding and Feng [20], and others.

In 2006, Park [21] introduce the concept of an abstract convex space, which is a topological space without any convexity structure and linear structure. Moreover, abstract convex space include topological vector spaces, H-space, G-convex space, and FC-space as special cases (see Park [21–23]). Abstract convex space will be the framework of this paper.

In this paper, we will introduce the new class of $\mathcal{A}_{C,\theta}$ mapping and $\mathcal{A}_{C,\theta}$-majorized mapping in abstract convex space. Some new fixed point theorems for set-valued mappings are proved under very weak coercive conditions. Next, two existence theorems of maximal elements for class of $\mathcal{A}_{C,\theta}$ mapping and $\mathcal{A}_{C,\theta}$-majorized mapping are obtained. As in applications, we establish new equilibria existence theorems for qualitative games and generalized games. Our results generalized and improve the corresponding results due to Ding and Feng [20], Ding and Wang [6], Park [22, 24], Yuan [25], Chowdhury et al. [4], Tan and Yuan [26], Borglin-Keiding [1], Yannelis [27], and so forth.

2. Preliminaries

Let X be a nonempty subset of topological space E. We shall denote by 2^X the family of all subsets of X, by $\langle X\rangle$ the family of all nonempty finite subsets of X, by $\operatorname{int}_E(X)$, the interior of X in E, and by $\operatorname{cl}_E(X)$ the closure of X in E.

If Y is a topological space and $T, S : X \to 2^Y$ are two mappings, for any $D \subset X$ and $y \in Y$, let $S(D) = \cup_{x\in D} S(x)$ and $S^-(y) = \{x \in X : y \in S(x)\}$. The dom S denotes the domain of S, that is, $\operatorname{dom} S = \{x \in X : S(x) \neq \emptyset\}$, and $T\cap S : X \to 2^Y$ is a mapping defined by $(T\cap S)(x) = T(x) \cap S(x)$ for each $x \in X$. The graph of T is the set $\operatorname{Gr}(T) = \{(x, y) \in X \times Y : y \in T(x)\}$, and the mapping $\overline{T} : X \to 2^Y$ is defined by $\overline{T}(x) = \{y \in Y : (x, y) \in \operatorname{cl}_{X\times Y}(\operatorname{Gr}(T))\}$. The mapping $\operatorname{cl} T : X \to 2^Y$ is defined by $(\operatorname{cl} T)(x) = \operatorname{cl}_Y(T(x))$ for each $x \in X$.

A subset A of E is said to be compactly open (resp., compactly closed) in E if for each nonempty compact subset C of E, $A \cap C$ is open (resp., closed) in E. Ding [28] define the compact interior and the compact closure of A denoted by $\operatorname{cint}(A)$ and $\operatorname{ccl}(A)$ as

$$\begin{gathered}\operatorname{cint}(A) = \bigcup\{B \subset E : B \subset A \text{ and } B \text{ is compactly open in } E\},\\ \operatorname{ccl}(A) = \bigcap\{B \subset E : A \subset B \text{ and } B \text{ is compactly closed in } E\}.\end{gathered} \tag{2.1}$$

It is easy to see that for any nonempty compact subset K of E, we have $\operatorname{cint}(A)\cap K = \operatorname{int}_K(A \cap K)$, $\operatorname{ccl}(A) \cap K = \operatorname{cl}_K(A \cap K)$, and hence $\operatorname{cint}(A)$ (resp., $\operatorname{ccl}(A)$) is compactly open (resp., compactly closed) in E. By the definitions, a subset A of E is compactly open (resp., compactly closed) in E if and only if $\operatorname{cint}(A) = A$ (resp., $\operatorname{ccl}(A) = A$).

Definition 2.1 (see [21, 22]). An abstract convex space $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a mapping $\Gamma : \langle D\rangle \to 2^E$ with nonempty values $\Gamma_A = \Gamma(A)$ for each $A \in \langle D\rangle$.

For any $D' \subset D$, the Γ-convex hull of D' is denoted and defined by

$$\operatorname{co}_\Gamma D' = \cup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E. \tag{2.2}$$

A subset X of E is called a Γ-convex subset of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\operatorname{co}_\Gamma D' \subset X$. Then, $(X, D'; \Gamma|_{\langle D'\rangle})$ is called a Γ-convex subspace of $(E, D; \Gamma)$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such a case, a subset X of E is said to be Γ-convex if $\operatorname{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' = X \cap D$. If $E = D$, let $(E; \Gamma) = (E, E; \Gamma)$.

Definition 2.2 (see [22]). Let $(E, D; \Gamma)$ be an abstract convex space. If a mapping $G : D \to 2^E$ satisfies

$$\Gamma_A \subset G(A) = \bigcup_{y \in A} G(y) \quad \forall A \in \langle D \rangle, \tag{2.3}$$

Then, G is called a KKM mapping.

Definition 2.3 (see [22]). The partial KKM principle for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM mapping $G : D \to 2^E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property.

Definition 2.4. Let X be a topological space, and Y be a nonempty subset of an abstract convex space $(E; \Gamma)$. Let $\theta : X \to Y$ be a single-valued mapping. Then, the mappings $\psi, \phi : X \to 2^Y$ are said to be an $\mathcal{A}_{C,\theta}$-pair if

(a) for each $x \in X$, $\theta(x) \notin \operatorname{co}_\Gamma(\phi(x)) \subset Y$ and $\psi(x) \subset \phi(x)$,

(b) the mapping $\psi^- : Y \to 2^X$ is compactly open valued on Y.

Definition 2.5. Let X be a topological space, and Y be a nonempty subset of an abstract convex space $(E; \Gamma)$. Let $\theta : X \to Y$ be a single-valued mapping, and $P : X \to 2^Y$ be a set-valued mapping. Then,

(i) P is said to be of class $\mathcal{A}_{C,\theta}$ if there exists an $\mathcal{A}_{C,\theta}$-pair such that

  (a) $\operatorname{dom} P \subset \operatorname{dom} \psi$,

  (b) for each $x \in X$, $P(x) \subset \phi(x)$.

(ii) $(\psi_x, \phi_x; N_x)$ is said to be an $\mathcal{A}_{C,\theta}$-majorant of P at $x \in X$ if N_x is an open neighborhood of x in X and the mapping $\phi_x, \psi_x : X \to 2^Y$ such that

  (a) for each $z \in N_x$, $P(z) \subset \phi_x(z)$, $\theta(z) \notin \operatorname{co}_\Gamma(\phi_x(z))$ and $\{z \in N_x : P(z) \neq \emptyset\} \subset \{z \in N_x : \psi_x(z) \neq \emptyset\}$,

  (b) for each $z \in X$, $\psi_x(z) \subset \phi_x(z)$ and $\operatorname{co}_\Gamma(\phi_x(z)) \subset Y$,

  (c) the mapping $\psi_x^- : Y \to 2^X$ is compactly open valued on Y.

(iii) P is said to be an $\mathcal{A}_{C,\theta}$-majorized mapping if for each $x \in X$ with $P(x) \neq \emptyset$, there exists an $\mathcal{A}_{C,\theta}$-majorant $(\psi_x, \phi_x; N_x)$ of P at x, and for any nonempty finite subset $A \in \langle \operatorname{dom} P \rangle$, the set $\{z \in \cap_{x \in A} N_x : P(z) \neq \emptyset\} \subset \{z \in \cap_{x \in A} N_x : \psi_x(z) \neq \emptyset\}$.

Remark 2.6. We note that our notions of the mapping P being of class $\mathcal{A}_{C,\theta}$ (resp., $\mathcal{A}_{C,\theta}$-majorized) improve notions of mapping of class $\mathcal{L}^*_{\theta,F}$ (resp., $\mathcal{L}^*_{\theta,F}$-majorized), respectively introduced by Ding and Feng [20] from FC-space to abstract convex space, which in turn generalize the corresponding notions in Ding and Wang [6], Chowdhury et al. [4], Yuan [25], Ding et al. [29], and Ding [30].

In this paper, we shall deal mainly with either the case (I) $X = Y$, and X is an abstract convex space, and $\theta = I_X$, which is the identity mapping on X or the case (II) $X = \Pi_{i \in I} X_i$, and $\theta = \pi_i : X \to X_i$ is the projection of X onto X_i and X_i is an abstract convex space. In both cases (I) and (II), we shall write $\mathcal{A}_C$ in place of $\mathcal{A}_{C,\theta}$.

Lemma 2.7 (see, [24]). *Let $(E, D; \Gamma)$ be an abstract convex space and $(X, D'; \Gamma')$ be a subspace. If $(E, D; \Gamma)$ satisfies the partial KKM principle, then so does $(X, D'; \Gamma')$.*

Lemma 2.8 (see, [22]). *Let $(E, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle, and $S : D \to 2^E$ be a mapping such that:*

(i) *for each $z \in X$, $S(z)$ is open,*

(ii) *$E = \bigcup_{z\in M} S(z)$ for some $M \in \langle D\rangle$.*

then, there exists an $N \in \langle D\rangle$ such that $\Gamma_N \cap \bigcap_{z\in N} S(z) \neq \emptyset$.

3. Fixed Point Theorems

Theorem 3.1. *Let $(E, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle, and K be a nonempty compact subset of E. Suppose that $F : E \to 2^D$, $G : E \to 2^E$ be mappings such that*

(i) *$F(x) \subset G(x)$ for each $x \in E$;*

(ii) *for each $y \in D$, $F^-(y)$ is compactly open in E and for each $x \in K$, $F(x) \neq \emptyset$,*

(iii) *for each $N \in \langle D\rangle$, there exists a compact abstract convex subset L_N of E containing N such that*

$$L_N \setminus K \subset \bigcup \{\operatorname{cint}(G^-(y)) : y \in L_N\}. \tag{3.1}$$

Then, there exists a point $\widehat{x} \in E$, such that $\widehat{x} \in \operatorname{co}_\Gamma(G(\widehat{x}))$.

Proof. Since for each $x \in K$, $F(x) \neq \emptyset$, then $K \subset \cup\{F^-(y) : y \in D\}$. By (i) and (ii), then $K \subset \{\operatorname{cint}(G^-(y)) : y \in D\}$. Since K is a nonempty compact subset of E, there exists a finite set $N \in \langle D\rangle$ such that

$$K \subset \bigcup \{\operatorname{cint}(G^-(y)) : y \in N\}. \tag{3.2}$$

By (iii), there exists a compact abstract convex subset L_N of E containing N such that

$$L_N \setminus K \subset \bigcup \{\operatorname{cint}(G^-(y)) : y \in L_N\}. \tag{3.3}$$

By (3.2) and $N \subset L_N$, then

$$L_N \cap K \subset \bigcup \{\operatorname{cint}(G^-(y)) : y \in N\} \subset \bigcup \{\operatorname{cint}(G^-(y)) : y \in L_N\}. \tag{3.4}$$

By (3.3) and (3.4), we have

$$L_N \subset \bigcup \{\operatorname{cint}(G^-(y)) : y \in L_N\}. \tag{3.5}$$

Since L_N is compact, there exists a finite set $B = \{z_1, z_2, \ldots, z_m\} \in \langle L_N\rangle$, such that

$$L_N = \bigcup_{i=1}^{m} \operatorname{cint}(G^-(z_i)) \cap L_N = \bigcup_{i=1}^{m} \operatorname{int}_{L_N}(G^-(z_i) \cap L_N). \tag{3.6}$$

Let $D' = L_N \cap D$ and define $\Gamma' : \langle D' \rangle \to 2^{L_N}$ by $\Gamma'_A = \Gamma_A \cap L_N$ for each $A \in \langle D' \rangle$, then $(L_N, D'; \Gamma')$ is an abstract convex space. Since $(E, D; \Gamma)$ satisfies the partial KKM principle, so does $(L_N, D'; \Gamma')$ by Lemma 2.7.

Define $S : L_N \to 2^{L_N}$ by $S(z) = \operatorname{cint}(G^-(z)) \cap L_N$ for each $z \in L_N$. It is easy to prove that the all the hypotheses of Lemma 2.8 are satisfied. By Lemma 2.8, there exists a finite set $M \in \langle L_N \rangle$ such that $\Gamma'_M \cap \bigcap_{z \in M} S(z) \neq \emptyset$. Let $\widehat{x} \in \Gamma'_M \cap \bigcap_{z \in M} S(z)$, then for each $z \in M$, $\widehat{x} \in S(z) = \operatorname{cint}(G^-(z)) \cap L_N \subset G^-(z)$, that is $M \subset G(\widehat{x})$ and $\Gamma_M \subset \operatorname{co}_\Gamma(G(\widehat{x}))$. Since $\widehat{x} \in \Gamma'_M = \Gamma_M \cap L_N \subset \Gamma_M$, thus $\widehat{x} \in \operatorname{co}_\Gamma(G(\widehat{x}))$. This completes the proof. □

Remark 3.2. Theorem 3.1 generalizes Theorem 3.1 of Ding and Feng [20] and Theorem 3.1 of Ding and Wang [6] from FC-space to abstract convex space, and the coercive condition (iii) in Theorem 2.1 is weaker than the condition (3) in Theorem 3.1 of Ding and Wang [6].

Corollary 3.3. *Let $(E, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle, and K be a nonempty compact subset of E. Suppose that $F : E \to 2^D$, $G : E \to 2^E$ be mappings such that*

(i) *$F(x) \subset G(x)$ for each $x \in E$,*

(ii) *for each $y \in D$, $F^-(y)$ is compactly open in E and for each $x \in K$, $F(x) \neq \emptyset$,*

(iii) *for each $N \in \langle D \rangle$, there exists a compact abstract convex subset L_N of E containing N such that for each $x \in L_N \setminus K$, there exists a point $\overline{y} \in L_N$ such that $x \notin \operatorname{cl}_X[X \setminus G^-(\overline{y})]$.*

Then, there exists a point $\widehat{x} \in E$, such that $\widehat{x} \in \operatorname{co}_\Gamma(G(\widehat{x}))$.

Proof. By (iii), for each $N \in \langle D \rangle$, there exists a compact abstract convex subset L_N of E containing N such that for each $x \in L_N \setminus K$, there exists $\overline{y} \in L_N$ such that $x \notin \operatorname{cl}_X[X \setminus G^-(\overline{y})]$, then $x \in (\operatorname{cl}_X[X \setminus G^-(\overline{y})])^c = \operatorname{int}(G^-(\overline{y}))$, thus $x \in \cup\{\operatorname{int}(G^-(y)) : y \in L_N\} \subset \{\operatorname{cint}(G^-(y)) : y \in L_N\}$, that is $L_N \setminus K \subset \{\operatorname{cint}(G^-(y)) : y \in L_N\}$. Hence, all the hypotheses of Theorem 3.1 are satisfied. By Theorem 3.1, there exists a point $\widehat{x} \in E$, such that $\widehat{x} \in \operatorname{co}_\Gamma(G(\widehat{x}))$. This completes the proof. □

Remark 3.4. Corollary 3.3 generalizes Theorem 3.2 of Ding and Feng [20] from FC-space to abstract convex space. Moreover, Corollary 3.3 improves the corresponding result of Park [22, 24].

4. Existence of Maximal Elements

Let X be a topological space, and $P : X \to 2^X$ be a mapping. A point $\widehat{x} \in X$ is called a maximal element of T if $T(\widehat{x}) = \emptyset$.

In the section, We firstly prove a selection theorem for $\mathcal{A}_C$-majorized mapping. Next, we will establish some new existence theorems of maximal elements for class $\mathcal{A}_C$ mapping and $\mathcal{A}_C$-majorized mapping defined on noncompact abstract convex space.

Lemma 4.1. *Let X be a regular topological space, and Y be a nonempty subset of an abstract convex space $(E; \Gamma)$. Let $\theta : X \to E$, and $P : X \to 2^Y$ be an $\mathcal{A}_{C,\theta}$-majorized mapping. If each open subset of X containing the set $B = \operatorname{dom} P$ is paracompact, then there exists a $\mathcal{A}_{C,\theta}$-pair $\psi, \phi : X \to 2^Y$ such that $P(x) \subset \phi(x)$ for each $x \in X$ and $\operatorname{dom} P \subset \operatorname{dom} \psi$.*

Proof. Since P is an $\mathcal{A}_{C,\theta}$-majorized mapping, for each $x \in B$, let N_x be an open neighborhood of x in X, and $\psi_x, \phi_x : X \to 2^Y$ be mappings such that

(1) for each $z \in N_x$, $P(z) \subset \phi_x(z)$, $\theta(z) \notin \mathrm{co}_\Gamma(\phi_x(z))$ and $\{z \in N_x : P(z) \neq \emptyset\} \subset \{z \in N_x : \psi_x(z) \neq \emptyset\}$,

(2) for each $z \in X$, $\psi_x(z) \subset \phi_x(z)$ and $\mathrm{co}_\Gamma(\phi_x(z)) \subset Y$,

(3) the mapping $\psi_x^- : Y \to 2^X$ is compactly open valued on Y,

(4) for each $A \in \langle \mathrm{dom} P\rangle$, $\{z \in \cap_{x\in A} N_x : P(z) \neq \emptyset\} \subset \{z \in \cap_{x\in A} N_x : \cap_{x\in A}\psi_x(z) \neq \emptyset\}$.

Since X is regular, for each $x \in B$, there exists an open neighborhood G_x of x in X such that $\mathrm{cl}_X G_x \subset N_x$. Let $G = \cup_{x\in B} G_x$, then G is an open subset of X containing B, so that G is paracompact by assumption. By Theorem VIII.1.4 of Dugundji [31], the open covering $\{G_x\}$ of G has an open precise neighborhood finite refinement G'_x. Given any $x \in B$, we define the mappings $\psi'_x, \phi'_x : G \to 2^Y$ by

$$\psi'_x(z) = \begin{cases} \psi_x(z) & \text{if } z \in G \cap \mathrm{cl}_X G'_x, \\ Y & \text{if } z \in G \setminus \mathrm{cl}_X G'_x, \end{cases}$$
$$\phi'_x(z) = \begin{cases} \phi_x(z) & \text{if } z \in G \cap \mathrm{cl}_X G'_x, \\ Y & \text{if } z \in G \setminus \mathrm{cl}_X G'_x, \end{cases} \tag{4.1}$$

then we have

(i) by (2), for each $z \in G$, $\psi'_x(z) \subset \phi'_x(z)$,

(ii) by (1), $\mathrm{dom}\, P \subset \mathrm{dom}\, \psi'_x$, and $P(z) \subset \phi'_x(z)$ for each $z \in G$,

(iii) for each $y \in Y$, the set

$$\begin{aligned} (\psi'_x)^-(y) &= \{z \in G \cap \mathrm{cl}_X G'_x : y \in \psi_x(z)\} \cup \{z \in G \setminus \mathrm{cl}_X G'_x : y \in Y\} \\ &= [(G \cap \mathrm{cl}_X G'_x) \cap \psi_x^-(y)] \cup (G \setminus \mathrm{cl}_X G'_x) \\ &= [G \cap \psi_x^-(y)] \cup (G \setminus \mathrm{cl}_X G'_x). \end{aligned} \tag{4.2}$$

It follows that for each nonempty compact subset C of X, the set

$$(\psi'_x)^-(y) \cap C = [G \cap \psi_x^-(y) \cap C] \cup [(G \setminus \mathrm{cl}_X G'_x) \cap C] \tag{4.3}$$

is open in C by (3). Thus, the mapping $(\psi'_x)^- : Y \to 2^G$ is compactly open valued on Y. Now, define $\psi, \phi : X \to 2^Y$ by

$$\psi(z) = \begin{cases} \bigcap_{x\in B} \psi'_x(z) & \text{if } z \in G, \\ \emptyset & \text{if } z \in X \setminus G, \end{cases}$$
$$\phi(z) = \begin{cases} \bigcap_{x\in B} \phi'_x(z) & \text{if } z \in G, \\ \emptyset & \text{if } z \in X \setminus G. \end{cases} \tag{4.4}$$

(a) For each $z \in X$, by (i), $\psi(z) \subset \phi(z)$ and $\mathrm{co}_\Gamma(\phi(z)) \subset Y$. If $z \in X \setminus G$, then $\phi(z) = \emptyset$, so that $\theta(z) \notin \mathrm{co}_\Gamma(\phi(z))$; if $z \in G$, then $z \in G \cap \mathrm{cl}_X G'_x$ for some $x \in B$, so that $\phi'_x(z) = \phi_x(z)$, and hence $\phi(z) \subset \phi_x(z)$, by (1), we have $\theta(z) \notin \mathrm{co}_\Gamma(\phi(z))$. Therefore, $\theta(z) \notin \mathrm{co}_\Gamma(\phi(z))$ for all $z \in X$.

(b) Now, we show that the mapping $\psi^- : Y \to 2^X$ is compactly open valued on Y. Indeed, let $y \in Y$ be such that $\psi^-(y) \neq \emptyset$ and C be a nonempty compact subset of X. Given a point u,

$$u \in \psi^-(y) \cap C = \{z \in X : y \in \psi(z)\} \cap C = \{z \in G : y \in \psi(z)\} \cap C. \tag{4.5}$$

Since $\{G'_x\}$ is a neighborhood finite refinement, there exists an open neighborhood M_u of u in G such that $\{x \in B : M_u \cap G'_x \neq \emptyset\} = \{x_1, x_2, \ldots, x_n\}$. Note that for each $x \in B$ with $x \notin \{x_1, x_2, \ldots, x_n\}$, $\emptyset = M_u \cap G'_x = M_u \cap \mathrm{cl}_X G'_x$, so that $\psi'_x(z) = Y$ for $z \in M_u$. Then, we have $\psi(z) = \cap_{x \in B} \psi'_x(z) = \cap_{i=1}^n \psi'_{x_i}(z)$ for all $z \in M_u$. It follows that

$$\begin{aligned}
\psi^-(y) &= \{z \in X : y \in \psi(z)\} = \left\{z \in G : y \in \bigcap_{x \in B} \psi'_x(z)\right\} \\
&\supset \left\{z \in M_u : y \in \bigcap_{x \in B} \psi'_x(z)\right\} \\
&= \left\{z \in M_u : y \in \bigcap_{i=1}^{n} \psi'_{x_i}(z)\right\} \\
&= M_u \cap \left\{z \in G : y \in \bigcap_{i=1}^{n} \psi'_{x_i}(z)\right\} \\
&= M_u \cap \left[\bigcap_{i=1}^{n} \left(\psi'_{x_i}\right)^-(y)\right].
\end{aligned} \tag{4.6}$$

Since $(\psi'_{x_i})^-(y)$ is compactly open in X by (iii), then $M'_u = M_u \cap [\cap_{i=1}^n (\psi'_{x_i})^-(y)] \cap C$ is an open neighborhood of u in C such that $M'_u \subset \psi^-(y) \cap C$. This shows that $\psi^- : Y \to 2^X$ is compactly open valued on Y.

By (a) and (b), thus (ψ, ϕ) is an $\mathcal{A}_{C,\theta}$-pair.

Next, we claim that $\mathrm{dom}\, P \subset \mathrm{dom}\, \psi$, indeed, for each $w \in \mathrm{dom}\, P$, we must have $w \in G$. Since $\{G'_x\}$ is neighborhood finite, then the set $\{x \in B : w \in \mathrm{cl}_X G'_x\} = \{x'_1, x'_2, \ldots, x'_m\}$. If $x \notin \{x'_1, x'_2, \ldots, x'_m\}$, $w \in G \setminus \mathrm{cl}_X G'_x$, and $\psi'_x(w) = Y$, thus we have $\psi(w) = \cap_{x \in B} \psi'_x(w) = \cap_{i=1}^m \psi'_{x'_i}(w)$. Since $w \in \cap_{i=1}^m \mathrm{cl}_X G'_{x'_i} \subset \cap_{i=1}^m N_{x'_i}$, by (4), $\psi(w) \neq \emptyset$. Hence, $\mathrm{dom}\, P \subset \mathrm{dom}\, \psi$.

Finally, we prove that $P(z) \subset \phi(z)$ for each $z \in X$. Indeed, let $z \in X$ with $P(z) \neq \emptyset$, then $z \in G$. For each $x \in B$, if $z \in G \setminus \mathrm{cl}_X G'_x$, then $\phi'_x(z) = Y$, and so $P(z) \subset \phi'_x(z)$, and if $z \in G \cap \mathrm{cl}_X G'_x$, we have $z \in \mathrm{cl}_X G'_x \subset \mathrm{cl}_X G_x \subset N_x$, so that by (1), $P(z) \subset \phi_x(z) \subset \phi'_x(z)$. It follows that $P(z) \subset \phi'_x(z)$ for all $x \in B$ so that $P(z) \subset \cap_{x \in B} \phi'_x(z) = \phi(z)$. This completes the proof. □

Remark 4.2. Lemma 4.1 generalizes Lemma 4.1 of Ding and Feng [20], Lemma 5.1 of Ding and Wang [15], Theorem 3.1 of Yuan [25], Lemma 3.1 of Chowdhury et al. [4], and Lemma 3.1 of Tan and Yuan [26].

Theorem 4.3. *Let $(X;\Gamma)$ be an abstract convex space satisfying the partial KKM principle, and let K be a nonempty compact subset of X. Suppose that the mapping $P : X \to 2^X$ is of class $\mathcal{A}_C$ and satisfy*

(i) *for each $N \in \langle X \rangle$, there exists a compact abstract convex subset L_N of X containing N such that*

$$L_N \setminus K \subset \bigcup \{\operatorname{cint}(P^-(y)) : y \in L_N\}. \tag{4.7}$$

Then, there exists a point $\widehat{x} \in K$ such that $P(\widehat{x}) = \emptyset$.

Proof. Since P is of class $\mathcal{A}_C$, then there exists an $\mathcal{A}_C$-pair (ϕ, ψ) such that

(a) $\operatorname{dom} P \subset \operatorname{dom} \psi$, $P(x) \subset \phi(x)$ for each $x \in X$,

(b) for each $x \in X$, $x \notin \operatorname{co}_\Gamma(\phi(x)) \subset X$ and $\psi(x) \subset \phi(x)$,

(c) the mapping $\psi^- : X \to 2^X$ is compactly open valued on X.

Suppose that for each $x \in K$, $P(x) \neq \emptyset$. By (a), $\psi(x) \neq \emptyset$ for each $x \in K$. By (i) and (a), for each $N \in \langle X \rangle$, there exists a compact abstract convex subset L_N of X containing N such that $L_N \setminus K \subset \cup\{\operatorname{cint}(P^-(y)) : y \in L_N\} \subset \cup\{\operatorname{cint}(\phi^-(y)) : y \in L_N\}$. Therefore, ψ and ϕ satisfy all the hypotheses of Theorem 3.1. By Theorem 3.1, there exists a point $\overline{x} \in K$ such that $\overline{x} \in \operatorname{co}_\Gamma(\phi(\overline{x}))$. Which contradicts with condition (b). Hence, there exists a point $\widehat{x} \in K$ such that $P(\widehat{x}) = \emptyset$. This completes the proof. □

Remark 4.4. Theorem 4.3 generalizes most existence theorems of maximal elements in the literature, for example; see Theorem 4.1 of Ding and Feng [20], Theorem 3.2 of Yuan [25], Theorem 3.1 of Chowdhury et al. [4], Theorem 3.2 of Tan and Yuan [26], Theorem 5.2 of Ding and Wang [6], and so on.

As an application of Lemma 4.1 and Theorem 4.3, we have the following existence theorem maximal elements.

Theorem 4.5. *Let $(X;\Gamma)$ be a paracompact abstract convex space satisfying the partial KKM principle, and let K be a nonempty compact subset of X. Let $P : X \to 2^X$ be an $\mathcal{A}_C$ majorized mapping and satisfy*

(i) *for each $N \in \langle X \rangle$, there exists a compact abstract convex subset L_N of X containing N such that*

$$L_N \setminus K \subset \bigcup \{\operatorname{cint}(P^-(y)) : y \in L_N\}. \tag{4.8}$$

Then, there exists a point $\widehat{x} \in K$ such that $P(\widehat{x}) = \emptyset$.

Proof. Suppose that $P(x) \neq \emptyset$ for all $x \in X$, then $\operatorname{dom} P = X$ is paracompact. By Lemma 4.1 P is of class $\mathcal{A}_C$. Therefore, all the hypotheses of Theorem 4.3 are satisfied. By Theorem 4.3, there exists a point $\widehat{x} \in K$ such that $P(\widehat{x}) = \emptyset$. Which is a contradiction. Thus, there exists a point $\widehat{x} \in X$ such that $P(\widehat{x}) = \emptyset$. By the assumptions, $\widehat{x}$ must in K. This completes the proof. □

Remark 4.6. Theorem 4.5 generalizes Theorem 5.3 of Ding and Wang [6], Theorem 3.2 of Chowdhury et al. [4], Theorem 3.3 of Tan and Yuan [26], Corollary 1 of Borglin-Keiding [1], and Theorem 2 of Yannelis [27].

5. Existence of Equilibrium of Points

Let I be a (finite or infinite) set of players. For each $i \in I$, let its strategy set X and Y_i $(i \in I)$ be nonempty set, and let $Y = \prod_{i\in I} Y_i$. $P_i : X \to 2^{Y_i}$ be the preference correspondence of i-th player. The collection $\Lambda = (X; Y_i; P_i)_{i\in I}$ will be called a qualitative game. A point $\hat{x} \in X$ is said to be an equilibrium of the qualitative game, if $P_i(\hat{x}) = \emptyset$ for each $i \in I$.

A generalized game is a quintuple family $\Lambda = (X; Y_i; A_i; B_i; P_i; \theta_i)_{i\in I}$, where X is a nonempty set, I is a (finite or infinite) set of players such that for each $i \in I$, Y_i is a nonempty set and $Y = \prod_{i\in I} Y_i$. $A_i, B_i : X \to 2^{Y_i}$, $\theta_i : X \to Y_i$ are the constraint correspondences, and $P_i : X \to 2^{Y_i}$ is the preference correspondence. An equilibrium of the generalized game Λ is a point $\hat{x} \in X$ such that for each $i \in I, \theta_i(\hat{x}) \in \overline{B}_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$. If $\theta_i = \pi_i : X \to X_i$ is the projection of X onto X_i, then our definition of an equilibrium point coincides with the standard definition given by Chowdhury et al. [4], and if, in addition, $A_i = B_i$ for each $i \in I$, our definition of an equilibrium point generalizes the standard definition, for example, Borglin and Keiding [1] and Gale and Mas-Colell [32].

Lemma 5.1 (see [33]). *Let $\{(X_i, D_i; \Gamma_i)\}_{i\in I}$ be a family of abstract convex spaces. Let $X := \Pi_{i\in I} X_i$ be equipped with the product topology, and let $D := \Pi_{i\in I} D_i$. For each $i \in I$, let $\pi_i : D \to D_i$ be the projection. For each $A \in \langle D \rangle$, define $\Gamma(A) := \Pi_{i\in I} \Gamma_i(\pi_i(A))$. Then, $(X, D; \Gamma)$ is an abstract convex space.*

As an application of Theorem 4.3, we prove the following existence theorem of equilibrium points for one person game in abstract convex space.

Theorem 5.2. *Let $(X; \Gamma)$ be an abstract convex space satisfying the partial KKM principle and K be a closed and compact subset of X, and let $K = \mathrm{co}_\Gamma K$. Suppose that the mappings A, B, and $P : X \to 2^X$ satisfy*

(i) *for each $x \in X$, $\mathrm{co}_\Gamma(\mathrm{A(x)}) \subset \overline{\mathrm{B}}(\mathrm{x})$ and $A(x) \cap P(x) \subset K$,*

(ii) *the mapping $A^- : X \to 2^X$ is compactly open valued on X,*

(iii) *the mapping $A \cap P : X \to 2^X$ is of class $\mathcal{A}_C$ and $A(x) \neq \emptyset$ for each $x \in K$,*

(iv) *for each $N \in \langle X \rangle$, there exists a compact abstract convex subset L_N of X containing N such that*

$$L_N \setminus K \subset \bigcup \{\mathrm{cint}\big((A \cap P)^-(y)\big) : y \in L_N\}. \tag{5.1}$$

Then, there exists a point $\hat{x} \in K$ such that $\hat{x} \in \overline{B}(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$.

Proof. Let $W = \{x \in X : x \notin \overline{B}(x)\}$, then W is open in X. Define $Q : X \to 2^X$ by

$$Q(x) = \begin{cases} K & \text{if } x \in X \setminus K, \\ A(x) & \text{if } x \in K \cap W, \\ A(x) \cap P(x) & \text{if } x \in K \setminus W. \end{cases} \tag{5.2}$$

By (iii), $A \cap P$ is of class $\mathscr{A}_C$, then there exist two mappings $\psi, \phi : X \to 2^X$ such that

(a) for each $x \in X$, $A(x) \cap P(x) \subset \phi(x)$ and $\operatorname{dom}(A \cap P) \subset \operatorname{dom}\psi$,

(b) for each $x \in X$, $x \notin \operatorname{co}_\Gamma(\phi(x))$ and $\psi(x) \subset \phi(x)$,

(c) the mapping $\psi^- : X \to 2^X$ is compactly open valued on X.

Define $\Psi, \Phi : X \to 2^X$ by

$$\Psi(x) = \begin{cases} K & \text{if } x \in X \setminus K, \\ A(x) & \text{if } x \in K \cap W, \\ \psi(x) & \text{if } x \in K \setminus W, \end{cases}$$
$$\Phi(x) = \begin{cases} K & \text{if } x \in X \setminus K, \\ A(x) & \text{if } x \in K \cap W, \\ \phi(x) & \text{if } x \in K \setminus W. \end{cases} \tag{5.3}$$

Then, we have

(a′) for each $x \in X$, $Q(x) \subset \Phi(x)$ by (a) and $\operatorname{dom} Q \subset \operatorname{dom}\Psi$,

(b′) for each $x \in X$, $\Psi(x) \subset \Phi(x)$ by (b), if $x \in X \setminus K$, $x \notin K = \operatorname{co}_\Gamma K = \operatorname{co}_\Gamma \Phi(x)$ and if $x \in K \cap W$, $x \notin \overline{B}(x)$ by (i), and $x \notin \operatorname{co}_\Gamma(A(x)) = \operatorname{co}_\Gamma(\Phi(x))$ and if $x \in K \setminus W$, $x \notin \operatorname{co}_\Gamma(\phi(x)) = \operatorname{co}_\Gamma(\Phi(x))$ by (b), thus $x \notin \operatorname{co}_\Gamma(\Phi(x))$ for each $x \in X$,

(c′) for each $y \in X$, then it is easy to verify that the set

$$\Psi^-(y) = \{x \in X : y \in \Psi(x)\} = (X \setminus K) \cup \left[A^-(y) \cap K \cap W\right] \cup \left[\psi^-(y) \cap (K \setminus W)\right] \tag{5.4}$$

is compactly open in X by (ii) and (c). This shows that Q is of class $\mathscr{A}_C$. By the definition of Q, condition (i) and (iv), for each $N \in \langle X \rangle$, there exists a compact abstract convex subset L_N of X containing N such that

$$L_N \setminus K \subset \bigcup\{\operatorname{cint}((A \cap P)^-(y)) : y \in L_N\} \subset \bigcup\{\operatorname{cint}(Q^-(y)) : y \in L_N\}. \tag{5.5}$$

Hence, all the hypotheses of Theorem 4.3 are satisfied. By Theorem 4.3, there exists a point $\widehat{x} \in K$ such that $Q(\widehat{x}) = \emptyset$. By the definition of Q and condition (iii), $\widehat{x} \in K \setminus W \subset K$, that is, $\widehat{x} \in \overline{B}(\widehat{x})$ and $A(\widehat{x}) \cap P(\widehat{x}) = \emptyset$. This completes the proof. □

Remark 5.3. Theorem 5.2 improves and generalizes Theorem 5.1 of Ding and Feng [20], Theorem 6.1 of Ding and Wang [6], Theorem 4.1 of Yuan [25], Theorem 4.1 of Chowdhury et al. [4], and Theorem 4.1 of Tan and Yuan [26].

As another application of Theorem 4.5, we can obtain the following existence of equilibria for qualitative games.

Theorem 5.4. *Let $\Lambda = (X; X_i; P_i)_{i\in I}$ be a qualitative game, For each $i \in I$, suppose that the following conditions are satisfied*

(i) $(X_i; \Gamma_i)_{i\in I}$ *is a family of paracompact abstract convex space such that $(X; \Gamma)$ satisfies the partial KKM principle, and K is a nonempty compact subset of X,*

(ii) $P_i : X \to 2^{X_i}$ *is an $\mathcal{A}_C$-majorized mapping,*

(iii) $W_i = \{x \in X : P_i(x) \neq \emptyset\}$ *is open in X,*

(iv) *for each $N \in \langle X\rangle$, there exists a compact abstract convex subset L_N of X containing N such that*

$$L_N \setminus K \subset \bigcup\{\operatorname{cint}(P_i^-(\pi_i(y))) : y \in L_N\}. \tag{5.6}$$

Then, Λ has an equilibrium point in K.

Proof. For each $x \in X$, let $I(x) = \{i \in I : P_i(x) \neq \emptyset\}$. Define a mapping $P_i' : X \to 2^X$ by $P_i'(x) = \pi_i^-(P_i(x)) = \Pi_{j\in I, j\neq i} X_i \otimes P_i(x)$ for each $x \in X$, where the mapping $\pi_i : X \to X_i$ is projection of X onto X_i. Furthermore, define the mapping $P : X \to 2^X$ by

$$P(x) = \begin{cases} \bigcap\limits_{i\in I(x)} P_i'(x) & \text{if } I(x) \neq \emptyset, \\ \emptyset & \text{if } I(x) = \emptyset. \end{cases} \tag{5.7}$$

Then, for each $x \in X$, $P(x) \neq \emptyset$ if and only if $I(x) \neq \emptyset$. We will show that P is an $\mathcal{A}_C$-majorized mapping. For each $x \in X$ with $P(x) \neq \emptyset$, let $i \in I(x)$ with $P_i(x) \neq \emptyset$, by (ii), and let N_x be an open neighborhood of x in X, and $\psi_{i,x}, \phi_{i,x} : X \to 2^{X_i}$ be mappings such that

(a) for each $z \in N_x$, $P_i(z) \subset \phi_{i,x}(z)$, $z_i \notin \operatorname{co}_\Gamma(\phi_{i,x}(z))$, and $\{z \in N_x : P_i(z) \neq \emptyset\} \subset \{z \in N_x : \psi_{i,x}(z) \neq \emptyset\}$,

(b) for each $z \in X$, $\psi_{i,x}(z) \subset \phi_{i,x}(z)$ and $\operatorname{co}_\Gamma(\phi_{i,x}(z)) \subset X_i$,

(c) the mapping $\psi_{i,x}^- : X_i \to 2^X$ is compactly open valued on X_i,

(d) for each finite subset $A \in \langle \operatorname{dom} P_i\rangle$, $\{z \in \cap_{x\in A} N_x : P_i(z) \neq \emptyset\} \subset \{z \in \cap_{x\in A} N_x : \cap_{x\in A}\psi_{i,x}(z) \neq \emptyset\}$.

By (iii), we may assume that $N_x \subset W_i$, hence $P_i(z) \neq \emptyset$ and $i \in I(z)$ for all $z \in N_x$. Now, define two mappings $\psi_x, \phi_x : X \to 2^X$ by

$$\psi_x(z) = \pi_i^-(\psi_{i,x}(z)), \quad \phi_x(z) = \pi_i^-(\phi_{i,x}(z)) \quad \text{for each } z \in X. \tag{5.8}$$

Then, we have

(a′) for each $z \in N_x$, by (a), $P(z) = \cap_{i\in I(z)} P_i'(z) \subset P_i'(z) = \pi_i^-(P_i(z)) \subset \pi_i^-(\phi_{i,x}(z)) = \phi_x(z)$ and $z \notin \operatorname{co}_\Gamma(\phi_x(z))$,

(b′) for each $z \in X$, $\psi_x(z) \subset \phi_x(z)$ by (b),

(c′) for each $y \in X$, $\psi_x^-(y) = \{z \in X : y \in \psi_x(z)\} = \{z \in X : y_i \in \psi_{i,x}(z)\} = \psi_{i,x}^-(y)$ is compactly open in X by (c), (d′) for each finite subset $A = \{x_1, x_2, \ldots, x_n\} \in \langle \operatorname{dom} P \rangle$, put $\cup\{B : B \subset A$ and $\cap_{x \in B} I(x) \neq \emptyset\} = \{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\}$. For each $i \in \cap_{k=1}^{m} I(x_{i_k})$, then

$$\bigcap_{x \in A} \psi_x(z) = \bigcap_{x \in A} \pi_i^{-}(\psi_{i,x}(z)) = \prod_{j \neq i_k, k=1,\ldots,n} X_j \otimes \bigcap_{k=1}^{m} \psi_{i,x_{i_k}}(z) \otimes \prod_{k=m+1}^{n} \psi_{i_k, x_{i_k}}(z). \tag{5.9}$$

For each $z \in \cap_{x \in A} N_x$, if $\cap_{x \in A} \psi_x(z) = \emptyset$, then there exists a set $A_1 = \{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\} \subset A$, such that $\cap_{x \in A_1} \psi_{i,x}(z) = \emptyset$, by (d), $P_i(z) = \emptyset$. Thus, $\{z \in \cap_{x \in A} N_x : P(z) \neq \emptyset\} \subset \{z \in \cap_{x \in A} N_x : \cap_{x \in A} \psi_x(z) \neq \emptyset\}$. This shows that P is an $\mathcal{A}_C$-majorized mapping. By $P^{-}(y) = P_i^{-}(\pi_i(y))$ and condition (iv), for each $N \in \langle X \rangle$, there exists a compact abstract convex subset L_N of X containing N such that

$$L_N \setminus K \subset \bigcup\{\operatorname{cint}\left(P_i^{-}(\pi_i(y))\right) : y \in L_N\} = \bigcup\{\operatorname{cint}(P^{-}(y)) : y \in L_N\}. \tag{5.10}$$

Hence, all the hypotheses of Theorem 4.5 are satisfied. By Theorem 4.5, there exists a point $\widehat{x} \in K$ such that $P(\widehat{x}) = \emptyset$. This implies that $I(\widehat{x}) = \emptyset$ and therefore $P_i(\widehat{x}) = \emptyset$ for each $i \in I$, that is, $\widehat{x}$ is an equilibrium point of Λ. □

Remark 5.5. Theorem 5.4 improves and generalizes Theorem 5.2 of Ding and Feng [20], Theorem 6.2 of Ding and Wang [6], Theorem 4.2 of Yuan [25], Theorem 4.2 of Chowdhury et al. [4], and Theorem 4.2 of Tan and Yuan [26].

Applying Theorem 5.4, we prove that the following equilibria existence theorem for a noncompact generalized games.

Theorem 5.6. *Let $\Lambda = (X; X_i; A_i; B_i; P_i, \pi)_{i \in I}$ be a generalized game. Let K be a compact and closed subset of X and $\operatorname{co}_\Gamma(\pi_i(K)) = \pi_i(K)$. Suppose that for each $i \in I$,*

(i) *(X_i, Γ_i) is a paracompact abstract convex space such that $(X; \Gamma)$ satisfies the partial KKM principle,*

(ii) *for each $x \in X$, $\operatorname{co}_\Gamma(A_i(x)) \subset \overline{B}_i(x)$, $A_i(x) \cap P_i(x) \subset \pi_i(K)$ and $\operatorname{dom} A_i = X$,*

(iii) *for each $y \in X_i$, $A_i^{-}(y)$ is compactly open in X,*

(iv) *$W_i = \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$ is open in X,*

(v) *$A_i \cap P_i : X \to 2^{X_i}$ is an $\mathcal{A}_C$-majorized mapping,*

(vi) *for each $N \in \langle X \rangle$, there exists a compact abstract convex subset L_N of X containing N such that*

$$L_N \setminus K \subset \bigcup\{\operatorname{cint}\left((A_i \cap P_i)^{-}(y)\right) : y \in L_N\}. \tag{5.11}$$

Then, Λ has an equilibrium point $\widehat{x}$ in K.

Proof. For each $i \in I$, let $F_i = \{x \in X : x_i \notin \overline{B}_i(x)\}$, then F_i is open in X. Define $Q_i : X \to 2^{X_i}$ by

$$Q_i(x) = \begin{cases} \pi_i(K) & \text{if } x \in X \setminus K, \\ (A_i \cap P_i)(x) & \text{if } x \in K \setminus F_i, \\ A_i(x) & \text{if } x \in F_i \cap K. \end{cases} \tag{5.12}$$

We will prove that the qualitative game $\Lambda' = (X; X_i, Q_i)_{i \in I}$ satisfies all the hypotheses of Theorem 5.4. For each $i \in I$, we have that the set

$$\begin{aligned} \{x \in X : Q_i(x) \neq \emptyset\} &= \{x \in X \setminus K : \pi_i(K) \neq \emptyset\} \cup \{x \in K \setminus F_i : (A_i \cap P_i)(x) \neq \emptyset\} \\ &\quad \cup \{x \in K \cap F_i : A_i(x) \neq \emptyset\} \\ &= (X \setminus K) \cup [(K \setminus F_i) \cap W_i] \cup (K_i \cap F_i) \\ &= (X \setminus K) \cup [K \cap (W_i \cup F_i)] \\ &= (X \setminus K) \cup W_i \cup F_i \end{aligned} \tag{5.13}$$

is open in X and hence the condition (iii) of Theorem 5.4 is satisfied. By (v), for each $x \in W_i$, there exist an open neighborhood N_x of x in X and two mappings $\varphi_{i,x}, \phi_{i,x} : X \to 2^{X_i}$ such that

(a) for each $z \in N_x$, $(A_i \cap P_i)(z) \subset \phi_{i,x}(z)$, $z_i \notin \mathrm{co}_\Gamma(\phi_{i,x}(z))$ and $\{z \in N_x : (A_i \cap P_i)(z) \neq \emptyset\} \subset \{z \in N_x : \varphi_{i,x}(z) \neq \emptyset\}$,

(b) for each $z \in X$, $\varphi_{i,x}(z) \subset \phi_{i,x}(z)$ and $\mathrm{co}_\Gamma(\phi_{i,x}(z)) \subset X_i$,

(c) the mapping $\varphi_{i,x}^{-} : X_i \to 2^X$ is compactly open valued on X_i,

(d) for each nonempty finite subset $A \in \langle \mathrm{dom}(A_i \cap P_i) \rangle$,

$$\left\{ z \in \bigcap_{x \in A} N_x : (A_i \cap P_i)(z) \neq \emptyset \right\} \subset \left\{ z \in \bigcap_{x \in A} N_x : \bigcap_{x \in A} \varphi_{i,x}(z) \neq \emptyset \right\}. \tag{5.14}$$

Define $\Psi_{i,x}, \Phi_{i,x} : X \to 2^{X_i}$ by

$$\begin{gathered} \Psi_{i,x}(z) = \begin{cases} \pi_i(K) & \text{if } z \in X \setminus K, \\ \varphi_{i,x}(z) & \text{if } z \in K \setminus F_i, \\ A_i(z) & \text{if } z \in F_i \cap K, \end{cases} \\ \Phi_{i,x}(z) = \begin{cases} \pi_i(K) & \text{if } z \in X \setminus K, \\ \phi_{i,x}(z) & \text{if } z \in K \setminus F_i, \\ A_i(z) & \text{if } z \in F_i \cap K. \end{cases} \end{gathered} \tag{5.15}$$

Now for each $x \in X$ with $Q_i(x) \neq \emptyset$, the set

$$U_x = \begin{cases} X \setminus K & \text{if } x \in X \setminus K, \\ N_x & \text{if } x \in K \setminus F_i, \\ F_i & \text{if } x \in F_i \cap K \end{cases} \tag{5.16}$$

is open in X. Then,

(a′) for each $z \in U_x$, $Q_i(z) \subset \Phi_{i,x}(z)$ by (a) if $z \in X \setminus K$, $z_i \notin \mathrm{co}_\Gamma(\pi_i(K)) = \mathrm{co}_\Gamma(\Phi_{i,x}(z))$ and if $z \in N_x$, $z \notin \mathrm{co}_\Gamma(\phi_{i,x}(z)) = \mathrm{co}_\Gamma(\Phi_{i,x}(z))$ by (a) if $z \in F_i$, $z \notin \mathrm{co}_\Gamma(A_i(z)) = \mathrm{co}_\Gamma(\Phi_{i,x}(z))$ by (ii), that is, $z \notin \mathrm{co}_\Gamma(\Phi_{i,x}(z))$ for each $z \in X$, and $\{z \in U_x : Q_i(z) \neq \emptyset\} \subset \{z \in U_x : \Psi_{i,x}(z) \neq \emptyset\}$ by (a),

(b′) for each $z \in X$, $\Psi_{i,x}(z) \subset \Phi_{i,x}(z)$ by (b) and $\mathrm{co}_\Gamma(\Psi_{i,x}(z)) \subset X_i$,

(c′) for each $y \in X_i$,

$$\begin{aligned} \Psi_{i,x}^{-}(y) &= \left\{z \in X : y \in \Psi_{i,x}(z)\right\} = \left\{z \in X \setminus K : y \in \pi_i(K)\right\} \\ &\quad \cup \left\{z \in K \setminus F_i : y \in \varphi_{i,x}(z)\right\} \cup \left\{z \in K \cap F_i : y \in A_i(z)\right\} \\ &= (X \setminus K) \cup \left[\varphi_{i,x}^{-}(y) \cap (K \cap F_i)\right] \cup \left[A_i^{-}(y) \cap K \cap F_i\right] \end{aligned} \tag{5.17}$$

is compactly open by (c) and (iii),

(d′) for each $A \in \langle \operatorname{dom} Q_i \rangle$, put

$$A = [A \cap (X \setminus K)] \cup [A \cap (K \setminus F_i)] \cup [A \cap K \cap F_i] = A_1 \cup A_2 \cup A_3. \tag{5.18}$$

Case I. If $A_3 \neq \emptyset$, $\cap_{x\in A} U_x \subset F_i \cap K$, then

$$\begin{aligned} \left\{z \in \bigcap_{x\in A} U_x : \cap_{x\in A} \Psi_{i,x}(z) \neq \emptyset\right\} &= \left\{z \in \bigcap_{x\in A} U_x : A_i(z) \neq \emptyset\right\} \\ &= \left\{z \in \bigcap_{x\in A} U_x : Q_i(z) \neq \emptyset\right\}. \end{aligned} \tag{5.19}$$

Case II. If $A_3 = \emptyset$, then

(1) if $A_1 \neq \emptyset$, $A_2 \neq \emptyset$, then $\cap_{x\in A} U_x = (\cap_{x\in A_2} N_x) \cap (X \setminus K) \subset X \setminus K$, and

$$\begin{aligned} \left\{z \in \bigcap_{x\in A} U_x : \bigcap_{x\in A} \Psi_{i,x}(z) \neq \emptyset\right\} &= \left\{z \in \bigcap_{x\in A} U_x : \pi_i(K) \neq \emptyset\right\} \\ &= \left\{z \in \bigcap_{x\in A} U_x : Q_i(z) \neq \emptyset\right\}. \end{aligned} \tag{5.20}$$

(2) If $A_2 = \emptyset,\ A_1 \neq \emptyset, \cap_{x\in A} U_x = X \setminus K$, that is similar to the condition (1).

(3) If $A_1 = \emptyset,\ A_2 \neq \emptyset$, then

$$\begin{aligned}
\left\{z \in \bigcap_{x\in A} U_x : \bigcap_{x\in A} \Psi_{i,x}(z) \neq \emptyset\right\} &= \left\{z \in \left(\bigcap_{x\in A_2} N_x\right) \cap (X \setminus K) : \Psi_{i,x}(z) \neq \emptyset\right\} \\
&\cup \left\{z \in \left(\bigcap_{x\in A_2} N_x\right) \cap (K \setminus F_i) : \Psi_{i,x}(z) \neq \emptyset\right\} \\
&\cup \left\{z \in \left(\bigcap_{x\in A_2} N_x\right) \cap (K \cap F_i) : \Psi_{i,x}(z) \neq \emptyset\right\} \\
&= \left\{z \in \left(\bigcap_{x\in A_2} N_x\right) \cap (X \setminus K) : \pi_i(K) \neq \emptyset\right\} \\
&\cup \left\{z \in \left(\bigcap_{x\in A_2} N_x\right) \cap (K \setminus F_i) : \bigcap_{x\in A} \varphi_{i,x}(z) \neq \emptyset\right\} \\
&\cup \left\{z \in \left(\bigcap_{x\in A_2} N_x\right) \cap (K \cap F_i) : A_i(z) \neq \emptyset\right\} \\
&\supset \left\{z \in \left(\bigcap_{x\in A_2} N_x\right) \cap (X \setminus K) : \pi_i(K) \neq \emptyset\right\} \\
&\cup \left\{z \in \left(\bigcap_{x\in A_2} N_x\right) \cap (K \setminus F_i) : (A_i \cap P_i)(z) \neq \emptyset\right\} \\
&\cup \left\{z \in \left(\bigcap_{x\in A_2} N_x\right) \cap (K \cap F_i) : A_i(z) \neq \emptyset\right\} \\
&= \left\{z \in \bigcap_{x\in A_2} N_x : Q_i(z) \neq \emptyset\right\}.
\end{aligned} \tag{5.21}$$

Thus, Q_i is an $\mathcal{A}_C$-majorized mapping. By the definition of Q and condition (vi) for each $N \in \langle X\rangle$, there exists a compact abstract convex subset L_N of X containing N such that

$$L_N \setminus K \subset \bigcup\left\{\text{cint } \left((A_i \cap P_i)^-(y)\right) : y \in L_N\right\} \subset \bigcup\left\{\text{cint}\left(Q_i^-(y)\right) : y \in L_N\right\}. \tag{5.22}$$

Hence, all the hypotheses of Theorem 5.4 are satisfied. By Theorem 5.4, there exists a point $\widehat{x} \in K$ such that $Q_i(\widehat{x}) = \emptyset\ (i \in I)$. By the definition of Q_i, $\widehat{x} \in K \setminus F_i \subset K$, that is for all $i \in I$, $\widehat{x}_i \in \overline{B}_i(\widehat{x})$, and $A_i(\widehat{x}) \cap P_i(\widehat{x}) = \emptyset$. This completes the proof. □

Remark 5.7. Theorem 5.6 generalized Theorem 5.3 of Ding and Feng [20], Theorem 6.3 of Ding and Wang [6], Theorem 4.4 of Chowdhury et al. [4], and Theorem 4.3 of Tan and Yuan [26].

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