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Research Article

Convergence Theorems for a Common Point of Solutions of Equilibrium and Fixed Point of Relatively Nonexpansive Multivalued Mapping Problems

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We introduce an iterative process which converges strongly to a common point of set of solutions of equilibrium problem and set of fixed points of finite family of relatively nonexpansive multi-valued mappings in Banach spaces.

1. Introduction

Let E be a real Banach space with dual E^*. The function $\phi : E \times E \to \mathbb{R}^+$, defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \text{for } x, y \in E, \tag{1.1}$$

is studied by Alber [1] and Reich [2], where J is the normalized duality mapping from E to 2^{E^*} defined by $Jx := \{f^* \in E^* : \langle x, f^*\rangle = \|x\|^2 = \|f^*\|^2\}$, where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. It is well known that E is smooth if and only if J is single valued and if E is uniformly smooth then J is uniformly continuous on bounded subsets of E. We note that in a Hilbert space H, J is the identity operator.

Let C be a nonempty closed convex subset of a Hilbert space H. It is well known that the metric projection of H onto C, $P_C : H \to C$, is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this direction, Alber [1] introduced a generalized projection operator Π_C in a Banach space E which is an analogue of metric projection in Hilbert spaces.

Let C be a nonempty closed and convex subset of a reflexive, strictly convex and smooth Banach space E. The *generalized projection mapping*, introduced by Alber [1], is a mapping $\Pi_C : E \to C$, that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem

$$\phi(\overline{x}, x) = \min\{\phi(y, x), y \in C\}. \tag{1.2}$$

Let C be a nonempty closed convex subset of a Banach space E. Let $T : C \to C$ be a single-valued mapping. An element $p \in C$ is called a *fixed point* of T if $T(p) = p$. The set of fixed points of T is denoted by $F(T)$. A point p in C is said to be an *asymptotic fixed point* of T (see [2]) if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$. T is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for each $x, y \in C$ and is called *relatively nonexpansive* if (A1) $F(T) \ne \emptyset$; (A2) $\phi(p, Tx) \le \phi(p, x)$ for $x \in C$ and $p \in F(T)$ and (A3) $F(T) = \widehat{F}(T)$.

Let C be a nonempty closed convex subset of a Banach space E and let $N(C)$ and $CB(C)$ denote the family of nonempty subsets and nonempty closed bounded subsets of C, respectively. Let H be the Hausdorff metric on $CB(C)$ defined by

$$H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}, \tag{1.3}$$

for all $A, B \in CB(C)$, where $d(a, B) = \inf\{\|a - b\| : b \in B\}$ is the distance from the point a to the subset B.

Let $T : C \to CB(C)$ be a multivalued mapping. T is said to be a *nonexpansive* if $H(Tx, Ty) \le \|x - y\|$, for $x, y \in C$. An element $p \in C$ is called a *fixed point* of T, if $p \in F(T)$, where $F(T) := \{p \in C : p \in T(p)\}$. A point $p \in C$ called an *asymptotic fixed point* of T, if there exists a sequence $\{x_n\}$ in C which converges weakly to p such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. T is said to be *relatively nonexpansive* if (B1) $F(T) \ne \emptyset$; (B2) $\phi(p, u) \le \phi(p, x)$ for $x \in C$, $u \in Tx$, $p \in F(T)$ and (B3) $F(T) = \widehat{F}(T)$, where $\widehat{F}(T)$ is the set of asymptotic fixed points of T.

We remark that the class of relatively nonexpansive single-valued mappings is contained in a class of relatively nonexpansive multi-valued mappings. An example of relatively nonexpansive multi-valued mapping by Homaeipour and Razani [3] is given below.

Example 1.1. Let $I = [0, 1]$, $X = L^p(I)$, $1 < p < \infty$ and $C = \{f \in X : f(x) \ge 0, \text{ for all } x \in I\}$. Let $T : C \to CB(C)$ be defined by

$$T(f) = \begin{cases} \left\{g \in C : f(x) - \dfrac{3}{4} \le g(x) \le f(x) - \dfrac{1}{4}, \ \forall x \in I\right\}, & \text{if } f(x) > 1, \ x \in I, \\ \{0\}, & \text{otherwise}. \end{cases} \tag{1.4}$$

It is shown in [3] that T is relatively nonexpansive multi-valued mapping which is not nonexpansive.

The study of fixed points for multi-valued nonexpansive mappings in relation to Hausdorff metric was introduced by Markin [4] (see also [5]). Since then a lot of activity in this area and fixed point theory for multi-valued nonexpansive mappings has been developed which has some nontrivial applications in pure and applied sciences including control theory, convex optimization, differential inclusion, and economics (see, e.g., [6] and references therein). Later, Lim [7] established the existence of fixed points for multi-valued nonexpansive mappings in uniformly convex Banach spaces.

It is well known that the normal *Mann's* iterative [8] algorithm has only weak convergence in an infinite-dimensional Hilbert space even for nonexpansive single-valued mappings. Consequently, in order to obtain strong convergence, one has to modify the normal Mann's iteration algorithm, the so called hybrid projection iteration method is such a modification. The hybrid projection iteration algorithm (HPIA) was introduced initially by Haugazeau [9] in 1968. For 40 years, (HPIA) has received rapid developments. For details, the readers are referred to papers [10–12] and the references therein.

In 2003, Nakajo and Takahashi [12] proposed the following modification of the Mann iteration method for a nonexpansive single-valued mapping T in a Hilbert space H:

$$\begin{aligned} &x_0 \in C, \text{ chosen arbitrary,} \\ &y_n = \alpha_n x_n + (1-\alpha_n) T x_n, \\ C_n &= \{ z \in C : \| y_n - z \| \le \| x_n - z \| \}, \\ Q_n &= \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0 \}, \\ &x_{n+1} = P_{C_n \cap Q_n}(x_0), \quad n \ge 0, \end{aligned} \tag{1.5}$$

where C is a closed convex subset of H, P_C denotes the metric projection from H onto C. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one then the sequence $\{x_n\}$ generated by (1.5) converges strongly to $P_{F(T)}(x_0)$.

In spaces more general than Hilbert spaces, Matsushita and Takahashi [11] proposed the following hybrid iteration method with generalized projection for relatively nonexpansive single-valued mapping T in a Banach space E:

$$\begin{aligned} &x_0 \in C, \text{ chosen arbitrary,} \\ &y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\ C_n &= \{ z \in C : \phi(z, y_n) \le \phi(z, x_n) \}, \\ Q_n &= \{ z \in C : \langle x_n - z, J x_0 - J x_n \rangle \ge 0 \}, \\ &x_{n+1} = \Pi_{C_n \cap Q_n}(x_0), \quad n \ge 0. \end{aligned} \tag{1.6}$$

They proved the following convergence theorem.

Theorem MT. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let T be a relatively nonexpansive single-valued mapping from C into itself, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n < 1$ and $\limsup_{n\to\infty} \alpha_n < 1$.*

Suppose that $\{x_n\}$ *is given by* (1.6), *where* J *is the duality mapping on* E. *If* $F(T)$ *is nonempty, then* $\{x_n\}$ *converges strongly to* $\Pi_{F(T)}x_0$, *where* $\Pi_{F(T)}(\cdot)$ *is the generalized projection from* E *onto* $F(T)$.

Let $f : C \times C \to \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of real numbers. The *equilibrium problem* for f is

$$\text{finding } x^* \in C \quad \text{such that } f(x^*, y) \geq 0, \quad \forall y \in C. \tag{1.7}$$

The solution set of (1.7) is denoted by $EP(f)$.

If $f(x, y) = \langle Ax, y - x \rangle$, where $A : C \to C$ is a monotone mapping, then the problem (1.7) reduces to the system of variational inequality problem

$$\text{find an element } x^* \in C \quad \text{such that } \langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{1.8}$$

That is, the problem (1.8) is a special case of (1.7). The set of solutions of inequality (1.8) is denoted by $VI(C, A)$.

For solving the equilibrium problem for a bifunction $f : C \times C \to R$, we assume that f satisfies the following conditions:

(A1) $f(x, x) = 0$, for all $x \in C$,

(A2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$, for all $x, y \in C$,

(A3) for each $x, y, z \in C$, $\lim_{t \to 0^+} f(tz + (1 - t)x, y) \leq f(x, y)$,

(A4) for each $x \in C$, $y \to f(x, y)$ is convex and lower semicontinuous.

Recently, many authors studied the problem of finding a common element of the set of fixed points of nonexpansive or relatively nonexpansive single-valued mapping and the set of solutions of an equilibrium problems in the frame work of Hilbert spaces and Banach spaces, respectively: see, for instance, [2, 13–21] and the references therein.

In [22], Kumam introduced the following iterative scheme in a Hilbert space:

$$\begin{gathered} x_0 \in H, \\ u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - y_n \rangle \geq 0, \quad \forall y \in C, \\ w_n = \alpha_n x_n + (1 - \alpha_n)Tu_n, \\ C_n = \{z \in H : \|w_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_{n+1} \cap Q_n}(x_0), \quad n \geq 0, \end{gathered} \tag{1.9}$$

for finding a common element of the set of fixed point of nonexpansive single-valued mapping T and set of solution of equilibrium problems.

In the case that E is a Banach space, Takahashi and Zembayashi [16] introduced the following iterative scheme which is called the shrinking projection method:

$$\begin{gathered} x_0 \in C, \quad \text{chosen arbitrary}, \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\ u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad n \geq 0, \end{gathered} \tag{1.10}$$

where J is the duality mapping on E, Π_C is the generalized projection from E onto C and T is relatively nonexpansive single-valued mapping. They proved that the sequence $\{x_n\}$ converges strongly to a common element of the set of fixed point of relatively nonexpansive single-valued mapping and set of solution of equilibrium problem under appropriate conditions.

We remark that the computation of x_{n+1} in (1.9) and (1.10) is not simple because of the involvement of computation of C_{n+1} from C_n for each $n \geq 0$.

More recently, Homaeipour and Razani [3] studied the following iterative scheme for a fixed point of relatively nonexpansive multi-valued mapping in uniformly convex and uniformly smooth Banach space E:

$$\begin{gathered} x_0 \in C, \quad \text{chosen arbitrary}, \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n J x_n + (1-\alpha_n) J z_n), \quad z_n \in T x_n, \ n \geq 0, \end{gathered} \tag{1.11}$$

where $\{\alpha_n\} \subset (0,1)$ for all $n \geq 0$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. They proved that if J is weakly sequentially continuous then the sequence $\{x_n\}$ converges *weakly* to a fixed point of T. Furthermore, it is shown that the scheme converges strongly to a fixed point of T if *interior of* $F(T)$ is *nonempty*.

But it is worth mentioning that the convergence of the scheme is either *weak* or it requires that the *interior* of $F(T)$ is *nonempty*.

In this paper, motivated by Kumam [22], Takahashi and Zembayashi [16], and Homaeipour and Razani [3], we construct an iterative scheme which converges strongly to a common point of set of solutions of equilibrium problem and set of fixed points of finite family of relatively nonexpansive multi-valued mappings in Banach spaces. Our scheme does not involve computation of C_n and Q_n, for each $n \geq 0$, and the requirement that the interior of F is nonempty is dispensed with. Our theorems improve and unify most of the results that have been proved for this important class of nonlinear operators.

2. Preliminaries

Let E be a normed linear space with $\dim E \geq 2$. The *modulus of smoothness* of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) := \sup\left\{ \frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1; \|y\| = \tau \right\}. \tag{2.1}$$

The space E is said to be *smooth* if $\rho_E(\tau) > 0$, for all $\tau > 0$ and E is called *uniformly smooth* if and only if $\lim_{t\to 0^+}(\rho_E(t)/t) = 0$.

The *modulus of convexity* of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) := \inf\left\{ 1 - \left\| \frac{x+y}{2} \right\| : \|x\| = \|y\| = 1; \epsilon = \|x-y\| \right\}. \tag{2.2}$$

E is called *uniformly convex* if and only if $\delta_E(\epsilon) > 0$, for every $\epsilon \in (0,2]$.

In the sequel, we will need the following results.

Lemma 2.1 (see [1]). *Let K be a nonempty closed and convex subset of a real reflexive, strictly convex, and smooth Banach space E and let $x \in E$. Then for all $y \in K$,*

$$\phi(y, \Pi_K x) + \phi(\Pi_K x, x) \leq \phi(y, x). \tag{2.3}$$

We make use of the function $V : E \times E^* \to \mathbb{R}$, defined by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2, \quad \forall x \in E, \ x^* \in E^*, \tag{2.4}$$

studied by Alber [1]. That is, $V(x, x^*) = \phi(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E^*$. We know the following lemma.

Lemma 2.2 (see [1]). *Let E be reflexive strictly convex and smooth Banach space with E^* as its dual. Then*

$$V(x, x^*) + 2\left\langle J^{-1}x^* - x, y^* \right\rangle \leq V(x, x^* + y^*), \tag{2.5}$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.3 (see [1]). *Let C be a convex subset of a real smooth Banach space E. Let $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle z - x_0, Jx - Jx_0 \rangle \leq 0, \quad \forall z \in C. \tag{2.6}$$

Lemma 2.4 (see [23]). *Let E be a uniformly convex Banach space and $B_R(0)$ be a closed ball of E. Then, there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_N x_N\|^2 \le \sum_{i=1}^{N} \alpha_i \|x_i\|^2 - \alpha_i \alpha_j g(\|x_i - x_j\|), \tag{2.7}$$

for $i, j \in \{1, \dots, N\}$, $\alpha_i \in (0,1)$ such that $\sum_{i=1}^{N} \alpha_i = 1$, and $x_i \in B_R(0) := \{x \in E : \|x\| \le R\}$, for $i = 1, 2, \dots, N$.

Lemma 2.5 (see [24]). *Let E be a real smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $x_n - y_n \to 0$, as $n \to \infty$.*

Proposition 2.6 (see [3]). *Let E be a strictly convex and smooth Banach space and C be a nonempty closed convex subset of E. Let $T : C \to N(C)$ be a relatively nonexpansive multi-valued mapping. Then $F(T)$ is closed and convex.*

Lemma 2.7 (see [16]). *Let C be a nonempty, closed and convex subset of a uniformly smooth, strictly convex and reflexive real Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)–(A4). For $r > 0$ and $x \in E$, define the mapping $F_r : E \to C$ as follows:*

$$F_r x := \left\{ z \in C : f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \ge 0,\ \forall y \in C \right\}. \tag{2.8}$$

Then the following statements hold:

(1) *F_r is single-valued,*

(2) *$F(F_r) = EP(f)$,*

(3) *$\phi(q, F_r x) + \phi(F_r x, x) \le \phi(q, x)$, for $q \in F(F_r)$,*

(4) *$EP(f)$ is closed and convex.*

Lemma 2.8 (see [25]). *Let $\{a_n\}$ be sequences of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \le a_{m_k+1}, \qquad a_k \le a_{m_k+1}. \tag{2.9}$$

In fact, $m_k = \max\{j \le k : a_j < a_{j+1}\}$.

Lemma 2.9 (see [26]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \le (1 - \beta_n) a_n + \beta_n \delta_n, \quad n \ge n_0,\ for\ some\ n_0 \in\ \mathbb{N}, \tag{2.10}$$

where $\{\beta_n\} \subset (0,1)$ and $\{\delta_n\} \subset R$ satisfying the following conditions: $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$, and $\limsup_{n\to\infty} \delta_n \le 0$. Then, $\lim_{n\to\infty} a_n = 0$.

3. Main Result

Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive real Banach space E with dual E^*. Let $f : C \times C \to \mathbb{R}$ be a bifunction. For the rest of this paper, $F_{r_n}x$ is a mapping defined as follows. For $x \in E$, let $F_{r_n} : E \to C$ be given by

$$F_{r_n}x := \left\{ z \in C : f(z, y) + \frac{1}{r_n}\langle y - z, Jz - Jx\rangle \geq 0,\ \forall y \in C \right\}, \tag{3.1}$$

where $\{r_n\}_{n\in\mathbb{N}} \subset [c_1, \infty)$, for some $c_1 > 0$.

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space E. Let $f : C \times C \to \mathbb{R}$, be a bifunction which satisfies conditions (A1)–(A4). Let $T_i : C \to CB(C)$, for $i = 1, 2, \ldots, N$, be a finite family of relatively nonexpansive multi-valued mappings. Assume that $F := \cap_{i=1}^{N} F(T_i) \cap EP(f)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} x_0 = w \in C, &\quad \textit{chosen arbitrarily}, \\ w_n &= F_{r_n}x_n, \\ y_n &= \Pi_C J^{-1}(\alpha_n Jw + (1 - \alpha_n)Jw_n), \\ x_{n+1} = J^{-1}&\left(\beta_{n,0}Jw_n + \sum_{i=1}^{N}\beta_{n,i}Ju_{n,i}\right), \quad u_{n,i} \in T_i y_n,\ n \geq 0, \end{aligned} \tag{3.2}$$

where $\alpha_n \in (0, 1)$ such that $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_{n,i}\} \subset [a, b] \subset (0, 1)$, for $i = 1, 2, \ldots, N$, satisfying $\beta_{n,0} + \beta_{n,1} + \cdots + \beta_{n,N} = 1$, for each $n \geq 0$. Then $\{x_n\}$ converges strongly to an element of F.

Proof. Since F is nonempty closed and convex, put $x^* := \Pi_F w$. Now from (3.2), Lemma 2.7(3) and property of ϕ, we get that

$$\begin{aligned} \phi(x^*, y_n) &= \phi\left(x^*, \Pi_C J^{-1}(\alpha_n Jw + (1 - \alpha_n)Jw_n)\right) \\ &\leq \phi\left(x^*, J^{-1}(\alpha_n Jw + (1 - \alpha_n)Jw_n)\right) \\ &= \|x^*\|^2 - 2\langle x^*, \alpha_n Jw + (1 - \alpha_n)Jw_n\rangle + \|\alpha_n Jw + (1 - \alpha_n)Jw_n\|^2 \\ &\leq \|x^*\|^2 - 2\alpha_n\langle x^*, Jw\rangle - 2(1 - \alpha_n)\langle x^*, Jw_n\rangle \\ &\quad + \alpha_n\|w\|^2 + (1 - \alpha_n)\|w_n\|^2 \\ &\leq \alpha_n\phi(x^*, w) + (1 - \alpha_n)\phi(x^*, w_n) \\ &= \alpha_n\phi(x^*, w) + (1 - \alpha_n)\phi(x^*, F_{r_n}x_n) \\ &\leq \alpha_n\phi(x^*, w) + (1 - \alpha_n)\phi(x^*, x_n). \end{aligned} \tag{3.3}$$

Now, from (3.2), Lemma 2.7(3), relatively nonexpansiveness of T_i, property of ϕ and (3.3), we have that

$$\begin{aligned}
\phi(x^*, x_{n+1}) &= \phi\left(x^*, J^{-1}\left(\beta_{n,0} J w_n + \sum_{i=1}^{N} \beta_{n,i} J u_{n,i}\right)\right) \\
&\leq \beta_{n,0}\phi(x^*, w_n) + \sum_{i=1}^{N} \beta_{n,i}\phi(x^*, u_{n,i}) \\
&= \beta_{n,0}\phi(x^*, F_{r_n} x_n) + \sum_{i=1}^{N} \beta_{n,i}\phi(x^*, u_{n,i}) \\
&\leq \beta_{n,0}\phi(x^*, x_n) + (1-\beta_{n,0})\phi(x^*, y_n) \\
&\leq \beta_{n,0}\phi(x^*, x_n) + (1-\beta_{n,0})\left[\alpha_n\phi(x^*, w) + (1-\alpha_n)\phi(x^*, x_n)\right] \\
&\leq \delta_n\phi(x^*, w) + (1-\delta_n)\phi(x^*, x_n),
\end{aligned} \tag{3.4}$$

where $\delta_n = (1-\beta_{n,0})\alpha_n$. Thus, by induction,

$$\phi(x^*, x_{n+1}) \leq \max\{\phi(x^*, x_0), \phi(x^*, w)\}, \quad \forall n \geq 0, \tag{3.5}$$

which implies that $\{x_n\}$ is bounded and hence $\{y_n\}$ and $\{w_n\}$ are bounded. Now let $z_n = J^{-1}(\alpha_n J w + (1-\alpha_n) J w_n)$. Then we have that $y_n = \Pi_C z_n$. Using Lemma 2.2 and property of ϕ, we obtain that

$$\begin{aligned}
\phi(x^*, y_n) \leq \phi(x^*, z_n) &= V(x^*, J z_n) \\
&\leq V(x^*, J z_n - \alpha_n(J w - J x^*)) - 2\langle z_n - x^*, -\alpha_n(J w - J x^*)\rangle \\
&= \phi\left(x^*, J^{-1}(\alpha_n J x^* + (1-\alpha_n) J w_n) + 2\alpha_n\langle z_n - x^*, J w - J x^*\rangle\right) \\
&\leq \alpha_n\phi(x^*, x^*) + (1-\alpha_n)\phi(x^*, w_n) + 2\alpha_n\langle z_n - x^*, J w - J x^*\rangle \\
&= (1-\alpha_n)\phi(x^*, w_n) + 2\alpha_n\langle z_n - x^*, J w - J x^*\rangle \\
&\leq (1-\alpha_n)\phi(x^*, x_n) + 2\alpha_n\langle z_n - x^*, J w - J x^*\rangle.
\end{aligned} \tag{3.6}$$

Furthermore, from (3.2), Lemma 2.4, relatively nonexpansiveness of T_i, for each $i = 1, 2, \dots, N$, Lemma 2.7(3), and (3.6) we have that

$$
\begin{aligned}
\phi(x^*, x_{n+1}) &= \phi\left(x^*, J^{-1}\left(\beta_{n,0} J w_n + \sum_{i=1}^{N} \beta_{n,i} J u_{n,i}\right)\right) \\
&\le \beta_{n,0}\phi(x^*, w_n) + \sum_{i=1}^{N} \beta_{n,i}\phi(x^*, u_{n,i}) \\
&\quad - \beta_{n,0}\beta_{n,i} g(\|Jw_n - Ju_{n,i}\|) \\
&= \beta_{n,0}\phi(x^*, F_{r_n} x_n) + \sum_{i=1}^{N} \beta_{n,i}\phi(x^*, u_{n,i}) \\
&\quad - \beta_{n,0}\beta_{n,i} g(\|Jw_n - Ju_{n,i}\|) \\
&\le \beta_{n,0}\left(\phi(x^*, x_n) - \phi(x_n, w_n)\right) + (1 - \beta_{n,0})\phi(x^*, y_n) \\
&\quad - \beta_{n,0}\beta_{n,i} g(\|Jw_n - Ju_{n,i}\|) \le \beta_{n,0}\phi(x^*, x_n) - \beta_{n,0}\phi(x_n, w_n) + (1 - \beta_{n,0}) \\
&\quad \times \left[(1 - \alpha_n)\phi(x^*, x_n) + 2\alpha_n \langle z_n - x^*, Jw - Jx^* \rangle\right] - \beta_{n,0}\beta_{n,i} g(\|Jw_n - Ju_{n,i}\|) \\
&= (1 - \delta_n)\phi(x^*, x_n) + 2\delta_n \langle z_n - x^*, Jw - Jx^* \rangle \\
&\quad - \beta_{n,0}\phi(x_n, w_n) - \beta_{n,0}\beta_{n,i} g(\|Jw_n - Ju_{n,i}\|),
\end{aligned}
\tag{3.7}
$$

and hence

$$
\phi(x^*, x_{n+1}) \le (1 - \delta_n)\phi(x^*, x_n) + 2\delta_n \langle z_n - x^*, Jw - Jx^* \rangle, \tag{3.8}
$$

where $\delta_n := \alpha_n(1 - \beta_{n,0})$, for all $n \in \mathbb{N}$. Note that δ_n satisfies $\lim_n \delta_n = 0$ and $\sum_{n=1}^{\infty} \delta_n = \infty$. □

Now, we consider two cases.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\phi(x^*, x_n)\}$ is nonincreasing for all $n \ge n_0$. In this situation, $\{\phi(x^*, x_n)\}$ is then convergent. Then from (3.7), we have that $\phi(x_n, w_n) \to 0$ and hence Lemma 2.5 implies that

$$
x_n - w_n \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.9}
$$

Moreover, from (3.7), we have that $\beta_{n,0}\beta_{n,i} g(\|Jw_n - Ju_{n,i}\|) \to 0$, as $n \to \infty$, which implies by the property of g that $Jw_n - Ju_{n,i} \to 0$, as $n \to \infty$, for each $i \in \{1, 2, \dots, N\}$, and hence, since J^{-1} uniformly continuous on bounded sets, we obtain that

$$
w_n - u_{n,i} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.10}
$$

Furthermore, by Lemma 2.1, property of ϕ and the fact that $\alpha_n \to 0$, as $n \to \infty$, imply that

$$\begin{aligned}\phi(w_n, y_n) &= \phi(w_n, \Pi_C z_n) \leq \phi(w_n, z_n)\\ &= \phi\Big(w_n, J^{-1}(\alpha_n Jw + (1-\alpha_n)Jw_n)\Big)\\ &\leq \alpha_n \phi(w_n, w) + (1-\alpha_n)\phi(w_n, w_n) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty,\end{aligned} \tag{3.11}$$

and hence

$$w_n - y_n \longrightarrow 0, \quad w_n - z_n \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.12}$$

Therefore, from (3.9), (3.10), and (3.12), we obtain that

$$x_n - z_n \longrightarrow 0, \quad y_n - x_n \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{3.13}$$

$$d(y_n, T_i y_n) \leq \|y_n - u_{n,i}\| \leq \|y_n - w_n\| + \|w_n - u_{n,i}\| \longrightarrow 0, \tag{3.14}$$

as $n \to \infty$, for each $i \in \{1, 2, \ldots, N\}$.

Let $\{z_{n_i}\}$ be a subsequence of $\{z_n\}$ such that $z_{n_i} \rightharpoonup z$ and $\limsup_{n\to\infty}\langle z_n - x^*, Jw - Jx^*\rangle = \lim_{i\to\infty}\langle z_{n_i} - x^*, Jw - Jx^*\rangle$. Then, from (3.12), (3.13), and the uniform continuity of J, we get that

$$x_{n_i}, w_{n_i}, y_{n_i} \rightharpoonup z, \quad Jx_n - Jw_n \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.15}$$

Now, we show that $z \in EP(f)$. But, from the definition of w_n and (A2) we note that

$$\frac{1}{r_{n_i}}\langle y - w_{n_i}, Jw_{n_i} - Jx_{n_i}\rangle \geq -f(w_{n_i}, y) \geq f(y, w_{n_i}), \quad \forall y \in C. \tag{3.16}$$

Letting $i \to \infty$, we have from (3.15) and (A4) that $f(y, z) \leq 0$, for all $y \in C$. Now, for $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)z$. Since $y \in C$ and $z \in C$, we have $y_t \in C$ and hence $f(y_t, z) \leq 0$. So, from the convexity of the equilibrium bifunction $f(x, y)$ on the second variable y, we have

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, z) \leq tf(y_t, y), \tag{3.17}$$

and hence $f(y_t, y) \geq 0$. Now, letting $t \to 0$ and condition (A3), we obtain that $f(z, y) \geq 0$, for all $y \in C$, and hence $z \in EP(f)$.

Next, we show that $z \in \cap_{i=1}^N F(T_i)$. But, since each T_i satisfies condition (B3) we obtain from (3.13) and (3.15) that $z \in F(T_i)$, for each $i = 1, 2, \ldots, N$, and hence $z \in \cap_{i=1}^N F(T_i)$. Thus, from the above discussions we obtain that $z \in F := \cap_{i=1}^N F(T_i) \cap EP(f)$. Therefore, by Lemma 2.3, we immediately obtain that $\limsup_{n\to\infty}\langle z_n - x^*, Jw - Jx^*\rangle = \lim_{i\to\infty}\langle z_{n_i} - x^*, Jw - Jx^*\rangle = \langle z - x^*, Jw - Jx^*\rangle \leq 0$. It follows from (3.8) and Lemma 2.9 that $\phi(x^*, x_n) \to 0$, as $n \to \infty$. Consequently, $x_n \to x^*$ by Lemma 2.5.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\phi(x^*, x_{n_i}) < \phi(x^*, x_{n_i+1}) \tag{3.18}$$

for all $i \in \mathbb{N}$. Then, by Lemma 2.8, there exist a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, $\phi(x^*, x_{m_k}) \le \phi(x^*, x_{m_k+1})$ and $\phi(x^*, x_k) \le \phi(x^*, x_{m_k+1})$, for all $k \in \mathbb{N}$. Now, from (3.7) and the fact that $\delta_n \to 0$, we have

$$\begin{aligned} &\beta_{m_k,0}\phi(x_{m_k}, w_{m_k}) + \beta_{m_k,0}\beta_{m_k,i}g(\|Jw_{m_k} - Ju_{m_k,i}\|) \\ &\quad \le \left(\phi(x^*, x_{m_k}) - \phi(x^*, x_{m_k+1})\right) - \delta_{m_k}\phi(x^*, x_{m_k}) + 2\delta_{m_k}\langle z_{m_k} - x^*, Jw - Jx^*\rangle, \end{aligned} \tag{3.19}$$

as $k \to \infty$. Thus, using the same proof of Case 1, we obtain that $x_{m_k} - w_{m_k} \to 0$ and $w_{m_k} - u_{m_k,i} \to 0$, as $k \to \infty$, for each $i = 1, 2, \ldots, N$ and hence

$$\limsup_{n\to\infty}\langle z_{m_k} - x^*, Jw - Jx^*\rangle \le 0. \tag{3.20}$$

Then from (3.8), we have that

$$\phi(x^*, x_{m_k+1}) \le (1 - \delta_{m_k})\phi(x^*, x_{m_k}) + 2\delta_{m_k}\langle z_{m_k} - x^*, Jw - Jx^*\rangle. \tag{3.21}$$

Since $\phi(x^*, x_{m_k}) \le \phi(x^*, x_{m_k+1})$, (3.21) implies that

$$\begin{aligned} \delta_{m_k}\phi(x^*, x_{m_k}) &\le \phi(x^*, x_{m_k}) - \phi(x^*, x_{m_k+1}) + 2\delta_{m_k}\langle z_{m_k} - x^*, Jw - Jx^*\rangle \\ &\le 2\delta_{m_k}\langle z_{m_k} - x^*, Jw - Jx^*\rangle. \end{aligned} \tag{3.22}$$

In particular, since $\delta_{m_k} > 0$, we get

$$\phi(x^*, x_{m_k}) \le 2\langle z_{m_k} - x^*, Jw - Jx^*\rangle. \tag{3.23}$$

Then, from (3.20), we obtain that $\phi(x^*, x_{m_k}) \to 0$, as $k \to \infty$. This together with (3.21) gives $\phi(x^*, x_{m_k+1}) \to 0$, as $k \to \infty$. But $\phi(x^*, x_k) \le \phi(x^*, x_{m_k+1})$ for all $k \in \mathbb{N}$, thus we obtain that $x_k \to x^*$. Therefore, from the above two cases, we can conclude that $\{x_n\}$ converges strongly to x^* and the proof is complete.

If in Theorem 3.1, we assume that $f(x, y) = \langle Ax, y - x\rangle$, for A continuous monotone mapping, then we obtain the following corollary.

Corollary 3.2. *Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space E. Let $A : C \to E^*$ be a continuous monotone mapping.*

Let $T_i : C \to CB(C)$, for $i = 1, 2, \ldots, N$, be a finite family of relatively nonexpansive multi-valued mappings. Assume that $F := \cap_{i=1}^{N} F(T_i) \cap VI(C, A)$ is nonempty. Let $\{x_n\}$ be a sequence generated by

$$\begin{aligned}
&x_0 = w \in C, \quad \text{chosen arbitrarily}, \\
&w_n \in C \quad \text{such that } \langle Aw_n, y - w_n \rangle + \frac{1}{r_n} \langle y - w_n, Jw_n - Jx_n \rangle \geq 0, \quad \forall y \in C, \\
&y_n = \Pi_C J^{-1}(\alpha_n Jw + (1 - \alpha_n) Jw_n), \\
&x_{n+1} = J^{-1}\left(\beta_{n,0} Jw_n + \sum_{i=1}^{N} \beta_{n,i} Ju_{n,i}\right), \quad u_{n,i} \in T_i y_n, \ n \geq 0,
\end{aligned} \tag{3.24}$$

where $\alpha_n \in (0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_{n,i}\} \subset [a, b] \subset (0, 1)$, for $i = 1, 2, \ldots, N$, satisfying $\beta_{n,0} + \beta_{n,1} + \cdots + \beta_{n,N} = 1$, for each $n \geq 0$. Then $\{x_n\}$ converges strongly to an element of F.

Proof. Let $f(x, y) = \langle Ax, y - x \rangle$. Since A is monotone and continuous, we get that a bifunction f satisfies conditions (A1)–(A4). Thus, the conclusion follows from Theorem 3.1. □

If in Theorem 3.1, we assume that $N = 1$, then we get the following theorem.

Corollary 3.3. *Let C be a nonempty, closed, and convex subset of a uniformly smooth and uniformly convex real Banach space E. Let $f : C \times C \to \mathbb{R}$, be a bifunction which satisfies conditions (A1)–(A4). Let $T : C \to CB(C)$ be a relatively nonexpansive multi-valued mapping. Assume that $F := F(T) \cap EP(f)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned}
&x_0 = w \in C, \quad \text{chosen arbitrarily}, \\
&w_n = F_{r_n} x_n, \\
&y_n = \Pi_C J^{-1}(\alpha_n Jw + (1 - \alpha_n) Jw_n), \\
&x_{n+1} = J^{-1}(\beta_n Jw_n + (1 - \beta_n) Ju_n), \quad u_n \in Ty_n, \ n \geq 0,
\end{aligned} \tag{3.25}$$

where $\alpha_n \in (0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_n\} \subset [a, b] \subset (0, 1)$, for each $n \geq 0$. Then $\{x_n\}$ converges strongly to an element of F.

Proof. The proof follows from Theorem 3.1 with $N = 1$. □

If in Theorem 3.1, we assume that $f \equiv 0$, we get the following corollary.

Corollary 3.4. *Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space E. Let $T_i : C \to CB(C)$, for $i = 1, 2, \ldots, N$, be a finite family of relatively* *nonexpansive multi-valued mappings. Assume that* $F := \cap_{i=1}^{N} F(T_i)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{aligned} x_0 &= w \in C, \quad \text{chosen arbitrarily}, \\ y_n &= \Pi_C J^{-1}(\alpha_n J w + (1-\alpha_n) J x_n), \\ x_{n+1} &= J^{-1}\left(\beta_{n,0} J x_n + \sum_{i=1}^{N} \beta_{n,i} J u_{n,i}\right), \quad u_{n,i} \in T_i y_n, \; n \ge 0, \end{aligned} \tag{3.26}$$

where $\alpha_n \in (0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_{n,i}\} \subset [a,b] \subset (0,1)$, *for* $i = 1,2,\ldots,N$ *satisfying* $\beta_{n,0} + \beta_{n,1} + \cdots + \beta_{n,N} = 1$, *for each* $n \ge 0$. *Then* $\{x_n\}$ *converges strongly to an element of* F.

If in Theorem 3.1, we assume that each T_i, $i = 1,2,\ldots,N$ is single valued, we get the following corollary.

Corollary 3.5. *Let* C *be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space* E. *Let* $f : C \times C \to \mathbb{R}$, *be a bifunction which satisfies conditions (A1)–(A4). Let* $T_i : C \to C$, *for* $i = 1,2,\ldots,N$, *be a finite family of relatively nonexpansive single-valued mappings. Assume that* $F := \cap_{i=1}^{N} F(T_i) \cap EP(f)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{aligned} x_0 &= w \in C, \quad \text{chosen arbitrarily}, \\ w_n &= F_{r_n} x_n, \\ y_n &= \Pi_C J^{-1}(\alpha_n J w + (1-\alpha_n) J w_n), \\ x_{n+1} &= J^{-1}\left(\beta_{n,0} J w_n + \sum_{i=1}^{N} \beta_{n,i} J T_i y_n\right), \quad n \ge 0, \end{aligned} \tag{3.27}$$

where $\alpha_n \in (0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_{n,i}\} \subset [a,b] \subset (0,1)$, *for* $i = 1,2,\ldots,N$, *satisfying* $\beta_{n,0} + \beta_{n,1} + \cdots + \beta_{n,N} = 1$, *for each* $n \ge 0$. *Then* $\{x_n\}$ *converges strongly to an element of* F.

If $E = H$, a real Hilbert space, then E is uniformly convex and uniformly smooth real Banach space. In this case, $J = I$, identity map on H and $\Pi_C = P_C$, projection mapping from H onto C. Thus, the following corollary holds.

Corollary 3.6. *Let* C *be a nonempty, closed, and convex subset of a Hilbert space* H. *Let* $f : C \times C \to \mathbb{R}$, *be a bifunction which satisfies conditions (A1)–(A4). Let* $T_i : C \to CB(C)$, *for* $i = 1,2,\ldots,N$, *be* *a finite family of relatively nonexpansive multi-valued mappings. Assume that* $F := \cap_{i=1}^{N} F(T_i) \cap EP(f)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{aligned} x_0 &= w \in C, \quad \textit{chosen arbitrarily}, \\ w_n &= F_{r_n} x_n, \\ y_n &= P_C(\alpha_n w + (1-\alpha_n) w_n), \\ x_{n+1} &= \beta_{n,0} w_n + \sum_{i=1}^{N} \beta_{n,i} u_{n,i}, \quad u_{n,i} \in T_i y_n, \ n \ge 0, \end{aligned} \tag{3.28}$$

where $\alpha_n \in (0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_{n,i}\} \subset [a,b] \subset (0,1)$, *for* $i = 1, 2, \ldots, N$, *satisfying* $\beta_{n,0} + \beta_{n,1} + \cdots + \beta_{n,N} = 1$, *for each* $n \ge 0$. *Then* $\{x_n\}$ *converges strongly to an element of* F.

Remark 3.7. (1) Theorem 3.1 improves and extends the corresponding results of Kumanm [22] and Takahashi and Zembayashi [16] in the sense that either our scheme does not require computation of C_{n+1}, for each $n \ge 1$, or the space considered is more general.

(2) Theorem 3.1 improves the corresponding results of Homaeipour and Razani [3] in the sense that our convergence is strong and the requirement that the interior of F is nonempty is dispensed with.

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