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*Research Article*

# Travelling Wave Solutions to the Benney-Luke and the Higher-Order Improved Boussinesq Equations of Sobolev Type

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By using the tanh-coth method, we obtained some travelling wave solutions of two well-known nonlinear Sobolev type partial differential equations, namely, the Benney-Luke equation and the higher-order improved Boussinesq equation. We show that the tanh-coth method is a useful, reliable, and concise method to solve these types of equations.

## 1. Introduction

The term "Sobolev equation" is used in the Russian literature to refer to any equation with spatial derivatives on the highest order time derivative [1]. In other words, they are characterized by having mixed time and space derivatives appearing in the highest-order terms of the equation and were studied by Sobolev [2]. Equations of Sobolev type describe many physical phenomena [3–7]. In recent years considerable attention has been paid to the study of equations of Sobolev type. For more details we refer the reader to [8] and references therein.

The Benney-Luke equation is as follows:

$$u_{tt} - u_{xx} + a u_{xxxx} - b u_{xxtt} + u_t u_{xx} + 2 u_x u_{xt} = 0, \tag{1.1}$$

where $a$ and $b$ are positive numbers, such that $a - b = \sigma - 1/3$ is a Sobolev type equation and studied for a very long time. The dimensionless parameter $\sigma$ is named the Bond number, which captures the effects of surface tension and gravity force and is a formally valid approximation for describing two-way water wave propagation in the presence of surface tension [9]. In [10] Pego and Quintero studied the propagation of long water waves with small amplitude. They showed that in the presence of a surface tension, the propagation of such waves is governed by (1.1), originally derived by Benney and Luke [11]. There are many studies concerning with this equation. Amongst them the stability analysis [9, 12], Cauchy problem [13–15], existence and analyticity of solutions [16], and travelling wave solutions [17] can be mentioned.

In [18], Schneider and Wayne showed that in the longwave limit the water wave problem without surface tension can be described approximately by two decoupled KdV equations. They considered a class of Boussinesq equation which models the water wave problem with surface tension as follows:

$$-u_{xxxxtt} + u_{xxtt} - u_{tt} + u_{xx} + \mu u_{xxxx} + \left(u^2\right)_{xx} = 0, \tag{1.2}$$

where $x, t, \mu \in \mathbb{R}$ and $u(x,t) \in \mathbb{R}$. Duruk et al. investigated the well posedness of the Cauchy problem

$$-\beta u_{xxxxtt} + u_{xxtt} - u_{tt} + u_{xx} + \left(g(u)\right)_{xx} = 0, \quad x \in \mathbb{R}, t > 0 \tag{1.3}$$

and showed that under certain conditions the Cauchy problem is globally well posed [19]. Nevertheless, several types of the improved Boussinesq equation were investigated by many researchers and found exact solutions by using exp-function method [20], modified extended tanh-function method [21], sine-cosine method [22], improved G′/G-expansion method [22], the standard tanh and the extended tanh method [23], and so forth.

The tanh-coth is a powerful and reliable technique for finding exact travelling wave solutions for nonlinear equations. This method has been used extensively, and it was subjected by some modifications using the Riccati equation. The main features of the tanh-coth method will be outlined in the subsequent section, and this method will be applied to the the Benney-Luke and the Higher-order improved Boussinesq equations. The main purpose of this work is to obtain travelling wave solutions of the above-mentioned equations and to show that the tanh-coth method can be easily applied to Sobolev type equations. Throughout the work, Maple is used to deal with the tedious algebraic operations.

## 2. Outline of the Tanh-Coth Method

Wazwaz has summarized the tanh method in the following manner.

(i) First consider a general form of nonlinear equation

$$P(u, u_t, u_x, u_{xx}, \ldots) = 0. \tag{2.1}$$

(ii) To find the traveling wave solution of (2.1), the wave variable $\xi = x - Vt$ is introduced, so that

$$u(x,t) = U(\mu\xi). \tag{2.2}$$

Based on this one may use the following changes:

$$\begin{aligned}\frac{\partial}{\partial t} &= -V\frac{d}{d\xi},\\ \frac{\partial}{\partial x} &= \mu\frac{d}{d\xi},\\ \frac{\partial^2}{\partial x^2} &= \mu^2\frac{d^2}{d\xi^2},\\ \frac{\partial^3}{\partial x^3} &= \mu^3\frac{d^3}{d\xi^3},\end{aligned} \tag{2.3}$$

and so on for other derivatives. Using (2.3) changes the PDE (2.1) to an ODE as follows:

$$Q(U,U',U'',\ldots) = 0. \tag{2.4}$$

(iii) If all terms of the resulting ODE contain derivatives in $\xi$, then by integrating this equation and by considering the constant of integration to be zero, one obtains a simplified ODE.

(iv) A new independent variable

$$Y = \tanh(\mu\xi) \tag{2.5}$$

is introduced that leads to the change of derivatives:

$$\begin{aligned}\frac{d}{d\xi} &= \mu\left(1-Y^2\right)\frac{d}{dY},\\ \frac{d^2}{d\xi^2} &= -2\mu^2 Y\left(1-Y^2\right)\frac{d}{dY} + \mu^2\left(1-Y^2\right)^2\frac{d^2}{dY^2},\\ \frac{d^3}{d\xi^3} &= 2\mu^3\left(1-Y^2\right)\left(3Y^2-1\right)\frac{d}{dY} - 6\mu^3 Y\left(1-Y^2\right)^2\frac{d^2}{dY^2} + \mu^3\left(1-Y^2\right)^3\frac{d^3}{dY^3},\\ \frac{d^4}{d\xi^4} &= -8\mu^4 Y\left(1-Y^2\right)\left(3Y^2-2\right)\frac{d}{dY} + 4\mu^4\left(1-Y^2\right)^2\left(9Y^2-2\right)\frac{d^2}{dY^2}\\ &\quad - 12\mu^4 Y\left(1-Y^2\right)^3\frac{d^3}{dY^3} + \mu^4\left(1-Y^2\right)^4\frac{d^4}{dY^4},\end{aligned} \tag{2.6}$$

where other derivatives can be derived in a similar manner.

(v) The ansatz of the form

$$U(\mu\xi) = S(Y) = \sum_{k=0}^{M} a_k Y^k + \sum_{k=1}^{M} b_k Y^{-k} \tag{2.7}$$

is introduced where $M$ is a positive integer, in most cases, that will be determined. If $M$ is not an integer, then a transformation formula is used to overcome this difficulty. Substituting (2.6) and (2.7) into the ODE, (2.4) yields an equation in powers of $Y$.

(vi) To determine the parameter $M$, the linear terms of highest order in the resulting equation with the highest order nonlinear terms are balanced. With $M$ determined, one collects the all coefficients of powers of $Y$ in the resulting equation where these coefficients have to vanish. This will give a system of algebraic equations involving the $a_k$ and $b_k$, $(k = 0, \ldots, M)$, $V$, and $\mu$. Having determined these parameters, knowing that $M$ is a positive integer in most cases, and using (2.7) one obtains an analytic solution in a closed form.

## 3. The Benney-Luke Equation

The Benney-Luke equation can be written as

$$u_{tt} - u_{xx} + au_{xxxx} - bu_{xxtt} + u_t u_{xx} + 2u_x u_{xt} = 0, \tag{3.1}$$

where $a$ and $b$ are positive numbers such that $a - b = \sigma - 1/3$ ($\sigma$ is named the Bond number). In order to solve (3.1) by the tanh-coth method, we use the wave transformation $u(x,t) = U(\mu\xi)$ with wave variable $\xi = x - Vt$; (3.1) takes on the form of an ordinary differential equation as follows:

$$\left(V^2 - 1\right)U'' + \left(a - bV^2\right)U'''' - 3VU'U'' = 0. \tag{3.2}$$

Balancing the order of $U''''$ with the order of $U'U''$ in (3.2) we find $M = 1$. Using the assumptions of the tanh-coth method (2.5)–(2.7) gives the solution in the form

$$U(\mu\xi) = S(Y) = \sum_{k=0}^{1} a_k Y^k + \sum_{k=1}^{1} b_k Y^{-k}. \tag{3.3}$$

Substituting (3.3) into (3.2), we obtain a system of algebraic equations for $a_0$, $a_1$, $b_1$, and $V$ in the following form:

$$\begin{aligned}
&Y^{10} : -24bV^2a_1\mu^4 + 24Va_1^2\mu^3 + 24aa_1\mu^4 = 0,\\
&Y^9 : 18Va_0a_1\mu^3 = 0,\\
&Y^8 : -40bV^2a_1\mu^4 - 2V^2a_1\mu^2 + 36Va_1^2\mu^3 - 12b_1Va_1\mu^3 + 40aa_1\mu^4 + 2a_1\mu^2 = 0,\\
&Y^7 : 24Va_0a_1\mu^3 = 0,\\
&Y^6 : -16bV^2a_1\mu^4 - 2V^2a_1\mu^2 + 12Va_1^2\mu^3 - 12b_1Va_1\mu^3 + 16aa_1\mu^4 + 2a_1\mu^2 = 0,\\
&Y^5 : 6Va_0a_1\mu^3 + 6Va_0b_1\mu^3 = 0,\\
&Y^4 : -16bV^2b_1\mu^4 - 2V^2b_1\mu^2 + 12Vb_1^2\mu^3 - 12a_1Vb_1\mu^3 + 16ab_1\mu^4 + 2b_1\mu^2 = 0,\\
&Y^3 : 24Va_0b_1\mu^3 = 0,\\
&Y^2 : -40bV^2b_1\mu^4 - 2V^2b_1\mu^2 + 36Vb_1^2\mu^3 - 12a_1Vb_1\mu^3 + 40ab_1\mu^4 + 2b_1\mu^2 = 0,\\
&Y^1 : 18Va_0b_1\mu^3 = 0,\\
&Y^0 : 24Vb_1^2\mu^3 + 24ab_1\mu^4 - 24V^2bb_1\mu^4 = 0.
\end{aligned} \tag{3.4}$$

From the output of the Maple packages we find three sets of solutions:

$$\begin{aligned}
&a_0 = b_1 = 0, \qquad a_1 = \frac{(-a+b)\mu}{V(2\mu^2b+1)}, \qquad V = \pm\sqrt{\frac{2\mu^2a+1}{2\mu^2b+1}},\\
&a_0 = a_1 = 0, \qquad b_1 = \frac{(-a+b)\mu}{V(2\mu^2b+1)}, \qquad V = \pm\sqrt{\frac{2\mu^2a+1}{2\mu^2b+1}},\\
&a_0 = 0, \qquad a_1 = b_1 = \frac{(-a+b)\mu}{V(8\mu^2b+1)}, \qquad V = \pm\sqrt{\frac{8\mu^2a+1}{8\mu^2b+1}},
\end{aligned} \tag{3.5}$$

where $\mu$ is left as a free parameter. The travelling wave solutions are as follows:

$$\begin{aligned}
u_1(x,t) &= \sqrt{\frac{2\mu^2b+1}{2\mu^2a+1}}\frac{(-a+b)\mu}{(2\mu^2b+1)}\tanh\mu\left(x \mp \sqrt{\frac{2\mu^2a+1}{2\mu^2b+1}}t\right),\\
u_2(x,t) &= \sqrt{\frac{2\mu^2b+1}{2\mu^2a+1}}\frac{(-a+b)\mu}{(2\mu^2b+1)}\coth\mu\left(x \mp \sqrt{\frac{2\mu^2a+1}{2\mu^2b+1}}t\right),\\
u_3(x,t) &= \sqrt{\frac{8\mu^2b+1}{8\mu^2a+1}}\frac{(-a+b)\mu}{(2\mu^2b+1)}\left(\tanh\mu\left(x \mp \sqrt{\frac{8\mu^2a+1}{8\mu^2b+1}}t\right) + \coth\mu\left(x \mp \sqrt{\frac{8\mu^2a+1}{8\mu^2b+1}}t\right)\right).
\end{aligned} \tag{3.6}$$

## 4. The Higher-Order Improved Boussinesq Equation

We consider the Higher-order improved Boussinesq equation as follows:

$$-\alpha u_{xxxxtt} + \beta u_{xxtt} - u_{tt} + u_{xx} + \left(u^2\right)_{xx} = 0, \tag{4.1}$$

where $\alpha$ and $\beta$ are arbitrary non zero real constants.

Using the wave transformation $u(x,t) = U(\mu\xi)$ with wave variable $\xi = x - Vt$ then by integrating this equation and considering the constant of integration to be zero, we obtain the ODE as follows:

$$-\alpha V^2 U'''' + \beta V^2 U'' + \left(1 - V^2\right)U + U^2 = 0. \tag{4.2}$$

Balancing the first term with the last term in (4.2) we find $M = 4$. Using the assumptions of the tanh-coth method (2.5)–(2.7) gives the solution in the form

$$U(\mu\xi) = S(Y) = \sum_{k=0}^{4} a_k Y^k + \sum_{k=1}^{4} b_k Y^{-k}. \tag{4.3}$$

Substituting (4.3) into (4.2), we obtain a system of algebraic equations for $a_0, a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, \mu$, and $V$ in the following form:

$$\begin{aligned}
Y^{16} &: a_4^2 - 840V^2 a_4 \alpha\mu^4 = 0,\\
Y^{15} &: 2a_4a_3 - 360V^2 a_3\alpha\mu^4 = 0,\\
Y^{14} &: 2a_4a_2 + a_3^2 + 20V^2 a_4\beta\mu^2 + 2080V^2 a_4\alpha\mu^4 - 120V^2 a_2\alpha\mu^4 = 0,\\
Y^{13} &: 2a_4a_1 + 2a_3a_2 + 12V^2 a_3\beta\mu^2 + 816V^2 a_3\alpha\mu^4 - 24V^2 a_1\alpha\mu^4 = 0,\\
Y^{12} &: 240\alpha V^2 a_2\mu^4 + 6\beta V^2 a_2\mu^2 - 1696a_4\alpha V^2\mu^4\\
&\quad - 32a_4\beta V^2\mu^2 - a_4V^2 + a_2^2 + a_4 + 2a_4a_0 + 2a_3a_1 = 0,\\
Y^{11} &: a_3 - V^2a_3 + 2a_4b_1 + 2a_3a_0 + 2a_1a_2 - 18V^2 a_3\beta\mu^2\\
&\quad - 576V^2 a_3\alpha\mu^4 + 2V^2 a_1\beta\mu^2 + 40V^2 a_1\alpha\mu^4 = 0,\\
Y^{10} &: a_2 - V^2a_2 + 2a_4b_2 + 2b_1a_3 + 2a_0a_2 + a_1^2\\
&\quad + 12V^2 a_4\beta\mu^2 + 480V^2 a_4\alpha\mu^4 - 8V^2 a_2\beta\mu^2 - 136V^2 a_2\alpha\mu^4 = 0,\\
Y^{9} &: a_1 - V^2a_1 + 2a_4b_3 + 2b_2a_3 + 2b_1a_2 + 2a_0a_1\\
&\quad + 6V^2 a_3\beta\mu^2 + 120V^2 a_3\alpha\mu^4 - 2V^2 a_1\beta\mu^2 - 16V^2 a_1\alpha\mu^4 = 0,
\end{aligned}$$

$$
\begin{aligned}
Y^8 :\ & a_0 - V^2 a_0 + 2a_4 b_4 + 2b_3 a_3 + 2b_1 a_1 + 2b_2 a_2 \\
& + a_0^2 - 24V^2 a_4 \alpha \mu^4 + 2V^2 b_2 \beta \mu^2 + 16V^2 b_2 \alpha \mu^4 - 24V^2 b_4 \alpha \mu^4 \\
& + 2V^2 a_2 \beta \mu^2 + 16V^2 a_2 \alpha \mu^4 + 16V^2 b_2 \alpha \mu^4 - 24V^2 b_4 \alpha \mu^4 + 2V^2 a_2 \beta \mu^2 + 16V^2 a_2 \alpha \mu^4 = 0, \\
Y^7 :\ & b_1 - V^2 b_1 + 2b_4 a_3 + 2b_1 a_0 + 2b_2 a_1 + 2b_3 a_2 \\
& - 2V^2 b_1 \beta \mu^2 - 16V^2 b_1 \alpha \mu^4 + 6V^2 b_3 \beta \mu^2 + 120V^2 b_3 \alpha \mu^4 = 0, \\
Y^6 :\ & b_2 - V^2 b_2 + 2b_2 a_0 + 2b_3 a_1 + 2b_4 a_2 + b_1^2 - 8V^2 b_2 \beta \mu^2 \\
& - 136V^2 b_2 \alpha \mu^4 + 12V^2 b_4 \beta \mu^2 + 480V^2 b_4 \alpha \mu^4 = 0, \\
Y^5 :\ & b_3 - V^2 b_3 + 2b_1 b_2 + 2b_3 a_0 + 2b_4 a_1 + 2V^2 b_1 \beta \mu^2 \\
& + 40V^2 b_1 \alpha \mu^4 - 18V^2 b_3 \beta \mu^2 - 576V^2 b_3 \alpha \mu^4 = 0, \\
Y^4 :\ & 240\alpha V^2 b_2 \mu^4 + 6\beta V^2 b_2 \mu^2 - 1696 b_4 \alpha V^2 \mu^4 \\
& - 32 b_4 \beta V^2 \mu^2 - b_4 V^2 + b_2^2 + b_4 + 2b_1 b_3 + 2b_4 a_0 = 0, \\
Y^3 :\ & 2b_1 b_4 + 2b_2 b_3 - 24V^2 b_1 \alpha \mu^4 + 12V^2 b_3 \beta \mu^2 + 816V^2 b_3 \alpha \mu^4 = 0, \\
Y^2 :\ & 2b_2 b_4 + b_3^2 - 120V^2 b_2 \alpha \mu^4 + 20V^2 b_4 \beta \mu^2 + 2080V^2 b_4 \alpha \mu^4 = 0, \\
Y^1 :\ & 2b_3 b_4 - 360V^2 b_3 \alpha \mu^4 = 0, \\
Y^0 :\ & b_4^2 - 840V^2 b_4 \alpha \mu^4 = 0.
\end{aligned}
\tag{4.4}
$$

Using Maple gives six sets of solutions:

$$
a_0 = a_4 = \frac{105\beta^2}{2(-36\beta^2 + 169\alpha)}, \qquad a_2 = \frac{-105\beta^2}{-36\beta^2 + 169\alpha}, \qquad a_1 = a_3 = b_1 = b_2 = b_3 = b_4 = 0,
$$

$$
V = \pm \frac{13}{\sqrt{-1872\mu^2\beta + 169}}, \qquad \mu = \pm \frac{1}{26}\sqrt{\frac{13\beta}{\alpha}},
$$

$$
a_0 = \frac{33\beta^2}{2(36\beta^2 + 169\alpha)}, \qquad a_2 = \frac{-105\beta^2}{36\beta^2 + 169\alpha}, \qquad a_4 = \frac{105\beta^2}{2(36\beta^2 + 169\alpha)},
$$

$$
a_1 = a_3 = b_1 = b_2 = b_3 = b_4 = 0,
$$

$$
V = \pm \frac{13}{\sqrt{1872\mu^2\beta + 169}}, \qquad \mu = \pm \frac{1}{26}\sqrt{\frac{13\beta}{\alpha}},
$$

$$
\begin{aligned}
&a_0 = \frac{105\beta^2}{2(-36\beta^2+169\alpha)}, \qquad b_2 = \frac{-105\beta^2}{-36\beta^2+169\alpha}, \qquad b_4 = \frac{105\beta^2}{2(-36\beta^2+169\alpha)},\\
&a_1 = a_2 = a_3 = a_4 = b_1 = b_3 = 0,\\
&V = \pm\frac{13}{\sqrt{-1872\mu^2\beta+169}}, \qquad \mu = \pm\frac{1}{26}\sqrt{\frac{13\beta}{\alpha}},\\
&a_0 = \frac{33\beta^2}{2(36\beta^2+169\alpha)}, \qquad b_2 = \frac{-105\beta^2}{36\beta^2+169\alpha}, \qquad b_4 = \frac{105\beta^2}{2(36\beta^2+169\alpha)},\\
&a_1 = a_2 = a_3 = a_4 = b_1 = b_3 = 0,\\
&V = \pm\frac{13}{\sqrt{1872\mu^2\beta+169}}, \qquad \mu = \pm\frac{1}{26}\sqrt{\frac{13\beta}{\alpha}},\\
&a_0 = \frac{315\beta^2}{16(-36\beta^2+169\alpha)}, \qquad a_2 = b_2 = \frac{-105\beta^2}{8(-36\beta^2+169\alpha)}, \qquad a_4 = b_4 = \frac{105\beta^2}{32(-36\beta^2+169\alpha)},\\
&a_1 = a_3 = b_1 = b_3 = 0,\\
&V = \pm\frac{13}{\sqrt{-7488\mu^2\beta+169}}, \qquad \mu = \pm\frac{1}{52}\sqrt{\frac{13\beta}{\alpha}},\\
&a_0 = \frac{-261\beta^2}{16(36\beta^2+169\alpha)}, \qquad a_2 = b_2 = \frac{-105\beta^2}{8(36\beta^2+169\alpha)}, \qquad a_4 = b_4 = \frac{105\beta^2}{32(36\beta^2+169\alpha)},\\
&a_1 = a_3 = b_1 = b_3 = 0,\\
&V = \pm\frac{13}{\sqrt{7488\mu^2\beta+169}}, \qquad \mu = \pm\frac{1}{52}\sqrt{\frac{13\beta}{\alpha}},
\end{aligned}
\tag{4.5}
$$

The travelling wave solutions are as follows:

$$
\begin{aligned}
u_1(x,t) &= \frac{105\beta^2}{2(-36\beta^2+169\alpha)} - \frac{105\beta^2}{-36\beta^2+169\alpha}\tanh^2\mu\left(x \mp \frac{13}{\sqrt{-1872\mu^2\beta+169}}t\right),\\
u_2(x,t) &= \frac{33\beta^2}{2(36\beta^2+169\alpha)} - \frac{105\beta^2}{36\beta^2+169\alpha}\tanh^2\mu\left(x \mp \frac{13}{\sqrt{1872\mu^2\beta+169}}t\right)\\
&\quad + \frac{105\beta^2}{2(36\beta^2+169\alpha)}\tanh^4\mu\left(x \mp \frac{13}{\sqrt{1872\mu^2\beta+169}}t\right),
\end{aligned}
$$

$$u_3(x,t) = \frac{105\beta^2}{2(-36\beta^2+169\alpha)} - \frac{105\beta^2}{-36\beta^2+169\alpha}\coth^2\mu\left(x \mp \frac{13}{\sqrt{-1872\mu^2\beta+169}}t\right)$$
$$+ \frac{105\beta^2}{2(-36\beta^2+169\alpha)}\coth^4\mu\left(x \mp \frac{13}{\sqrt{-1872\mu^2\beta+169}}t\right),$$
$$u_4(x,t) = \frac{33\beta^2}{2(36\beta^2+169\alpha)} - \frac{105\beta^2}{36\beta^2+169\alpha}\coth^2\mu\left(x \mp \frac{13}{\sqrt{1872\mu^2\beta+169}}t\right)$$
$$+ \frac{105\beta^2}{2(36\beta^2+169\alpha)}\coth^4\mu\left(x \mp \frac{13}{\sqrt{1872\mu^2\beta+169}}t\right),$$
$$u_5(x,t) = \frac{315\beta^2}{16(-36\beta^2+169\alpha)} - \frac{105\beta^2}{8(-36\beta^2+169\alpha)}\tanh^2\mu\left(x \mp \frac{13}{\sqrt{-7488\mu^2\beta+169}}t\right)$$
$$- \frac{105\beta^2}{8(-36\beta^2+169\alpha)}\coth^2\mu\left(x \mp \frac{13}{\sqrt{-7488\mu^2\beta+169}}t\right)$$
$$+ \frac{105\beta^2}{32(-36\beta^2+169\alpha)}\tanh^4\mu\left(x \mp \frac{13}{\sqrt{-7488\mu^2\beta+169}}t\right)$$
$$+ \frac{105\beta^2}{32(-36\beta^2+169\alpha)}\coth^4\mu\left(x \mp \frac{13}{\sqrt{-7488\mu^2\beta+169}}t\right),$$
$$u_6(x,t) = -\frac{261\beta^2}{16(36\beta^2+169\alpha)} - \frac{105\beta^2}{8(36\beta^2+169\alpha)}\tanh^2\mu\left(x \mp \frac{13}{\sqrt{7488\mu^2\beta+169}}t\right)$$
$$- \frac{105\beta^2}{8(36\beta^2+169\alpha)}\coth^2\mu\left(x \mp \frac{13}{\sqrt{7488\mu^2\beta+169}}t\right)$$
$$+ \frac{105\beta^2}{32(36\beta^2+169\alpha)}\tanh^4\mu\left(x \mp \frac{13}{\sqrt{7488\mu^2\beta+169}}t\right)$$
$$+ \frac{105\beta^2}{32(36\beta^2+169\alpha)}\coth^4\mu\left(x \mp \frac{13}{\sqrt{7488\mu^2\beta+169}}t\right). \tag{4.6}$$

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