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Research Article

The Equivalence of Convergence Results of Modified Mann and Ishikawa Iterations with Errors without Bounded Range Assumption

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Let E be an arbitrary uniformly smooth real Banach space, let D be a nonempty closed convex subset of E, and let $T : D \to D$ be a uniformly generalized Lipschitz generalized asymptotically Φ-strongly pseudocontractive mapping with $q \in F(T) \neq \emptyset$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ be four real sequences in $[0,1]$ and satisfy the conditions: (i) $a_n + c_n \leq 1$, $b_n + d_n \leq 1$; (ii) $a_n, b_n, d_n \to 0$ as $n \to \infty$ and $c_n = o(a_n)$; (iii) $\Sigma_{n=0}^{\infty} a_n = \infty$. For some $x_0, z_0 \in D$, let $\{u_n\}, \{v_n\}, \{w_n\}$ be any bounded sequences in D, and let $\{x_n\}, \{z_n\}$ be the modified Ishikawa and Mann iterative sequences with errors, respectively. Then the convergence of $\{x_n\}$ is equivalent to that of $\{z_n\}$.

1. Introduction and Preliminary

Let E be a real Banach space and let E^* be its dual space. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\}, \quad \forall x \in E, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that

(i) if E is a smooth Banach space, then the mapping J is single-valued;

(ii) $J(\alpha x) = \alpha J(x)$ for all $x \in E$ and $\alpha \in \Re$;

(iii) if E is a uniformly smooth Banach space, then the mapping J is uniformly continuous on any bounded subset of E. Throughout this paper, we denote that

j is the single-valued normalized duality mapping, D is a nonempty closed convex subset of E, $T : D \to D$ is a mapping, and T^0 is the unit mapping I.

In 1972, Goebel and Kirk [1] introduced the class of asymptotically nonexpansive mappings as follows.

Definition 1.1. A mapping T is said to be asymptotically nonexpansive if for each $x, y \in D$

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall n \geq 0, \tag{1.2}$$

where $\{k_n\} \subset [1, +\infty)$ with $\lim_{n\to\infty} k_n = 1$.

Schu [2], in 1991, gave the definition of asymptotically pseudocontractive mappings and proved the correlation results.

Definition 1.2. The mapping T is called asymptotically pseudocontractive with the sequence $\{k_n\} \subset [1, +\infty)$ if and only if $\lim_{n\to\infty} k_n = 1$, and for all $n \in N$ and all $x, y \in D$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle T^n x - T^n y, j(x-y)\rangle \leq k_n \|x - y\|^2. \tag{1.3}$$

It is easy to find that every asymptotically nonexpansive mapping is asymptotically pseudocontractive. However, the converse is not true in general. See example of [3].

Recently, Colao [4] combined the proof ideas of the papers of Chang [5] and C. E. Chidume and C. O. Chidume [6] and then showed the equivalent theorem results of the convergence between Mann and Ishikawa iterations with errors for generalized strongly asymptotically ϕ-pseudocontractive mapping with bounded range. In fact, he proved the following theorem.

Theorem 1.3. *Let X be a uniformly smooth Banach space, and let $T : X \to X$ be generalized strongly asymptotically ϕ-pseudocontractive mapping with fixed point x^* and bounded range. Let $\{x_n\}$ and $\{z_n\}$ be the sequences defined by* (1.4) *and* (1.5)*, respectively,*

$$\begin{aligned} y_n &= (1 - \beta_n - \delta_n)x_n + \beta_n T^n x_n + \delta_n v_n, \quad n \geq 0, \\ x_{n+1} &= (1 - \alpha_n - \gamma_n)x_n + \alpha_n T^n y_n + \gamma_n u_n, \quad n \geq 0, \end{aligned} \tag{1.4}$$

$$z_{n+1} = (1 - \alpha_n - \gamma_n)z_n + \alpha_n T^n z_n + \gamma_n w_n, \quad n \geq 0, \tag{1.5}$$

where $\{\alpha_n\}, \{\gamma_n\}, \{\beta_n\}, \{\delta_n\} \subset [0, 1]$ satisfy

(H1) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \delta_n = 0$ *and* $\gamma_n = o(\alpha_n)$*,*

(H2) $\Sigma_{n=1}^{\infty} \alpha_n = \infty$*,*

and the sequences $\{u_n\}, \{v_n\}, \{w_n\}$ are bounded in X, then for any initial point $z_0, x_0 \in X$, the following two assertions are equivalent.

(1) *The modified Ishikawa iteration sequence with errors* (1.4) *converges to x^*;*

(2) *The modified Mann iteration sequence with errors* (1.5) *converges to x^*.*

The aim of this paper is to prove the equivalence of convergent results of above Ishikawa and Mann iterations with errors for generalized asymptotically Φ-strongly pseudocontractive mappings without bounded range assumptions in uniformly smooth real Banach spaces. For this, we need the following concepts and lemmas.

Definition 1.4 (see [4]). The mapping T is called generalized asymptotically Φ-strongly pseudocontractive if

$$\langle T^n x - T^n y, j(x-y)\rangle \le k_n\|x-y\|^2 - \Phi(\|x-y\|), \quad n \ge 0, \tag{1.6}$$

where $j(x-y) \in J(x-y)$, $\{k_n\} \subset [1,+\infty)$ is converging to one and $\Phi : [0,+\infty) \to [0,+\infty)$ is strictly increasing continuous function with $\Phi(0) = 0$.

Definition 1.5 (see [4]). For arbitrary given $x_0 \in D$, modified Ishikawa iterative process with errors $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} y_n &= (1-b_n-d_n)x_n + b_nT^nx_n + d_nw_n, \quad n \ge 0,\\ x_{n+1} &= (1-a_n-c_n)x_n + a_nT^ny_n + c_nv_n, \quad n \ge 0, \end{aligned} \tag{1.7}$$

where $\{v_n\},\{w_n\}$ are any bounded sequences in D; $\{a_n\},\{b_n\},\{c_n\},\{d_n\}$ are four real sequences in $[0,1]$ and satisfy $a_n + c_n \le 1, b_n + d_n \le 1$, for all $n \ge 0$. If $b_n = d_n = 0$, we define modified Mann iterative process with errors $\{z_n\}$ by

$$z_{n+1} = (1-a_n-c_n)z_n + a_nT^nz_n + c_nu_n, \quad n \ge 0, \tag{1.8}$$

where $\{u_n\}$ is any bounded sequence in D.

Lemma 1.6 (see [7]). *Let E be a uniformly smooth real Banach space and let $J : E \to 2^{E^*}$ be a normalized duality mapping. Then*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, J(x+y)\rangle, \tag{1.9}$$

for all $x, y \in E$.

Lemma 1.7 (see [8]). *Let $\{\rho_n\}_{n=0}^{\infty}$ be a nonnegative sequence which satisfies the following inequality:*

$$\rho_{n+1} \le (1-\lambda_n)\rho_n + \sigma_n, \quad n \ge 0, \tag{1.10}$$

where $\lambda_n \in [0,1]$ with $\sum_{n=0}^{\infty}\lambda_n = \infty$, $\sigma_n = o(\lambda_n)$. Then $\rho_n \to 0$ as $n \to \infty$.

2. Main Results

First of all, we give a new concept.

Definition 2.1. A mapping $T : D \to D$ is called uniformly generalized Lipschitz if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \le L(1 + \|x - y\|), \quad \forall x, y \in D, \forall n \ge 0. \tag{2.1}$$

It is mentioned to notice that if T has bounded range, then it is uniformly generalized Lipschitz. In fact, since $R(T^n) \subseteq R(T)$, then $\sup_{x\in D}\{\|T^n x\|\} \le \sup_{x\in D}\{\|Tx\|\} = M_1$, thus $\|T^n x - T^n y\| \le 2M_1 \le L(1 + \|x - y\|)$, where $L = 2M_1$. On the contrary, it is not true in general (See [6]).

In the following, we prove the main theorems of this paper.

Theorem 2.2. *Let E be an arbitrary uniformly smooth real Banach space, let D be a nonempty closed convex subset of E, and let $T : D \to D$ be a uniformly generalized Lipschitz generalized asymptotically Φ-strongly pseudocontractive mapping with $q \in F(T) \ne \emptyset$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ be four real sequences in $[0, 1]$ and satisfy the following conditions:*

(i) $a_n + c_n \le 1$, $b_n + d_n \le 1$;

(ii) $a_n, b_n, d_n \to 0$ *as* $n \to \infty$ *and* $c_n = o(a_n)$;

(iii) $\Sigma_{n=0}^{\infty} a_n = \infty$.

For some $x_0, z_0 \in D$, let $\{u_n\}, \{v_n\}, \{w_n\}$ be any bounded sequences in D, and let $\{x_n\}$ and $\{z_n\}$ be Ishikawa and Mann iterative sequences with errors defined by (1.7) *and* (1.8), *respectively. Then the following conclusions are equivalent:*

(1) *$\{x_n\}$ converges strongly to the unique fixed point q of T;*

(2) *$\{z_n\}$ converges strongly to the unique fixed point q of T.*

Proof. (1)⇒(2) is obvious, that is, let $b_n = d_n = 0$, (1.7) turns into (1.8). We only need to show that (2)⇒(1). Since $T : D \to D$ is a uniformly generalized Lipschitz generalized asymptotically Φ-strongly pseudocontractive mapping, then there exists a strictly increasing continuous function $\Phi : [0, +\infty) \to [0, +\infty)$ with $\Phi(0) = 0$ such that

$$\langle T^n x - T^n y, J(x - y)\rangle \le k_n\|x - y\|^2 - \Phi(\|x - y\|), \tag{2.2}$$

that is,

$$\langle (k_n I - T^n)x - (k_n I - T^n)y, J(x - y)\rangle \ge \Phi(\|x - y\|), \tag{2.3}$$

$$\|T^n x - T^n y\| \le L(1 + \|x - y\|), \tag{2.4}$$

for any $x, y \in D$. For convenience, denote $k = \sup_n\{k_n\}$.

Step 1. There exists $x_0 \in D$ and $x_0 \ne Tx_0$ such that $r_0 = (k+L)\|x_0 - q\|^2 + L\|x_0 - q\| \in R(\Phi)$ (range of Φ).

Indeed, if $\Phi(r) \to +\infty$ as $r \to +\infty$, then $r_0 \in R(\Phi)$; if $\sup\{\Phi(r) : r \in [0, +\infty)\} = r_1 < +\infty$ with $r_1 < r_0$, then, for $q \in D$, there exists a sequence $\{v_n\}$ in D such that $v_n \to q$ as $n \to \infty$ with $v_n \ne q$. Furthermore, there exists a natural number n_0 such that $(k+L)\|v_n - q\|^2 + L\|v_n - q\| <$

$r_1/2$ for $n \geq n_0$, then we redefine x_0, r_0 such that $x_0 = v_{n_0}, r_0 = (k+L)\|x_0-q\|^2+L\|x_0-q\| \in R(\Phi)$. Hence, it is to ensure that $\Phi^{-1}(r_0)$ is well defined.

Step 2. For any $n \geq 0$, $\{x_n\}$ is a bounded sequence.

Set $R = \Phi^{-1}(r_0)$. From (2.3), we have

$$\langle k_n(x_0 - q) - (T^n x_0 - q), J(x_0 - q)\rangle \geq \Phi(\|x_0 - q\|), \tag{2.5}$$

that is, $(k + L)\|x_0 - q\|^2 + L\|x_0 - q\| \geq \Phi(\|x_0 - q\|)$. Thus, we obtain that $\|x_0 - q\| \leq R$. Denote

$$\begin{aligned} B_1 &= \{x \in D : \|x - q\| \leq R\}, \\ B_2 &= \{x \in D : \|x - q\| \leq 2R\}, \\ M &= \sup_n\{\|v_n - q\|\} + \sup_n\{\|w_n - q\|\}. \end{aligned} \tag{2.6}$$

Next, we want to prove that $x_n \in B_1$ for any $n \geq 0$ by induction. If $n = 0$, then $x_0 \in B_1$. Now we assume that it holds for some n, that is, $x_n \in B_1$. We prove that $x_{n+1} \in B_1$. Suppose that it is not the case, then $\|x_{n+1} - q\| > R$. Since J is uniformly continuous on bounded subset of E, then, for $\epsilon_0 = \Phi(R/4)/24L(1 + 2R)$, there exists $\delta > 0$ such that $\|Jx - Jy\| < \epsilon_0$ when $\|x - y\| < \delta$, for all $x, y \in B_2$. Now denote

$$\begin{aligned} \tau_0 = \min\Bigg\{ &\frac{R}{2[L(1 + 2R) + 2R + M]}, \frac{R}{4[L(1 + R) + 2R + M]}, \\ &\frac{\delta}{2[L(1 + 2R) + 2R + M]}, \frac{\Phi(R/4)}{24R^2}, \frac{\Phi(R/4)}{24L(1 + 2R)}, \frac{\Phi(R/4)}{48MR} \Bigg\}. \end{aligned} \tag{2.7}$$

Since $a_n, b_n, c_n, d_n \to 0$ as $n \to \infty$, and $c_n = o(a_n)$, without loss of generality, we assume that $0 \leq a_n, b_n, c_n, d_n \leq \tau_0, c_n < a_n\tau_0$ for any $n \geq 0$. Then we obtain the following estimates:

$$\begin{aligned} \|T^n x_n - q\| &\leq L(1 + \|x_n - q\|) \\ &\leq L(1 + R), \\ \|y_n - q\| &\leq (1 - b_n - d_n)\|x_n - q\| + b_n\|T^n x_n - q\| + d_n\|w_n - q\| \\ &\leq R + b_n L(1 + \|x_n - q\|) + d_n M \\ &\leq R + b_n L(1 + R) + d_n M \\ &\leq R + \tau_0[L(1 + R) + M] \\ &\leq 2R, \\ \|T^n y_n - q\| &\leq L(1 + \|y_n - q\|) \\ &\leq L(1 + 2R), \end{aligned}$$

$$\begin{aligned}
\|x_n - T^n x_n\| &\le \|x_n - q\| + \|T^n x_n - q\| \\
&\le L + (1+L)\|x_n - q\| \\
&\le L + (1+L)R, \\
\|(x_n - q) - (y_n - q)\| &\le b_n\|x_n - T^n x_n\| + d_n\left[\|w_n - q\| + \|x_n - q\|\right] \\
&\le b_n[L + (1+L)R] + d_n(M + R) \\
&\le \tau_0[L(1+R) + 2R + M] \\
&\le \tau_0[L(1+2R) + 2R + M] \\
&\le \frac{\delta}{2} < \delta, \\
\|x_n - q\| &\ge \|x_{n+1} - q\| - a_n\|T^n y_n - x_n\| - c_n\|v_n - x_n\| \\
&\ge \|x_{n+1} - q\| - a_n\left[\|T^n y_n - q\| + \|x_n - q\|\right] - c_n\left[\|x_n - q\| + \|v_n - q\|\right] \\
&> R - a_n[L(1+2R) + R] - c_n(R + M) \\
&\ge R - \tau_0[L(1+2R) + M + 2R] \\
&\ge R - \frac{R}{2} = \frac{R}{2}, \\
\|y_n - q\| &\ge \|x_n - q\| - b_n\|T^n x_n - x_n\| - d_n\|x_n - w_n\| \\
&\ge \|x_n - q\| - b_n[L + (1+L)R] - d_n\left[\|x_n - q\| + \|w_n - q\|\right] \\
&\ge \|x_n - q\| - b_n[L + (1+L)R] - d_n(R + M) \\
&\ge \|x_n - q\| - \tau_0[L(1+R) + 2R + M] \\
&> \frac{R}{2} - \frac{R}{4} = \frac{R}{4}, \\
\|x_{n+1} - q\| &\le (1 - a_n - c_n)\|x_n - q\| + a_n\|T^n y_n - q\| + c_n\|v_n - q\| \\
&\le R + \tau_0[L(1+2R) + M] \\
&\le 2R, \\
\|(x_{n+1} - q) - (x_n - q)\| &\le a_n\|T^n y_n - x_n\| + c_n\|u_n - x_n\| \\
&\le a_n\left[\|T^n y_n - q\| + \|x_n - q\|\right] + c_n\left[\|v_n - q\| + \|x_n - q\|\right] \\
&\le a_n[L(1+2R) + R] + c_n(M + R) \\
&\le \tau_0[L(1+2R) + 2R + M] \\
&\le \frac{\delta}{2} < \delta.
\end{aligned} \tag{2.8}$$

Hence, $\|J(x_n - q) - J(y_n - q)\| < \epsilon_0$; $\|J(x_{n+1} - q) - J(x_n - q)\| < \epsilon_0$.

Using Lemma 1.6 and formulas above, we obtain

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\leq (1-a_n-c_n)^2\|x_n-q\|^2+2a_n\langle T^n y_n-q, J(x_{n+1}-q)\rangle \\
&\quad +2c_n\langle u_n-q, J(x_{n+1}-q)\rangle \\
&\leq (1-a_n)^2\|x_n-q\|^2+2a_n\langle T^n y_n-q, J(x_{n+1}-q)-J(x_n-q)\rangle \\
&\quad +2a_n\langle T^n y_n-q, J(x_n-q)-J(y_n-q)\rangle \\
&\quad +2a_n\langle T^n y_n-q, J(y_n-q)\rangle+2c_n\langle u_n-q, J(x_{n+1}-q)\rangle \\
&\leq (1-a_n)^2\|x_n-q\|^2+2a_n\|T^n y_n-q\|\cdot\|J(x_{n+1}-q)-J(x_n-q)\| \\
&\quad +2a_n\|T^n y_n-q\|\cdot\|J(x_n-q)-J(y_n-q)\| \\
&\quad +2a_n\left[\|y_n-q\|^2-\Phi(\|y_n-q\|)\right]+2c_n\|u_n-q\|\cdot\|x_{n+1}-q\| \\
&\leq (1-a_n)^2R^2+4a_nL(1+2R)\epsilon_0+2a_n\left[\|y_n-q\|^2-\Phi(\|y_n-q\|)\right] \\
&\quad +4c_nMR,
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
\|y_n-q\|^2 &\leq (1-b_n-d_n)^2\|x_n-q\|^2+2b_n\langle T^n x_n-q, J(y_n-q)\rangle \\
&\quad +2d_n\langle w_n-q, J(y_n-q)\rangle \\
&\leq \|x_n-q\|^2+2b_n\langle T^n x_n-q, J(y_n-q)-J(x_n-q)\rangle \\
&\quad +2b_n\langle T^n x_n-q, J(x_n-q)\rangle+2d_n\|w_n-q\|\cdot\|y_n-q\| \\
&\leq \|x_n-q\|^2+2b_n\|T_n^x-q\|\cdot\|J(y_n-q)-J(x_n-q)\| \\
&\quad +2b_n\left[\|x_n-q\|^2-\Phi(\|x_n-q\|)\right]+2d_n\|w_n-q\|\cdot\|y_n-q\| \\
&\leq R^2+2b_nL(1+R)\epsilon_0+2b_nR^2+4d_nMR.
\end{aligned} \tag{2.10}$$

Substitute (2.10) into (2.9)

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\leq (1-a_n)^2R^2+4a_nL(1+2R)\epsilon_0+2a_n\left[R^2+2b_nL(1+R)\epsilon_0+2b_nR^2+4d_nMR\right] \\
&\quad -2a_n\Phi(\|y_n-q\|)+4c_nMR \\
&\leq R^2+a_n^2R^2+4a_nL(1+2R)\epsilon_0+2a_n\left[2b_nL(1+R)\epsilon_0+2b_nR^2+4d_nMR\right] \\
&\quad -2a_n\Phi\left(\frac{R}{4}\right)+4c_nMR
\end{aligned}$$

$$
\begin{aligned}
&= R^2 + 2a_n\left[\frac{a_n}{2}R^2 + 2L(1+2R)\epsilon_0 + 2b_nL(1+R)\epsilon_0 + 2b_nR^2 + 4d_nMR + \frac{2c_nMR}{a_n}\right] \\
&\quad - 2a_n\Phi\left(\frac{R}{4}\right) \\
&\le R^2 + 2a_n\left[\frac{\Phi(R/4)}{2} - \Phi\left(\frac{R}{4}\right)\right] \\
&\le R^2 - \Phi\left(\frac{R}{4}\right)a_n \\
&\le R^2,
\end{aligned}
\tag{2.11}
$$

this is a contradiction. Thus $x_{n+1} \in B_1$, that is, $\{x_n\}$ is a bounded sequence. So $\{y_n\}$, $\{T^n y_n\}, \{T^n x_n\}$ are all bounded sequences. Since $\|z_n - q\| \to 0$ as $n \to \infty$, without loss of generality, we let $\|z_n - q\| \le 1$. Therefore, $\|x_n - z_n\|$ is also bounded.

Step 3. We want to prove $\|x_n - z_n\| \to 0$ as $n \to \infty$.

Set $M_0 = \max\{\sup_n\|T^n y_n - T^n z_n\|, \sup_n\|v_n - u_n\|, \sup_n\|x_n - z_n\|, \sup_n\|T^n x_n - x_n\|, \sup_n\|w_n - x_n\|, \sup_n\|y_n - z_n\|, \sup_n\|v_n - x_n\|\}$.

Again using Lemma 1.6, we have

$$
\begin{aligned}
\|x_{n+1} - z_{n+1}\|^2 &\le (1-a_n-c_n)^2\|x_n - z_n\|^2 + 2a_n\langle T^n y_n - T^n z_n, J(x_{n+1} - z_{n+1})\rangle \\
&\quad + 2c_n\langle v_n - u_n, J(x_{n+1} - z_{n+1})\rangle \\
&\le (1-a_n)^2\|x_n - z_n\|^2 + 2a_n\langle T^n y_n - T^n z_n, J(x_{n+1} - z_{n+1}) - J(x_n - z_n)\rangle \\
&\quad + 2a_n\langle T^n y_n - T^n z_n, J(x_n - z_n) - J(y_n - z_n)\rangle \\
&\quad + 2a_n\langle T^n y_n - T^n z_n, J(y_n - z_n)\rangle + 2c_n\|v_n - u_n\| \cdot \|x_{n+1} - z_{n+1}\| \\
&\le (1-a_n)^2\|x_n - z_n\|^2 + 2a_nM_0A_n + 2a_nM_0B_n \\
&\quad + 2a_n\left[\|y_n - z_n\|^2 - \Phi(\|y_n - z_n\|)\right] + 2c_nM_0^2,
\end{aligned}
\tag{2.12}
$$

$$
\begin{aligned}
\|y_n - z_n\|^2 &\le \|x_n - z_n\|^2 + 2b_n\langle T^n x_n - x_n, J(y_n - z_n)\rangle \\
&\quad + 2d_n\langle w_n - x_n, J(y_n - z_n)\rangle \\
&\le \|x_n - z_n\|^2 + 2b_nM_0^2 + 2d_nM_0^2,
\end{aligned}
\tag{2.13}
$$

where $A_n = \|J(x_{n+1} - z_{n+1}) - J(x_n - z_n)\|$, $B_n = \|J(x_n - z_n) - J(y_n - z_n)\|$, and $A_n, B_n \to 0$ as $n \to \infty$.

Taking place (2.13) into (2.12), we have

$$\begin{aligned}\|x_{n+1}-z_{n+1}\|^2 &\le (1-a_n)^2\|x_n-z_n\|^2+2a_nM_0A_n+2a_nM_0B_n\\&\quad+2a_n\left[\|x_n-z_n\|^2+2b_nM_0^2+2d_nM_0^2-\Phi(\|y_n-z_n\|)\right]+2c_nM_0^2\\&\le \|x_n-z_n\|^2+a_n^2M_0^2+2a_nM_0A_n+2a_nM_0B_n+4a_nb_nM_0^2+4a_nd_nM_0^2\\&\quad-2a_n\Phi(\|y_n-z_n\|)+2c_nM_0^2\\&=\|x_n-z_n\|^2+2a_n\left[C_n-2a_n\Phi(\|y_n-z_n\|)\right],\end{aligned}\tag{2.14}$$

where $C_n = a_nM_0^2/2 + M_0A_n + M_0B_n + 2b_nM_0^2 + 2d_nM_0^2 + c_nM_0^2/a_n \longrightarrow 0$ as $n \longrightarrow \infty$.

Set $\inf_{n\ge 0}\Phi(\|y_n - z_n\|)/(1 + \|x_{n+1} - z_{n+1}\|^2) = \lambda$, then $\lambda = 0$. If it is not the case, we assume that $\lambda > 0$. Let $0 < \gamma < \min\{1, \lambda\}$, then $\Phi(\|y_n - z_n\|)/(1 + \|x_{n+1} - z_{n+1}\|^2) \ge \gamma$, that is, $\Phi(\|y_n - z_n\|) \ge \gamma + \gamma\|x_{n+1} - z_{n+1}\|^2 \ge \gamma\|x_{n+1} - z_{n+1}\|^2$. Thus, from (2.14) that

$$\|x_{n+1}-z_{n+1}\|^2 \le \|x_n-z_n\|^2+2a_n\left(C_n-\gamma\|x_{n+1}-z_{n+1}\|^2\right),\tag{2.15}$$

which implies that

$$\begin{aligned}\|x_{n+1}-z_{n+1}\|^2 &\le \frac{1}{1+2a_n\gamma}\|x_n-z_n\|^2+\frac{2a_nC_n}{1+2a_n\gamma}\\&=\left(1-\frac{2a_n\gamma}{1+2a_n\gamma}\right)\|x_n-z_n\|^2+\frac{2a_nC_n}{1+2a_n\gamma}.\end{aligned}\tag{2.16}$$

Let $\rho_n = \|x_n - z_n\|^2, \lambda_n = 2a_n\gamma/(1 + 2a_n\gamma), \sigma_n = 2a_nC_n/(1 + 2a_n\gamma)$. Then we get that

$$\rho_{n+1} \le (1-\lambda_n)\rho_n+\sigma_n.\tag{2.17}$$

Applying Lemma 1.7, we get that $\rho_n \longrightarrow 0$ as $n \longrightarrow \infty$. This is a contradiction and so $\lambda = 0$. Therefore, there exists an infinite subsequence such that $\Phi(\|y_{n_i} - z_{n_i}\|)/(1+\|x_{n_i+1} - z_{n_i+1}\|^2) \longrightarrow 0$ as $i \longrightarrow \infty$. Since $0 \le \Phi(\|y_{n_i} - z_{n_i}\|)/(1 + M_0^2) \le \Phi(\|y_{n_i} - z_{n_i}\|)/(1 + \|x_{n_i+1} - z_{n_i+1}\|^2)$, then $\Phi(\|y_{n_i} - z_{n_i}\|) \longrightarrow 0$ as $i \longrightarrow \infty$. In view of the strictly increasing and continuity of Φ, we have $\|y_{n_i} - z_{n_i}\| \longrightarrow 0$ as $i \longrightarrow \infty$. From (1.7), we have

$$\|x_{n_i}-z_{n_i}\| \le \|y_{n_i}-z_{n_i}\|+b_{n_i}\|x_{n_i}-Tx_{n_i}\|+c_{n_i}\|x_{n_i}-w_{n_i}\| \longrightarrow 0,\tag{2.18}$$

as $i \longrightarrow \infty$. Next we want to prove $\|x_n - z_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. Let for all $\epsilon \in (0, 1)$, there exists n_{i_0} such that $\|x_{n_i} - z_{n_i}\| < \epsilon, a_n, a_{n_i} < \min\{\epsilon/4L(1+M_0), \epsilon/8M_0\}, c_n, c_{n_i} < \epsilon/16M_0, b_n, d_n, b_{n_i}, d_{n_i} < \epsilon/8M_0, C_n, C_{n_i} < \Phi(\epsilon/4)/2$, for any $n_i, n \ge n_{i_0}$. First, we want to prove $\|x_{n_i+1} - z_{n_i+1}\| < \epsilon$.

Suppose it is not this case, then $\|x_{n_i+1} - z_{n_i+1}\| \geq \epsilon$. Using (1.7), we may get the following estimates:

$$\begin{aligned} \|x_{n_i} - z_{n_i}\| &\geq \|x_{n_i+1} - z_{n_i+1}\| - a_{n_i}\left\|T^n y_{n_i} - T^n z_{n_i}\right\| - a_{n_i}\|x_{n_i} - z_{n_i}\| \\ &\quad - c_{n_i}\|v_{n_i} - u_{n_i}\| - c_{n_i}\|x_{n_i} - z_{n_i}\| \\ &\geq \epsilon - a_{n_i}L(1 + M_0) - (a_{n_i} + 2c_{n_i})M_0 \\ &> \frac{\epsilon}{2}, \end{aligned} \tag{2.19}$$

$$\begin{aligned} \left\|y_{n_i} - z_{n_i}\right\| &\geq \|x_{n_i} - z_{n_i}\| - b_{n_i}\|T^n x_{n_i} - x_{n_i}\| - d_{n_i}\|v_{n_i} - x_{n_i}\| \\ &\geq \frac{\epsilon}{2} - (b_{n_i} + d_{n_i})M_0 \\ &> \frac{\epsilon}{4}. \end{aligned} \tag{2.20}$$

Since Φ is strictly increasing, then (2.20) leads to $\Phi(\|y_{n_i} - z_{n_i}\|) \geq \Phi(\epsilon/4)$. From (2.14), we have

$$\begin{aligned} \|x_{n_i+1} - z_{n_i+1}\|^2 &\leq \|x_{n_i} - z_{n_i}\|^2 + 2a_{n_i}\left[C_{n_i} - \Phi\left(\left\|y_{n_i} - z_{n_i}\right\|\right)\right] \\ &< \epsilon^2 + 2a_{n_i}\left[\frac{1}{2}\Phi\left(\frac{\epsilon}{4}\right) - \Phi\left(\frac{\epsilon}{4}\right)\right] \\ &\leq \epsilon^2 - \Phi\left(\frac{\epsilon}{4}\right)a_{n_i} \\ &\leq \epsilon^2, \end{aligned} \tag{2.21}$$

is a contradiction. Hence, $\|x_{n_i+1} - z_{n_i+1}\| < \epsilon$. Suppose that $\|x_{n_i+m} - z_{n_i+m}\| < \epsilon$ holds. Repeating the above course, we can easily prove that $\|x_{n_i+m+1} - z_{n_i+m+1}\| < \epsilon$ holds. Therefore, for any m and $n_i \geq n_0$, we obtain that $\|x_{n_i+m} - z_{n_i+m}\| < \epsilon$, which means $\|x_n - z_n\| \to 0$ as $n \to \infty$. This completes the proof. □

In order to make the existence of Theorem 2.2 more meaningful, we give the following theorem.

Theorem 2.3. *Let E be an arbitrary uniformly smooth real Banach space, let D be a nonempty closed convex subset of E, and let $T : D \to D$ be a uniformly generalized Lipschitz generalized asymptotically Φ-strongly pseudocontractive mapping with $q \in F(T) \neq \emptyset$. Let $\{a_n\}, \{c_n\}$ be two real sequences in $[0,1]$ and satisfy the conditions (i) $a_n + c_n \leq 1$; (ii) $a_n \to 0$ as $n \to \infty$ and $c_n = o(a_n)$; (iii) $\Sigma_{n=0}^{\infty} a_n = \infty$. For some $z_0 \in D$, let $\{u_n\}$ be any bounded sequence in D and let $\{z_n\}$ be modified Mann iterative sequence with errors defined by (1.8). Then $\{z_n\}$ converges strongly to the unique fixed point q of T.*

Proof. Since $T : D \to D$ is a uniformly generalized Lipschitz generalized asymptotically Φ-strongly pseudocontractive mapping, then there exists a strictly increasing continuous function $\Phi : [0,+\infty) \to [0,+\infty)$ with $\Phi(0) = 0$ such that

$$\left\langle (k_n I - T^n)x - (k_n I - T^n)y, J(x-y) \right\rangle \geq \Phi(\|x-y\|), \tag{2.22}$$

$$\|T^n x - T^n y\| \leq L(1 + \|x-y\|), \tag{2.23}$$

for any $x, y \in D$.

Step 1. There exists $z_0 \in D$ and $z_0 \neq Tz_0$ such that $r_0 = (k+L)\|z_0 - q\|^2 + L\|z_0 - q\| \in R(\Phi)$, where $k = \sup_n\{k_n\}$. In fact, if $\Phi(r) \to +\infty$ as $r \to +\infty$, then $r_0 \in R(\Phi)$; if $\sup\{\Phi(r) : r \in [0,+\infty)\} = r_1 < +\infty$ with $r_1 < r_0$, then, for $q \in D$, there exists a sequence $\{v_n\}$ in D such that $v_n \to q$ as $n \to \infty$ with $v_n \neq q$. Furthermore, there exists a natural number n_0 such that $(k+L)\|v_n - q\|^2 + L\|v_n - q\| < (r_1/2)$ for $n \geq n_0$, then we redefine z_0, r_0 such that $z_0 = v_{n_0}, r_0 = (k+L)\|z_0 - q\|^2 + L\|z_0 - q\| \in R(\Phi)$.

Step 2. For any $n \geq 0$, $\{z_n\}$ is bounded.

Set $r = \Phi^{-1}(r_0)$, we have $\|x_0 - q\| \leq R$. Let $B_1' = \{z \in D : \|z - q\| \leq r\}, B_2' = \{z \in D : \|z - q\| \leq 2r\}, M' = \sup_n\{\|u_n - q\|\}$. Next, we prove that $z_n \in B_1'$ for any $n \geq 0$ by induction. First $z_0 \in B_1'$ is obvious. Suppose that $z_n \in B_1'$ holds. We prove that $z_{n+1} \in B_1'$. If it is not the case, then $\|z_{n+1} - q\| > r$. By uniformly continuity of J on bounded subset, we choose $\epsilon_0 = \Phi(r/2)/16L(1+2r)$, there exists $\delta > 0$ such that $\|Jx - Jy\| < \epsilon_0$ when $\|x - y\| < \delta$, for all $x, y \in B_2'$. Now denote

$$\tau_0 = \min\left\{ \frac{r}{2[L(1+r)+2r+M']}, \frac{\delta}{2[L(1+r)+2r+M']}, \frac{\Phi(r/2)}{8r^2}, \frac{\Phi(r/2)}{24L(1+2r)}, \frac{\Phi(r/2)}{16M'r} \right\}. \tag{2.24}$$

Since $a_n, c_n, k_n - 1 \to 0$ as $n \to \infty$, and $c_n = o(a_n)$, without loss of generality, let $0 \leq a_n, c_n, k_n - 1 \leq \tau_0, c_n < a_n\tau_0$ for any $n \geq 0$. Then we have the following estimates from (1.8):

$$\begin{aligned}
\|z_n - T^n z_n\| &\leq \|z_n - q\| + \|T^n z_n - q\| \\
&\leq r + L(1+r), \\
\|z_n - q\| &\geq \|z_{n+1} - q\| - a_n\|T^n z_n - z_n\| - c_n\|u_n - z_n\| \\
&> r - a_n[r + L(1+r)] - c_n(r + M') \\
&\geq r - \tau_0[L(1+r) + 2r + M'] \\
&\geq \frac{r}{2},
\end{aligned}$$

$$\begin{aligned}\left\|z_{n+1}-q\right\| &\leq (1-a_n-c_n)\left\|z_n-q\right\|+a_n\left\|T^n z_n-q\right\|+c_n\left\|u_n-q\right\| \\ &\leq r+\tau_0\left[L(1+r)+M'\right] \\ &\leq 2r, \\ \left\|(z_{n+1}-q)-(z_n-q)\right\| &\leq a_n\|T^n z_n-z_n\|+c_n\|u_n-z_n\| \\ &\leq a_n[r+L(1+r)]+c_n(r+M') \\ &\leq \tau_0\left[L(1+r)+2r+M'\right] \\ &\leq \frac{\delta}{2}<\delta.\end{aligned} \tag{2.25}$$

Therefore, $\|J(z_{n+1}-q)-J(z_n-q)\|<\epsilon_0$.

Using Lemma 1.6 and formulas above, we obtain

$$\begin{aligned}\left\|z_{n+1}-q\right\|^2 &\leq (1-a_n)^2\left\|z_n-q\right\|^2+2a_n\left\langle T^n z_n-q, J(z_{n+1}-q)-J(z_n-q)\right\rangle \\ &\quad +2a_n\left\langle T^n z_n-q, J(z_n-q)\right\rangle+2c_n\left\langle u_n-q, J(z_{n+1}-q)\right\rangle \\ &\leq (1-a_n)^2\left\|z_n-q\right\|^2+2a_n\left\|T^n z_n-q\right\|\cdot\left\|J(z_{n+1}-q)-J(z_n-q)\right\| \\ &\quad +2a_n\left[k_n\left\|z_n-q\right\|^2-\Phi\left(\left\|z_n-q\right\|\right)\right]+2c_n\left\|u_n-q\right\|\cdot\left\|z_{n+1}-q\right\| \\ &\leq (1-a_n)^2r^2+4a_nL(1+2r)\epsilon_0 \\ &\quad +2a_n\left[k_n\left\|z_n-q\right\|^2-\Phi\left(\left\|z_n-q\right\|\right)\right]+4c_nM'r \\ &\leq (1-a_n)^2r^2+4a_nL(1+2r)\epsilon_0+2a_n\left[k_nr^2-\Phi\left(\frac{r}{2}\right)\right]+4c_nM'r \\ &= r^2+2a_n\left[\frac{a_n}{2}r^2+2L(1+2r)\epsilon_0+(k_n-1)r^2+\frac{2c_nM'r}{a_n}\right]-2a_n\Phi\left(\frac{r}{2}\right) \\ &\leq r^2+2a_n\left[\frac{\Phi(r/2)}{2}-\Phi\left(\frac{r}{2}\right)\right] \\ &\leq r^2-a_n\Phi\left(\frac{r}{2}\right) \\ &\leq r^2,\end{aligned} \tag{2.26}$$

this is a contradiction. Thus $z_{n+1}\in B_1'$, that is, $\{z_n\}$ is a bounded sequence, so $\{T^n z_n\}$ is also bounded. Denote $M_0=\sup_n\{\|z_n-q\|\}+\sup_n\{\|T^n z_n-q\|\}+\sup_n\{\|u_n-q\|\}$.

Step 3. We prove $\|z_n-q\|\to 0$ as $n\to\infty$.

Again using Lemma 1.6, we have

$$\begin{aligned}
\|z_{n+1}-q\|^2 &\le (1-a_n-c_n)^2\|z_n-q\|^2+2a_n\langle T^n z_n-q, J(z_{n+1}-q)\rangle \\
&\quad +2c_n\langle u_n-q, J(z_{n+1}-q)\rangle \\
&\le (1-a_n)^2\|z_n-q\|^2+2a_n\langle T^n z_n-q, J(z_{n+1}-q)-J(z_n-q)\rangle \\
&\quad +2a_n\langle T^n z_n-q, J(z_n-q)\rangle+2c_n\|u_n-q\|\cdot\|z_{n+1}-q\| \\
&\le (1-a_n)^2\|z_n-q\|^2+2a_nM_0D_n \\
&\quad +2a_n\Big[k_n\|z_n-q\|^2-\Phi(\|z_n-q\|)\Big]+2c_nM_0^2 \\
&\le \|z_n-q\|^2+2a_n\left[(k_n-1)M_0^2+\frac{a_nM_0^2}{2}+M_0D_n+\frac{c_nM_0^2}{a_n}-\Phi(\|z_n-q\|)\right] \\
&\le \|z_n-q\|^2+2a_n[E_n-\Phi(\|z_n-q\|)],
\end{aligned} \tag{2.27}$$

where

$$D_n=\|J(z_{n+1}-q)-J(z_n-q)\|, \qquad E_n=(k_n-1)M_0^2+\frac{a_nM_0^2}{2}+M_0D_n+\frac{c_nM_0^2}{a_n}, \tag{2.28}$$

and $D_n, E_n \to 0$ as $n \to \infty$.

Set $\inf_{n\ge 0}\Phi(\|z_n-q\|)/(1+\|z_{n+1}-q\|^2)=\lambda$, then $\lambda=0$. If it is not the case, we assume that $\lambda>0$. Let $0<\gamma<\min\{1,\lambda\}$, then $\Phi(\|z_n-q\|)/(1+\|z_{n+1}-q\|^2)\ge\gamma$, that is, $\Phi(\|z_n-q\|)\ge \gamma+\gamma\|z_{n+1}-q\|^2\ge\gamma\|z_{n+1}-q\|^2$. Thus, from (2.14) that

$$\|z_{n+1}-q\|^2\le\|z_n-q\|^2+2a_n\Big(E_n-\gamma\|z_{n+1}-q\|^2\Big), \tag{2.29}$$

which implies that

$$\begin{aligned}
\|z_{n+1}-q\|^2 &\le \frac{1}{1+2a_n\gamma}\|z_n-q\|^2+\frac{2a_nE_n}{1+2a_n\gamma} \\
&=\left(1-\frac{2a_n\gamma}{1+2a_n\gamma}\right)\|z_n-q\|^2+\frac{2a_nE_n}{1+2a_n\gamma}.
\end{aligned} \tag{2.30}$$

Let $\rho_n=\|z_n-q\|^2, \lambda_n=2a_n\gamma/(1+2a_n\gamma), \sigma_n=2a_nE_n/(1+2a_n\gamma)$. Then we get that

$$\rho_{n+1}\le(1-\lambda_n)\rho_n+\sigma_n. \tag{2.31}$$

Applying Lemma 1.7, we get that $\rho_n\to 0$ as $n\to\infty$. This is a contradiction and so $\lambda=0$. Therefore, there exists an infinite subsequence such that $\Phi(\|z_{n_i}-q\|)/(1+\|z_{n_i+1}-q\|^2)\to 0$ as $i\to\infty$. Since $0\le\Phi(\|z_{n_i}-q\|)/(1+M_0^2)\le\Phi(\|z_{n_i}-q\|)/(1+\|z_{n_i+1}-q\|^2)$, then $\Phi(\|z_{n_i}-q\|)\to 0$

as $i \to \infty$. In view of the strictly increasing and continuity of Φ, we have $\|z_{n_i} - q\| \to 0$ as $i \to \infty$. Let $\varepsilon \in (0,1)$ be any given, there exists n_{i_0} such that $\|z_{n_i} - q\| < \epsilon, a_{n_i}, a_n < \min\{\epsilon/4L(1+M_0), \epsilon/8M_0\}$, $c_{n_i}, c_n < \epsilon/16M_0$, $E_{n_i}, E_n < \Phi(\epsilon/2)/2$, for any $n_i, n \geq n_{i_0}$. First, we want to prove $\|z_{n_i+1} - q\| < \epsilon$. Suppose it is not this case, then $\|z_{n_i+1} - q\| \geq \epsilon$. Using (1.8), we may get the following estimates:

$$\begin{aligned} \|z_{n_i} - q\| &\geq \|z_{n_i+1} - q\| - a_{n_i}\|T^n z_{n_i} - q\| - a_{n_i}\|z_{n_i} - q\| - c_{n_i}\|u_{n_i} - q\| \\ &\geq \epsilon - a_{n_i}L(1+M_0) - (a_{n_i} + 2c_{n_i})M_0 \\ &> \frac{\epsilon}{2}. \end{aligned} \tag{2.32}$$

Since Φ is strictly increasing, then (2.32) leads to $\Phi(\|z_{n_i} - q\|) \geq \Phi(\epsilon/2)$. From (2.27), we have

$$\begin{aligned} \|z_{n_i+1} - q\|^2 &\leq \|z_{n_i} - q\|^2 + 2a_{n_i}[E_{n_i} - \Phi(\|z_{n_i} - q\|)] \\ &< \epsilon^2 + 2a_{n_i}\left[\frac{1}{2}\Phi\left(\frac{\epsilon}{2}\right) - \Phi\left(\frac{\epsilon}{2}\right)\right] \\ &\leq \epsilon^2 - \Phi\left(\frac{\epsilon}{2}\right)a_{n_i} \\ &\leq \epsilon^2, \end{aligned} \tag{2.33}$$

is a contradiction. Hence, $\|z_{n_i+1} - q\| < \epsilon$. Suppose that $\|z_{n_i+m} - q\| < \epsilon$ holds. Repeating the above course, we can easily prove that $\|z_{n_i+m+1} - q\| < \epsilon$ holds. Therefore, for any m and $n_i \geq n_0$, we obtain that $\|z_{n_i+m} - q\| < \epsilon$, which means $\|z_n - q\| \to 0$ as $n \to \infty$. This completes the proof. □

Theorem 2.4. *Let E be an arbitrary uniformly smooth real Banach space, let D be a nonempty closed convex subset of E, and let $T : D \to D$ be a uniformly generalized Lipschitz generalized asymptotically Φ-strongly pseudocontractive mapping with $q \in F(T) \neq \emptyset$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ be four real sequences in $[0,1]$ and satisfy the conditions (i) $a_n + c_n \leq 1$, $b_n + d_n \leq 1$; (ii) $a_n, b_n, d_n \to 0$ as $n \to \infty$ and $c_n = o(a_n)$; (iii) $\sum_{n=0}^{\infty} a_n = \infty$. For some $x_0 \in D$, let $\{v_n\}, \{w_n\}$ be two arbitrary bounded sequences in D, and let $\{x_n\}$ be Ishikawa iterative sequence with errors defined by (1.7). Then (1.7) converges strongly to the unique fixed point q of T.*

Proof. By Theorems 2.3 and 2.2, we obtain directly the result of Theorem 2.4. □

Remark 2.5. Our Theorem 2.2 extends and improves Theorem 3.1 of [4] from the bounded range of T to uniformly generalized Lipschitz mapping, and the proof course of Theorem 2.2 is quite different from that of [4].

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