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Research Article

On the Hyers-Ulam Stability of a General Mixed Additive and Cubic Functional Equation in *n*-Banach Spaces

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The objective of the present paper is to determine the generalized Hyers-Ulam stability of the mixed additive-cubic functional equation in n-Banach spaces by the direct method. In addition, we show under some suitable conditions that an approximately mixed additive-cubic function can be approximated by a mixed additive and cubic mapping.

1. Introduction and Preliminaries

A basic question in the theory of functional equations is as follows: when is it true that a function, which approximately satisfies a functional equation, must be close to an exact solution of the equation?

If the problem accepts a unique solution, we say the equation is stable (see [1]). The study of stability problems for functional equations is related to a question of Ulam [2] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [3]. The result of Hyers was generalized by Aoki [4] for approximate additive mappings and by Rassias [5] for approximate linear mappings by allowing the Cauchy difference operator $\mathrm{CD}f(x,y) = f(x+y) - [f(x)+f(y)]$ to be controlled by $\epsilon(\|x\|^p + \|y\|^p)$. In 1994, a generalization of Rassias' theorem was obtained by Găvruţa [6], who replaced $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x,y)$. On the other hand, several further interesting discussions, modifications, extensions, and generalizations of the original problem of Ulam have been proposed (see, e.g. [7–12] and the references therein).

Recently, Park [9] investigated the approximate additive mappings, approximate Jensen mappings, and approximate quadratic mappings in 2-Banach spaces and proved the generalized Hyers-Ulam stability of the Cauchy functional equation, the Jensen functional equation, and the quadratic functional equation in 2-Banach spaces. This is the first result for the stability problem of functional equations in 2-Banach spaces.

In [11, 12], we introduced the following mixed additive-cubic functional equation for fixed integers k with $k \neq 0, \pm 1$:

$$f(kx+y) + f(kx-y) = kf(x+y) + kf(x-y) + 2f(kx) - 2kf(x), \tag{1.1}$$

with $f(0) = 0$, and investigated the generalized Hyers-Ulam stability of (1.1) in quasi-Banach spaces and non-Archimedean fuzzy normed spaces, respectively.

In this paper, we investigate, approximate mixed additive-cubic mappings in n-Banach spaces. That is, we prove the generalized Hyers-Ulam stability of a general mixed additive-cubic equation (1.1) in n-Banach spaces by the direct method.

The concept of 2-normed spaces was initially developed by Gähler [13, 14] in the middle of 1960s, while that of n-normed spaces can be found in [15, 16]. Since then, many others have studied this concept and obtained various results; see for instance [15, 17–19].

We recall some basic facts concerning n-normed spaces and some preliminary results.

Definition 1.1. Let $n \in \mathbb{N}$, and let X be a real linear space with $\dim X \geq n$ and $\|\cdot, \dots, \cdot\| : X^n \to \mathbb{R}$ a function satisfying the following properties:

(N1) $\|x_1, x_2, \dots, x_n\| = 0$ if and only if $x_1, x_2, \dots, x_n$ are linearly dependent,

(N2) $\|x_1, x_2, \dots, x_n\|$ is invariant under permutation,

(N3) $\|\alpha x_1, x_2, \dots, x_n\| = |\alpha| \|x_1, x_2, \dots, x_n\|$,

(N4) $\|x + y, x_2, \dots, x_n\| \leq \|x, x_2, \dots, x_n\| + \|y, x_2, \dots, x_n\|$

for all $\alpha \in \mathbb{R}$ and $x, y, x_1, x_2, \dots, x_n \in$ X. Then the function $\|\cdot, \dots, \cdot\|$ is called an n-norm on X and the pair $(X, \|\cdot, \dots, \cdot\|)$ is called an n-normed space.

Example 1.2. For $x_1, x_2, \dots, x_n \in \mathbb{R}^n$, the Euclidean n-norm $\|x_1, x_2, \dots, x_n\|_E$ is defined by

$$\|x_1, x_2, \dots, x_n\|_E = \left|\det(x_{ij})\right| = \text{abs}\left(\begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}\right), \tag{1.2}$$

where $x_i = (x_{i1}, \dots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \dots, n$.

Example 1.3. The standard n-norm on X, a real inner product space of dimension $\dim X \geq n$, is as follows:

$$\|x_1, x_2, \dots, x_n\|_S = \begin{vmatrix} \langle x_1, x_1 \rangle & \cdots & \langle x_1, x_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, x_1 \rangle & \cdots & \langle x_n, x_n \rangle \end{vmatrix}^{1/2}, \tag{1.3}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on X. If $X = \mathbb{R}^n$, then this n-norm is exactly the same as the Euclidean n-norm $\|x_1, x_2, \dots, x_n\|_E$ mentioned earlier. For $n = 1$, this n-norm is the usual norm $\|x_1\| = \langle x_1, x_1 \rangle^{1/2}$.

Definition 1.4. A sequence $\{x_k\}$ in an n-normed space X is said to converge to some $x \in X$ in the n-norm if

$$\lim_{k \to \infty} \|x_k - x, y_2, \ldots, y_n\| = 0, \tag{1.4}$$

for every $y_2, \ldots, y_n \in X$.

Definition 1.5. A sequence $\{x_k\}$ in an n-normed space X is said to be a Cauchy sequence with respect to the n-norm if

$$\lim_{k,l \to \infty} \|x_k - x_l, y_2, \ldots, y_n\| = 0, \tag{1.5}$$

for every $y_2, \ldots, y_n \in X$. If every Cauchy sequence in X converges to some $x \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be an n-Banach space.

Now we state the following results as lemma (see [9] for the details).

Lemma 1.6. *Let X be an n-normed space. Then,*

(1) *For $x_i \in X (i = 1, \ldots, n)$ and γ, a real number,*

$$\|x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n\| = \|x_1, \ldots, x_i, \ldots, x_j + \gamma x_i, \ldots, x_n\| \tag{1.6}$$

for all $1 \leq i \neq j \leq n$,

(2) $|\|x, y_2, \ldots, y_n\| - \|y, y_2, \ldots, y_n\|| \leq \|x - y, y_2, \ldots, y_n\|$ *for all $x, y, y_2, \ldots, y_n \in X$,*

(3) *if $\|x, y_2, \ldots, y_n\| = 0$ for all $y_2, \ldots, y_n \in X$, then $x = 0$,*

(4) *for a convergent sequence $\{x_j\}$ in X,*

$$\lim_{j \to \infty} \|x_j, y_2, \ldots, y_n\| = \left\| \lim_{j \to \infty} x_j, y_2, \ldots, y_n \right\| \tag{1.7}$$

for all $y_2, \ldots, y_n \in X$.

2. Approximate Mixed Additive-Cubic Mappings

In this section, we investigate the generalized Hyers-Ulam stability of the generalized mixed additive-cubic functional equation in n-Banach spaces. Let X be a linear space and Y an n-Banach space. For convenience, we use the following abbreviation for a given mapping $f : X \to Y$:

$$Df(x, y) := f(kx + y) + f(kx - y) - kf(x + y) - kf(x - y) - 2f(kx) + 2kf(x) \tag{2.1}$$

for all $x, y \in X$.

Theorem 2.1. *Let X be a linear space and Y an n-Banach space. Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : X^{n+1} \to [0,\infty)$ such that*

$$\sum_{j=0}^{\infty} \frac{1}{2^j} \varphi\left(2^j x, 2^j y, u_2, \ldots, u_n\right) < \infty, \tag{2.2}$$

$$\left\| Df(x,y), u_2, \ldots, u_n \right\|_Y \le \varphi(x, y, u_2, \ldots, u_n) \tag{2.3}$$

for all $x, y, u_2, \ldots, u_n \in X$. Then, there is a unique additive mapping $A : X \to Y$ such that

$$\left\| f(2x) - 8f(x) - A(x), u_2, \ldots, u_n \right\|_Y \le \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} \widetilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right) \tag{2.4}$$

for all $x, u_2, \ldots, u_n \in X$, where

$$\begin{aligned}
&\widetilde{\varphi}(x, u_2, \ldots, u_n) \\
&:= \frac{1}{|k^3 - k|} \Bigg\{ (|k| + 1) \left[\varphi(x, (2k+1)x, u_2, \ldots, u_n) + \varphi(x, (2k-1)x, u_2, \ldots, u_n) \right] \\
&\quad + \varphi(3x, x, u_2, \ldots, u_n) + \left(8k^2 + 1\right) \varphi(x, x, u_2, \ldots, u_n) + \varphi(x, 3kx, u_2, \ldots, u_n) \\
&\quad + \varphi(x, kx, u_2, \ldots, u_n) + k^2 \varphi(2x, 2x, u_2, \ldots, u_n) + \varphi(2x, 2kx, u_2, \ldots, u_n) \\
&\quad + 2\varphi(x, (k+1)x, u_2, \ldots, u_n) + 2\varphi(x, (k-1)x, u_2, \ldots, u_n) + 2\varphi(2x, x, u_2, \ldots, u_n) \\
&\quad + 2\varphi(2x, kx, u_2, \ldots, u_n) + 8\varphi\left(\frac{x}{2}, \frac{kx}{2}, u_2, \ldots, u_n\right) \\
&\quad + 8|k|\varphi\left(\frac{x}{2}, \frac{(2k-1)x}{2}, u_2, \ldots, u_n\right) + 8|k|\varphi\left(\frac{x}{2}, \frac{(2k+1)x}{2}, u_2, \ldots, u_n\right) \\
&\quad + 8\varphi\left(\frac{x}{2}, \frac{3kx}{2}, u_2, \ldots, u_n\right) + \frac{|k|+1}{|k-1|} \varphi(0, (k+1)x, u_2, \ldots, u_n) \\
&\quad + \frac{8k^2+1}{|k-1|} \varphi(0, (k-1)x, u_2, \ldots, u_n) + \frac{2}{|k-1|} \varphi(0, x, u_2, \ldots, u_n) \\
&\quad + \frac{|k|}{|k-1|} \varphi(0, (3k-1)x, u_2, \ldots, u_n) + \frac{k^2}{|k-1|} \varphi(0, 2(k-1)x, u_2, \ldots, u_n) \\
&\quad + \frac{k^2 + |k| - 1}{|k-1|} \varphi(0, 2kx, u_2, \ldots, u_n) \\
&\quad + \frac{8|k|}{|k-1|} \varphi\left(0, \frac{(3k-1)x}{2}, u_2, \ldots, u_n\right) + \frac{8|k|}{|k-1|} \varphi\left(0, \frac{(k+1)x}{2}, u_2, \ldots, u_n\right) \\
&\quad + \frac{8k^2 + 2|k| - 8}{|k-1|} \varphi(0, kx, u_2, \ldots, u_n) \Bigg\}.
\end{aligned} \tag{2.5}$$

Proof. Letting $x = 0$ in (2.3), we get

$$\|f(y) + f(-y), u_2, \dots, u_n\|_Y \leq \frac{1}{|k-1|}\varphi(0, y, u_2, \dots, u_n) \tag{2.6}$$

for all $y, u_2, \dots, u_n \in X$. Putting $y = x$ in (2.3), we have

$$\|f((k+1)x) + f((k-1)x) - kf(2x) - 2f(kx) + 2kf(x), u_2, \dots, u_n\|_Y \leq \varphi(x, x, u_2, \dots, u_n) \tag{2.7}$$

for all $x, u_2, \dots, u_n \in X$. Thus

$$\begin{aligned}&\|f(2(k+1)x) + f(2(k-1)x) - kf(4x) - 2f(2kx) + 2kf(2x), u_2, \dots, u_n\|_Y\\ &\quad\leq \varphi(2x, 2x, u_2, \dots, u_n)\end{aligned} \tag{2.8}$$

for all $x, u_2, \dots, u_n \in X$. Letting $y = kx$ in (2.3), we get

$$\|f(2kx) - kf((k+1)x) - kf(-(k-1)x) - 2f(kx) + 2kf(x), u_2, \dots, u_n\|_Y \leq \varphi(x, kx, u_2, \dots, u_n) \tag{2.9}$$

for all $x, u_2, \dots, u_n \in X$. Letting $y = (k+1)x$ in (2.3), we have

$$\begin{aligned}&\|f((2k+1)x) + f(-x) - kf((k+2)x) - kf(-kx) - 2f(kx) + 2kf(x), u_2, \dots, u_n\|_Y\\ &\quad\leq \varphi(x, (k+1)x, u_2, \dots, u_n)\end{aligned} \tag{2.10}$$

for all $x, u_2, \dots, u_n \in X$. Letting $y = (k-1)x$ in (2.3), we have

$$\begin{aligned}&\|f((2k-1)x) - (k+2)f(kx) - kf(-(k-2)x) + (2k+1)f(x), u_2, \dots, u_n\|_Y\\ &\quad\leq \varphi(x, (k-1)x, u_2, \dots, u_n)\end{aligned} \tag{2.11}$$

for all $x, u_2, \dots, u_n \in X$. Replacing x and y by $2x$ and x in (2.3), respectively, we get

$$\begin{aligned}&\|f((2k+1)x) + f((2k-1)x) - 2f(2kx) - kf(3x) + 2kf(2x) - kf(x), u_2, \dots, u_n\|_Y\\ &\quad\leq \varphi(2x, x, u_2, \dots, u_n)\end{aligned} \tag{2.12}$$

for all $x, u_2, \dots, u_n \in X$. Replacing x and y by $3x$ and x in (2.3), respectively, we get

$$\begin{aligned}&\|f((3k+1)x) + f((3k-1)x) - 2f(3kx) - kf(4x) - kf(2x) + 2kf(3x), u_2, \dots, u_n\|_Y\\ &\quad\leq \varphi(3x, x, u_2, \dots, u_n)\end{aligned} \tag{2.13}$$

for all $x, u_2, \ldots, u_n \in X$. Replacing x and y by $2x$ and kx in (2.3), respectively, we have

$$\begin{aligned}&\left\| f(3kx) + f(kx) - kf((k+2)x) - kf(-(k-2)x) - 2f(2kx) + 2kf(2x), u_2, \ldots, u_n \right\|_Y \\ &\quad \leq \varphi(2x, kx, u_2, \ldots, u_n)\end{aligned} \tag{2.14}$$

for all $x, u_2, \ldots, u_n \in X$. Setting $y = (2k+1)x$ in (2.3), we have

$$\begin{aligned}&\left\| f((3k+1)x) + f(-(k+1)x) - kf(2(k+1)x) - kf(-2kx) - 2f(kx) + 2kf(x), u_2, \ldots, u_n \right\|_Y \\ &\quad \leq \varphi(x, (2k+1)x, u_2, \ldots, u_n)\end{aligned} \tag{2.15}$$

for all $x, u_2, \ldots, u_n \in X$. Letting $y = (2k-1)x$ in (2.3), we have

$$\begin{aligned}&\left\| f((3k-1)x) + f(-(k-1)x) - kf(-2(k-1)x) - kf(2kx) - 2f(kx) + 2kf(x), u_2, \ldots, u_n \right\|_Y \\ &\quad \leq \varphi(x, (2k-1)x, u_2, \ldots, u_n)\end{aligned} \tag{2.16}$$

for all $x, u_2, \ldots, u_n \in X$. Letting $y = 3kx$ in (2.3), we have

$$\begin{aligned}&\left\| f(4\mathrm{k}x) + f(-2kx) - kf((3k+1)x) - kf(-(3k-1)x) - 2f(kx) + 2kf(x), u_2, \ldots, u_n \right\|_Y \\ &\quad \leq \varphi(x, 3kx, u_2, \ldots, u_n)\end{aligned} \tag{2.17}$$

for all $x, u_2, \ldots, u_n \in X$. By (2.6), (2.7), (2.13), (2.15), and (2.16), we get

$$\begin{aligned}&\left\| kf(2(k+1)x) + kf(-2(k-1)x) + 6f(kx) - 2f(3kx) - kf(4x) + 2kf(3x) - 6kf(x), u_2, \ldots, u_n \right\|_Y \\ &\quad \leq \varphi(x, (2k+1)x, u_2, \ldots, u_n) + \varphi(x, (2k-1)x, u_2, \ldots, u_n) + \varphi(3x, x, u_2, \ldots, u_n) \\ &\qquad + \varphi(x, x, u_2, \ldots, u_n) + \frac{1}{|k-1|}\varphi(0, (k+1)x, u_2, \ldots, u_n) \\ &\qquad + \frac{1}{|k-1|}\varphi(0, (k-1)x, u_2, \ldots, u_n) + \frac{|k|}{|k-1|}\varphi(0, 2kx, u_2, \ldots, u_n)\end{aligned} \tag{2.18}$$

for all $x, u_2, \ldots, u_n \in X$. By (2.6), (2.10), and (2.11), we have

$$\begin{aligned}&\left\| f((2k+1)x) + f((2k-1)x) - kf((k+2)x) - kf(-(k-2)x) - 4f(kx) + 4kf(x), u_2, \ldots, u_n \right\|_Y \\ &\quad \leq \varphi(x, (k+1)x, u_2, \ldots, u_n) + \varphi(x, (k-1)x, u_2, \ldots, u_n) + \frac{1}{|k-1|}\varphi(0, x, u_2, \ldots, u_n) \\ &\qquad + \left|\frac{k}{k-1}\right|\varphi(0, kx, u_2, \ldots, u_n)\end{aligned} \tag{2.19}$$

for all $x, u_2, \ldots, u_n \in X$. It follows from (2.12) and (2.19) that

$$\begin{aligned}
&\left\|kf((k+2)x) + kf(-(k-2)x) - 2f(2kx) + 4f(kx) - kf(3x) + 2kf(2x) - 5kf(x), u_2, \ldots, u_n\right\|_Y \\
&\quad \leq \varphi(x, (k+1)x, u_2, \ldots, u_n) + \varphi(x, (k-1)x, u_2, \ldots, u_n) + \varphi(2x, x, u_2, \ldots, u_n) \\
&\qquad + \frac{1}{|k-1|}\varphi(0, x, u_2, \ldots, u_n) + \left|\frac{k}{k-1}\right|\varphi(0, kx, u_2, \ldots, u_n)
\end{aligned} \tag{2.20}$$

for all $x, u_2, \ldots, u_n \in X$. By (2.14) and (2.20), we have

$$\begin{aligned}
&\left\|f(3kx) - 4f(2kx) + 5f(kx) - kf(3x) + 4kf(2x) - 5kf(x), u_2, \ldots, u_n\right\|_Y \\
&\quad \leq \varphi(x, (k+1)x, u_2, \ldots, u_n) + \varphi(x, (k-1)x, u_2, \ldots, u_n) + \varphi(2x, x, u_2, \ldots, u_n) \\
&\qquad + \varphi(2x, kx, u_2, \ldots, u_n) + \frac{1}{|k-1|}\varphi(0, x, u_2, \ldots, u_n) + \left|\frac{k}{k-1}\right|\varphi(0, kx, u_2, \ldots, u_n)
\end{aligned} \tag{2.21}$$

for all $x, u_2, \ldots, u_n \in X$. By (2.6), (2.15), (2.16), and (2.17), we have

$$\begin{aligned}
&\Big\|kf(-(k+1)x) - kf(-(k-1)x) - k^2 f(2(k+1)x) + k^2 f(-2(k-1)x) \\
&\qquad + k^2 f(2kx) - \left(k^2 - 1\right) f(-2kx) + f(4kx) - 2f(kx) + 2kf(x), u_2, \ldots, u_n\Big\|_Y \\
&\quad \leq |k|\varphi(x, (2k+1)x, u_2, \ldots, u_n) + |k|\varphi(x, (2k-1)x, u_2, \ldots, u_n) + \varphi(x, 3kx, u_2, \ldots, u_n) \\
&\qquad + \left|\frac{k}{k-1}\right|\varphi(0, (3k-1)x, u_2, \ldots, u_n)
\end{aligned} \tag{2.22}$$

for all $x, u_2, \ldots, u_n \in X$. It follows from (2.6), (2.8), (2.9), and (2.22) that

$$\begin{aligned}
&\left\|f(4kx) - 2f(2kx) - k^3 f(4x) + 2k^3 f(2x), u_2, \ldots, u_n\right\|_Y \\
&\quad \leq |k|\varphi(x, (2k+1)x, u_2, \ldots, u_n) + |k|\varphi(x, (2k-1)x, u_2, \ldots, u_n) + \varphi(x, 3kx, u_2, \ldots, u_n) \\
&\qquad + \varphi(x, kx, u_2, \ldots, u_n) + k^2\varphi(2x, 2x, u_2, \ldots, u_n) + \left|\frac{k}{k-1}\right|\varphi(0, (3k-1)x, u_2, \ldots, u_n) \\
&\qquad + \left|\frac{k}{k-1}\right|\varphi(0, (k+1)x, u_2, \ldots, u_n) + \frac{k^2}{|k-1|}\varphi(0, 2(k-1)x, u_2, \ldots, u_n) \\
&\qquad + \frac{k^2-1}{|k-1|}\varphi(0, 2kx, u_2, \ldots, u_n)
\end{aligned} \tag{2.23}$$

for all $x, u_2, \ldots, u_n \in X$. Hence,

$$\begin{aligned}
&\left\| f(2kx) - 2f(kx) - k^3 f(2x) + 2k^3 f(x), u_2, \ldots, u_n \right\|_Y \\
&\quad \leq |k|\varphi\left(\frac{x}{2}, \frac{(2k+1)x}{2}, u_2, \ldots, u_n\right) + |k|\varphi\left(\frac{x}{2}, \frac{(2k-1)x}{2}, u_2, \ldots, u_n\right) + \varphi\left(\frac{x}{2}, \frac{3kx}{2}, u_2, \ldots, u_n\right) \\
&\qquad + \varphi\left(\frac{x}{2}, \frac{kx}{2}, u_2, \ldots, u_n\right) + k^2\varphi(x, x, u_2, \ldots, u_n) + \left|\frac{k}{k-1}\right|\varphi\left(0, \frac{(3k-1)x}{2}, u_2, \ldots, u_n\right) \\
&\qquad + \left|\frac{k}{k-1}\right|\varphi\left(0, \frac{(k+1)x}{2}, u_2, \ldots, u_n\right) + \frac{k^2}{|k-1|}\varphi(0, (k-1)x, u_2, \ldots, u_n) \\
&\qquad + \frac{k^2-1}{|k-1|}\varphi(0, kx, u_2, \ldots, u_n)
\end{aligned} \tag{2.24}$$

for all $x, u_2, \ldots, u_n \in X$. By (2.9), we have

$$\begin{aligned}
&\| f(4kx) - kf(2(k+1)x) - kf(-2(k-1)x) - 2f(2kx) + 2kf(2x), u_2, \ldots, u_n \|_Y \\
&\quad \leq \varphi(2x, 2kx, u_2, \ldots, u_n)
\end{aligned} \tag{2.25}$$

for all $x, u_2, \ldots, u_n \in X$. From (2.23) and (2.25), we have

$$\begin{aligned}
&\left\| kf(2(k+1)x) + kf(-2(k-1)x) - k^3 f(4x) + \left(2k^3 - 2k\right) f(2x) \right\|_Y \\
&\quad \leq |k|\varphi(x, (2k+1)x, u_2, \ldots, u_n) + |k|\varphi(x, (2k-1)x, u_2, \ldots, u_n) + \varphi(x, 3kx, u_2, \ldots, u_n) \\
&\qquad + \varphi(x, kx, u_2, \ldots, u_n) + k^2\varphi(2x, 2x, u_2, \ldots, u_n) + \varphi(2x, 2kx, u_2, \ldots, u_n) \\
&\qquad + \left|\frac{k}{k-1}\right|\varphi(0, (3k-1)x, u_2, \ldots, u_n) + \left|\frac{k}{k-1}\right|\varphi(0, (k+1)x, u_2, \ldots, u_n) \\
&\qquad + \frac{k^2}{|k-1|}\varphi(0, 2(k-1)x, u_2, \ldots, u_n) + \frac{k^2-1}{|k-1|}\varphi(0, 2kx, u_2, \ldots, u_n)
\end{aligned} \tag{2.26}$$

for all $x, u_2, \ldots, u_n \in X$. Also, from (2.18) and (2.26), we get

$$\begin{aligned}
&\left\| 2f(3kx) - 6f(kx) + \left(k - k^3\right) f(4x) - 2kf(3x) + \left(2k^3 - 2k\right) f(2x) + 6kf(x), u_2, \ldots, u_n \right\|_Y \\
&\quad \leq (|k|+1)\left[\varphi(x, (2k+1)x, u_2, \ldots, u_n) + \varphi(x, (2k-1)x, u_2, \ldots, u_n)\right] + \varphi(3x, x, u_2, \ldots, u_n) \\
&\qquad + \varphi(x, x, u_2, \ldots, u_n) + \varphi(x, 3kx, u_2, \ldots, u_n) + \varphi(x, kx, u_2, \ldots, u_n) \\
&\qquad + k^2\varphi(2x, 2x, u_2, \ldots, u_n) + \varphi(2x, 2kx, u_2, \ldots, u_n) + \frac{|k|+1}{|k-1|}\varphi(0, (k+1)x, u_2, \ldots, u_n)
\end{aligned}$$

$$
+\frac{1}{|k-1|}\varphi(0,(k-1)x,u_2,\dots,u_n)+\frac{k^2+|k|-1}{|k-1|}\varphi(0,2kx,u_2,\dots,u_n)
+\left|\frac{k}{k-1}\right|\varphi(0,(3k-1)x,u_2,\dots,u_n)+\frac{k^2}{|k-1|}\varphi(0,2(k-1)x,u_2,\dots,u_n)
\tag{2.27}
$$

for all $x, u_2, \dots, u_n \in X$.

On the other hand, it follows from (2.21) and (2.27) that

$$
\begin{aligned}
&\left\|8f(2kx)-16f(kx)+\left(k-k^3\right)f(4x)+\left(2k^3-10k\right)f(2x)+16kf(x),u_2,\dots,u_n\right\|_Y \\
&\quad\le (|k|+1)\left[\varphi(x,(2k+1)x,u_2,\dots,u_n)+\varphi(x,(2k-1)x,u_2,\dots,u_n)\right]+\varphi(3x,x,u_2,\dots,u_n) \\
&\qquad+\varphi(x,x,u_2,\dots,u_n)+\varphi(x,3kx,u_2,\dots,u_n)+\varphi(x,kx,u_2,\dots,u_n) \\
&\qquad+k^2\varphi(2x,2x,u_2,\dots,u_n)+\varphi(2x,2kx,u_2,\dots,u_n)+2\varphi(x,(k+1)x,u_2,\dots,u_n) \\
&\qquad+2\varphi(x,(k-1)x,u_2,\dots,u_n)+2\varphi(2x,x,u_2,\dots,u_n)+2\varphi(2x,kx,u_2,\dots,u_n) \\
&\qquad+\frac{2}{|k-1|}\varphi(0,x,u_2,\dots,u_n)+\frac{2|k|}{|k-1|}\varphi(0,kx,u_2,\dots,u_n)+\frac{|k|+1}{|k-1|}\varphi(0,(k+1)x,u_2,\dots,u_n) \\
&\qquad+\frac{1}{|k-1|}\varphi(0,(k-1)x,u_2,\dots,u_n)+\frac{k^2+|k|-1}{|k-1|}\varphi(0,2kx,u_2,\dots,u_n) \\
&\qquad+\left|\frac{k}{k-1}\right|\varphi(0,(3k-1)x,u_2,\dots,u_n)+\frac{k^2}{|k-1|}\varphi(0,2(k-1)x,u_2,\dots,u_n)
\end{aligned}
\tag{2.28}
$$

for all $x, u_2, \dots, u_n \in X$. Therefore by (2.24) and (2.28), we get

$$
\begin{aligned}
&\left\|f(4x)-10f(2x)+16f(x),u_2,\dots,u_n\right\|_Y \\
&\quad\le\frac{1}{|k^3-k|} \\
&\qquad\times\Bigg\{(|k|+1)\left[\varphi(x,(2k+1)x,u_2,\dots,u_n)+\varphi(x,(2k-1)x,u_2,\dots,u_n)\right] \\
&\qquad\quad+\varphi(3x,x,u_2,\dots,u_n)+\left(8k^2+1\right)\varphi(x,x,u_2,\dots,u_n)+\varphi(x,3kx,u_2,\dots,u_n) \\
&\qquad\quad+\varphi(x,kx,u_2,\dots,u_n)+k^2\varphi(2x,2x,u_2,\dots,u_n)+\varphi(2x,2kx,u_2,\dots,u_n) \\
&\qquad\quad+2\varphi(x,(k+1)x,u_2,\dots,u_n)+2\varphi(x,(k-1)x,u_2,\dots,u_n)+2\varphi(2x,x,u_2,\dots,u_n) \\
&\qquad\quad+2\varphi(2x,kx,u_2,\dots,u_n)+8\varphi\left(\frac{x}{2},\frac{kx}{2},u_2,\dots,u_n\right)+8|k|\varphi\left(\frac{x}{2},\frac{(2k-1)x}{2},u_2,\dots,u_n\right) \\
&\qquad\quad+8|k|\varphi\left(\frac{x}{2},\frac{(2k+1)x}{2},u_2,\dots,u_n\right)+8\varphi\left(\frac{x}{2},\frac{3kx}{2},u_2,\dots,u_n\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{|k|+1}{|k-1|}\varphi(0,(k+1)x,u_2,\ldots,u_n)+\frac{8k^2+1}{|k-1|}\varphi(0,(k-1)x,u_2,\ldots,u_n)\\
&+\frac{2}{|k-1|}\varphi(0,x,u_2,\ldots,u_n)+\left|\frac{k}{k-1}\right|\varphi(0,(3k-1)x,u_2,\ldots,u_n)\\
&+\frac{k^2}{|k-1|}\varphi(0,2(k-1)x,u_2,\ldots,u_n)+\frac{k^2+|k|-1}{|k-1|}\varphi(0,2kx,u_2,\ldots,u_n)\\
&+\frac{8|k|}{|k-1|}\varphi\left(0,\frac{(3k-1)x}{2},u_2,\ldots,u_n\right)\\
&+\frac{8|k|}{|k-1|}\varphi\left(0,\frac{(k+1)x}{2},u_2,\ldots,u_n\right)+\frac{8k^2+2|k|-8}{|k-1|}\varphi(0,kx,u_2,\ldots,u_n)\Bigg\}\\
&:=\tilde{\varphi}(x,u_2,\ldots,u_n)
\end{aligned}
\tag{2.29}
$$

for all $x,u_2,\ldots,u_n\in X$.

Now, let $g : X \to Y$ be the mapping defined by $g(x) := f(2x) - 8f(x)$ for all $x,u_2,\ldots,u_n\in X$. Then, (2.29) means that

$$\|f(4x)-10f(2x)+16f(x),u_2,\ldots,u_n\|_Y\le\tilde{\varphi}(x,u_2,\ldots,u_n)\tag{2.30}$$

for all $x,u_2,\ldots,u_n\in X$. Also, we get

$$\|g(2x)-2g(x),u_2,\ldots,u_n\|_Y\le\tilde{\varphi}(x,u_2,\ldots,u_n)\tag{2.31}$$

for all $x\in X$. Replacing x by 2^jx in (2.31) and dividing both sides of (2.31) by 2^{j+1}, we get

$$\left\|\frac{1}{2^j}g(2^jx)-\frac{1}{2^{j+1}}g(2^{j+1}x),u_2,\ldots,u_n\right\|_Y\le\frac{1}{2^{j+1}}\tilde{\varphi}\left(2^jx,u_2,\ldots,u_n\right)\tag{2.32}$$

for all $x,u_2,\ldots,u_n\in X$ and all integers $j\ge0$. For all integers l,m with $0\le l<m$, we have

$$
\begin{aligned}
\left\|\frac{1}{2^l}g\left(2^lx\right)-\frac{1}{2^m}g(2^mx),u_2,\ldots,u_n\right\|_Y&\le\sum_{j=l}^{m-1}\left\|\frac{1}{2^j}g\left(2^jx\right)-\frac{1}{2^{j+1}}g\left(2^{j+1}x\right),u_2,\ldots,u_n\right\|_Y\\
&\le\sum_{j=l}^{m-1}\frac{1}{2^{j+1}}\tilde{\varphi}\left(2^jx,u_2,\ldots,u_n\right)
\end{aligned}
\tag{2.33}
$$

for all $x,u_2,\ldots,u_n\in X$. So, we get

$$\lim_{l,m\to\infty}\left\|\frac{1}{2^l}g(2^lx)-\frac{1}{2^m}g(2^mx),u_2,\ldots,u_n\right\|_Y=0\tag{2.34}$$

for all $x, u_2, \ldots, u_n \in X$. This shows that the sequence $\{(1/2^j)g(2^j x)\}$ is a Cauchy sequence in Y. Since Y is an n-Banach space, the sequence $\{(1/2^j)g(2^j x)\}$ converges. So, we can define a mapping $A : X \to Y$ by

$$A(x) := \lim_{j\to\infty} \frac{1}{2^j} g\left(2^j x\right) \tag{2.35}$$

for all $x \in X$. Putting $l = 0$, then passing the limit $m \to \infty$ in (2.33), and using Lemma 1.6(4), we get

$$\|g(x) - A(x), u_2, \ldots, u_n\|_Y \le \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} \widetilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right) \tag{2.36}$$

for all $x, u_2, \ldots, u_n \in X$.

Now we show that A is additive. By Lemma 1.6, (2.2), (2.32), and (2.35), we have

$$\begin{aligned} \|A(2x) - 2A(x), u_2, \ldots, u_n\|_Y &= \lim_{j\to\infty} \left\| \frac{1}{2^j} g\left(2^{j+1}x\right) - \frac{1}{2^{j-1}} g\left(2^j x\right), u_2, \ldots, u_n \right\|_Y \\ &= 2 \lim_{j\to\infty} \left\| \frac{1}{2^{j+1}} g\left(2^{j+1}x\right) - \frac{1}{2^j} g\left(2^j x\right), u_2, \ldots, u_n \right\|_Y \\ &\le \lim_{j\to\infty} \frac{1}{2^j} \widetilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right) = 0 \end{aligned} \tag{2.37}$$

for all $x, u_2, \ldots, u_n \in X$. By Lemma 1.6(3), $A(2x) = 2A(x)$ for all $x \in X$. Also, by Lemma 1.6(4), (2.2), (2.3), and (2.35), we get

$$\begin{aligned} &\|DA(x, y), u_2, \ldots, u_n\|_Y \\ &\quad = \lim_{j\to\infty} \frac{1}{2^j} \left\| Dg\left(2^j x, 2^j y\right), u_2, \ldots, u_n \right\|_Y \\ &\quad = \lim_{j\to\infty} \frac{1}{2^j} \left\| Df\left(2^{j+1}x, 2^{j+1}y\right) - 8Df\left(2^j x, 2^j y\right), u_2, \ldots, u_n \right\|_Y \\ &\quad \le \lim_{j\to\infty} \frac{1}{2^j} \left[\left\| Df\left(2^{j+1}x, 2^{j+1}y\right), u_2, \ldots, u_n \right\|_Y + 8 \left\| Df\left(2^j x, 2^j y\right), u_2, \ldots, u_n \right\|_Y \right] \\ &\quad \le \lim_{j\to\infty} \frac{1}{2^j} \left[\varphi\left(2^{j+1}x, 2^{j+1}y, u_2, \ldots, u_n\right) + 8\varphi\left(2^j x, 2^j y, u_2, \ldots, u_n\right) \right] = 0 \end{aligned} \tag{2.38}$$

for all $x, y, u_2, \ldots, u_n \in X$. By Lemma 1.6(3), $DA(x, y) = 0$ for all $x, y \in X$. Hence, the mapping A satisfies (1.1). By [11, Lemma 2.3], the mapping $x \to A(2x) - 8A(x)$ is additive. Therefore, $A(2x) = 2A(x)$ implies that the mapping A is additive.

To prove the uniqueness of A, let $B : X \to Y$ be another additive mapping satisfying (2.4). Fix $x \in X$. Clearly, $A(2^l x) = 2^l A(x)$ and $B(2^l x) = 2^l B(x)$ for all $l \in \mathbb{N}$. It follows from (2.4) that

$$\begin{aligned}
\|A(x) - B(x), u_2, \dots, u_n\|_Y &= \left\| \frac{A(2^l x)}{2^l} - \frac{B(2^l x)}{2^l}, u_2, \dots, u_n \right\|_Y \\
&\le \frac{1}{2^l}\Big[\left\| f\left(2^{l+1}x\right) - 8f\left(2^l x\right) - A\left(2^l x\right), u_2, \dots, u_n \right\|_Y \\
&\quad + \left\| B\left(2^l x\right) - f\left(2^{l+1}x\right) + 8f\left(2^l x\right), u_2, \dots, u_n \right\|_Y \Big] \\
&\le \frac{1}{2^l}\sum_{j=0}^{\infty}\frac{1}{2^j}\widetilde{\varphi}\left(2^{j+l}x, u_2, \dots, u_n\right) \\
&\le \sum_{j=0}^{\infty}\frac{1}{2^{j+l}}\widetilde{\varphi}\left(2^{j+l}x, u_2, \dots, u_n\right) = \sum_{j=l}^{\infty}\frac{1}{2^j}\widetilde{\varphi}\left(2^j x, u_2, \dots, u_n\right)
\end{aligned} \tag{2.39}$$

for all $x, u_2, \dots, u_n \in X$, and $l \in \mathbb{N}$. By (2.2), we see that the right-hand side of the above inequality tends to 0 as $l \to \infty$. Therefore, $\|A(x) - B(x), u_2, \dots, u_n\|_Y = 0$ for all $u_2, \dots, u_n \in X$. By Lemma 1.6, we can conclude that $A(x) = B(x)$ for all $x \in X$. So, $A = B$. This proves the uniqueness of A. □

Theorem 2.2. *Let X be a linear space and Y an n-Banach space. Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : X^{n+1} \to [0, \infty)$ such that*

$$\begin{gathered}
\sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, u_2, \dots, u_n\right) < \infty, \\
\left\| Df(x,y), u_2, \dots, u_n \right\|_Y \le \varphi(x, y, u_2, \dots, u_n)
\end{gathered} \tag{2.40}$$

for all $x, y, u_2, \dots, u_n \in X$. Then, there is a unique additive mapping $A : X \to Y$ such that

$$\left\| f(2x) - 8f(x) - A(x), u_2, \dots, u_n \right\|_Y \le \sum_{j=1}^{\infty} 2^{j-1}\widetilde{\varphi}\left(\frac{x}{2^j}, u_2, \dots, u_n\right) \tag{2.41}$$

for all $x, u_2, \dots, u_n \in X$, where $\widetilde{\varphi}(x, u_2, \dots, u_n)$ is defined as in Theorem 2.1.

Proof. The proof is similar to the proof of Theorem 2.1. □

Corollary 2.3. *Let X be a normed space and Y an n-Banach space. Let $\theta \in [0, \infty), p, r_2, \dots, r_n \in (0, \infty)$ such that $p \neq 1$, and let $f : X \to Y$ be a mapping with $f(0) = 0$ such that*

$$\left\| Df(x,y), u_2, \dots, u_n \right\|_Y \le \theta\left(\|x\|_X^p + \|y\|_X^p \right) \|u_2\|_X^{r_2} \cdots \|u_n\|_X^{r_n} \tag{2.42}$$

for all $x, y, u_2, \ldots, u_n \in X$. Then, there exists a unique additive mapping $A : X \to Y$ such that

$$\left\| f(2x) - 8f(x) - A(x), u_2, \ldots, u_n \right\|_Y \le \frac{\theta\epsilon \|x\|_X^p \|u_2\|_X^{r_2} \cdots \|u_n\|_X^{r_n}}{|(2-2^p)(k^3-k)|} \tag{2.43}$$

for all $x, u_2, \ldots, u_n \in X$, where

$$\begin{aligned}\epsilon = {} & \left(1 + |k| + 2^{3-p}|k|\right)\left[(2k+1)^p + (2k-1)^p\right] + 2|k| + 13 + 3^p + 3|k|^p + 16k^2 + 3^p|k|^p + 2^{p+1}k^2 \\ & + 2^p\left(5 + |k|^p\right) + 2|k+1|^p + 2|k-1|^p + 2^{3-p}\left(2 + |k| + |k|^p + 3^p|k|^p\right) + \frac{(|k|+1)|k+1|^p}{|k-1|} \\ & + \frac{2^{3-p}|k|}{|k-1|}|k+1|^p + \left(1 + 8k^2 + 2^pk^2\right)|k-1|^{p-1} + \frac{2^p|k|^p\left(k^2+|k|-1\right)}{|k-1|} \\ & + \frac{2}{|k-1|} + \frac{|k|\left(2^{3-p}+1\right)}{|k-1|}|3k-1|^p + \frac{8k^2+2|k|-8}{|k-1|}|k|^p.\end{aligned} \tag{2.44}$$

Proof. Define $\varphi(x,y) = \theta(\|x\|_X^p + \|y\|_X^p)\|u_2\|_X^{r_2} \cdots \|u_n\|_X^{r_n}$ for all $x, y, u_2, \ldots, u_n \in X$, and apply Theorems 2.1 and 2.2. □

The following example shows that the assumption $p \neq 1$ cannot be omitted in Corollary 2.3.

Example 2.4. Let $X = \mathbb{C}$ be a linear space over $\mathbb{R}$. Define $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ by $\|x_1, x_2\| = |a_1b_2 - a_2b_1|$, where $x_j = a_j + b_ji \in \mathbb{C}$, $a_j, b_j \in \mathbb{R}$, $j = 1, 2$ ($i = \sqrt{-1}$ is the imaginary unit). Then, $(X, \|\cdot,\cdot\|)$ is a 2-normed linear space.

Let $\phi : \mathbb{C} \to \mathbb{C}$ defined by

$$\phi(x) = \begin{cases} x, & \text{for } |x| < 1, \\ 1, & \text{for } |x| \ge 1. \end{cases} \tag{2.45}$$

Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(x) = \sum_{m=0}^{\infty} \alpha^{-m}\phi(\alpha^m x) \tag{2.46}$$

for all $x \in \mathbb{C}$, where $\alpha > |k|$. Then, f satisfies the functional inequality

$$\|Df(x,y), u\| \le \frac{4\alpha^2(|k|+1)}{\alpha - 1}(|x| + |y|)|u| \tag{2.47}$$

for all $x, y, u \in \mathbb{C}$, but there do not exist an additive mapping $A : \mathbb{C} \to \mathbb{C}$ and a constant $d > 0$ such that $\|f(x) - A(x), u\| \le d\ |x||u|$ for all $x, u \in \mathbb{C}$.

It is clear that $|f(x)| \le \alpha/(\alpha-1)$ for all $x \in \mathbb{C}$. If $|x|+|y|=0$ or $|x|+|y| \ge 1/\alpha$ for all $x, y \in \mathbb{C}$, then the inequality (2.47) holds. Now suppose that $0 < |x|+|y| < 1/\alpha$. Then, there exists an integer $n \ge 1$ such that

$$\frac{1}{\alpha^{n+1}} \le |x| + |y| < \frac{1}{\alpha^n}. \tag{2.48}$$

Hence, $\alpha^m|kx \pm y| < 1, \alpha^m|x \pm y| < 1, \alpha^m|x| < 1$ for all $m = 0, 1, \ldots, n-1$. From the definition of f and (2.48), we obtain that

$$\begin{aligned}
&\|Df(x,y),u\| \\
&\quad= \Bigg\| \sum_{m=n}^{\infty} \alpha^{-m}\phi(\alpha^m(kx+y)) + \sum_{m=n}^{\infty} \alpha^{-m}\phi(\alpha^m(kx-y)) \ - k\sum_{m=n}^{\infty} \alpha^{-m}\phi(\alpha^m(x+y)) \\
&\qquad -k\sum_{m=n}^{\infty} \alpha^{-m}\phi(\alpha^m(x-y)) \ - 2\sum_{m=n}^{\infty} \alpha^{-m}\phi(\alpha^m kx) + 2k\sum_{m=n}^{\infty} \alpha^{-m}\phi(\alpha^m x), u \Bigg\| \\
&\quad\le \frac{4\alpha^2(|k|+1)}{\alpha-1}(|x|+|y|)|u|.
\end{aligned} \tag{2.49}$$

Therefore, f satisfies (2.47). Now, we claim that the functional equation (1.1) is not stable for $p = 1$ in Corollary 2.3. Suppose on the contrary that there exist an additive mapping $A : \mathbb{C} \to \mathbb{C}$ and a constant $d > 0$ such that $\|f(x) - A(x), u\| \le d\ |x||u|$ for all $x, u \in \mathbb{C}$. Then, there exists a constant $c \in \mathbb{C}$ such that $A(x) = cx$ for all rational numbers x. So, we obtain that

$$\|f(x), u\| \le (d + |c|)\ |x||u| \tag{2.50}$$

for all rational numbers x and all $u \in \mathbb{C}$. Let $s \in \mathbb{N}$ with $s+1 > d+|c|$. If x is a rational number in $(0, \alpha^{-s})$ and $u = bi$ $(b \in \mathbb{R})$, then $\alpha^m x \in (0,1)$ for all $m = 0, 1, \ldots, s$, and we get

$$\|f(x), u\| = \left\| \sum_{m=0}^{\infty} \frac{\phi(\alpha^m x)}{\alpha^m}, u \right\| \ge \sum_{m=0}^{s} \frac{\phi(\alpha^m x)}{\alpha^m}|b| = (s+1)x|b| > (d+|c|)x|b| = (d+|c|)|x||u|, \tag{2.51}$$

which contradicts (2.50).

Theorem 2.5. *Let X be a linear space and Y an n-Banach space. Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : X^{n+1} \to [0, \infty)$ such that*

$$\sum_{j=0}^{\infty} \frac{1}{8^j}\varphi\left(2^j x, 2^j y, u_2, \ldots, u_n\right) < \infty, \tag{2.52}$$

$$\|Df(x,y), u_2, \ldots, u_n\|_Y \le \varphi(x, y, u_2, \ldots, u_n) \tag{2.53}$$

for all $x, y, u_2, \ldots, u_n \in X$. Then, there is a unique cubic mapping $C : X \to Y$ such that

$$\|f(2x) - 2f(x) - C(x), u_2, \ldots, u_n\|_Y \leq \sum_{j=0}^{\infty} \frac{1}{8^{j+1}} \widetilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right) \tag{2.54}$$

for all $x, u_2, \ldots, u_n \in X$, where $\widetilde{\varphi}(x, u_2, \ldots, u_n)$ is defined as in Theorem 2.1.

Proof. As in the proof of Theorem 2.1, we have

$$\|f(4x) - 10f(2x) + 16f(x), u_2, \ldots, u_n\|_Y \leq \widetilde{\varphi}(x, u_2, \ldots, u_n) \tag{2.55}$$

for all $x \in X$, where $\widetilde{\varphi}(x, u_2, \ldots, u_n)$ is defined as in Theorem 2.1.

Now, let $h : X \to Y$ be the mapping defined by $h(x) := f(2x) - 2f(x)$. By (2.55), we have

$$\|h(2x) - 8h(x), u_2, \ldots, u_n\|_Y \leq \widetilde{\varphi}(x, u_2, \ldots, u_n) \tag{2.56}$$

for all $x \in X$. Replacing x by $2^j x$ in (2.56) and dividing both sides of (2.56) by 8^{j+1}, we get

$$\left\|\frac{1}{8^j} h(2^j x) - \frac{1}{8^{j+1}} h(2^{j+1} x), u_2, \ldots, u_n\right\|_Y \leq \frac{1}{8^{j+1}} \widetilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right) \tag{2.57}$$

for all $x, u_2, \ldots, u_n \in X$ and all integers $j \geq 0$. For all integers l, m with $0 \leq l < m$, we have

$$\begin{aligned}\left\|\frac{1}{8^l} h(2^l x) - \frac{1}{8^m} h(2^m x), u_2, \ldots, u_n\right\|_Y &\leq \sum_{j=l}^{m-1} \left\|\frac{1}{8^j} h(2^j x) - \frac{1}{8^{j+1}} h(2^{j+1} x), u_2, \ldots, u_n\right\|_Y \\ &\leq \sum_{j=l}^{m-1} \frac{1}{8^{j+1}} \widetilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right)\end{aligned} \tag{2.58}$$

for all $x, u_2, \ldots, u_n \in X$. So, we get

$$\lim_{l,m \to \infty} \left\|\frac{1}{8^l} h(2^l x) - \frac{1}{8^m} h(2^m x), u_2, \ldots, u_n\right\|_Y = 0 \tag{2.59}$$

for all $x, u_2, \ldots, u_n \in X$. This shows that the sequence $\{(1/8^j) h(2^j x)\}$ is a Cauchy sequence in Y. Since Y is an n-Banach space, the sequence $\{(1/8^j) h(2^j x)\}$ converges. So, we can define a mapping $C : X \to Y$ by

$$C(x) := \lim_{j \to \infty} \frac{1}{8^j} h\left(2^j x\right) \tag{2.60}$$

for all $x \in X$. Putting $l = 0$, then passing the limit $m \to \infty$ in (2.58), and using Lemma 1.6(4), we get

$$\|h(x) - C(x), u_2, \ldots, u_n\|_Y \leq \sum_{j=0}^{\infty} \frac{1}{8^{j+1}} \tilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right) \tag{2.61}$$

for all $x, u_2, \ldots, u_n \in X$.

Now we show that C is cubic. By Lemma 1.6, (2.52), (2.58), and (2.60), we have

$$\begin{aligned}
\|C(2x) - 8C(x), u_2, \ldots, u_n\|_Y &= \lim_{j\to\infty} \left\| \frac{1}{8^j} h(2^{j+1}x) - \frac{1}{8^{j-1}} h(2^j x), u_2, \ldots, u_n \right\|_Y \\
&= 8 \lim_{j\to\infty} \left\| \frac{1}{8^{j+1}} h(2^{j+1}x) - \frac{1}{8^j} h(2^j x), u_2, \ldots, u_n \right\|_Y \\
&\leq \lim_{j\to\infty} \frac{1}{8^j} \tilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right) = 0
\end{aligned} \tag{2.62}$$

for all $x, u_2, \ldots, u_n \in X$. By Lemma 1.6(3), $C(2x) = 8C(x)$ for all $x \in X$. Also, by Lemma 1.6(4), (2.52), (2.53), and (2.60), we get

$$\begin{aligned}
&\left\| DC(x, y), u_2, \ldots, u_n \right\|_Y \\
&\quad = \lim_{j\to\infty} \frac{1}{8^j} \left\| Dh(2^j x, 2^j y), u_2, \ldots, u_n \right\|_Y \\
&\quad = \lim_{j\to\infty} \frac{1}{8^j} \left\| Df(2^{j+1}x, 2^{j+1}y) - 2Df(2^j x, 2^j y), u_2, \ldots, u_n \right\|_Y \\
&\quad \leq \lim_{j\to\infty} \frac{1}{8^j} \left[\left\| Df(2^{j+1}x, 2^{j+1}y), u_2, \ldots, u_n \right\|_Y + 2\left\| Df(2^j x, 2^j y), u_2, \ldots, u_n \right\|_Y \right] \\
&\quad \leq \lim_{j\to\infty} \frac{1}{8^j} \left[\varphi\left(2^{j+1}x, 2^{j+1}y, u_2, \ldots, u_n\right) + 2\varphi\left(2^j x, 2^j y, u_2, \ldots, u_n\right) \right] = 0
\end{aligned} \tag{2.63}$$

for all $x, y, u_2, \ldots, u_n \in X$. By Lemma 1.6(3), $DC(x, y) = 0$ for all $x, y \in X$. Hence the mapping C satisfies (1.1). By [11, Lemma 2.3], the mapping $x \to C(2x) - 2C(x)$ is cubic. Therefore, $C(2x) = 8C(x)$ implies that the mapping C is cubic.

To prove the uniqueness of C, let $S : X \to Y$ be another cubic mapping satisfying (2.54). Fix $x \in X$. Clearly, $C(2^l x) = 8^l A(x)$ and $S(2^l x) = 8^l S(x)$ for all $l \in \mathbb{N}$. It follows from (2.54) that

$$\begin{aligned}
\|C(x) - S(x), u_2, \ldots, u_n\|_Y &= \left\| \frac{C(2^l x)}{8^l} - \frac{S(2^l x)}{8^l}, u_2, \ldots, u_n \right\|_Y \\
&\le \frac{1}{8^l}\Big[\left\| f(2^{l+1}x) - 2f(2^l x) - C(2^l x), u_2, \ldots, u_n \right\|_Y \\
&\quad + \left\| S(2^l x) - f(2^{l+1}x) + 2f(2^l x), u_2, \ldots, u_n \right\|_Y \Big] \\
&\le \frac{1}{8^l} \sum_{j=0}^{\infty} \frac{1}{8^j} \widetilde{\varphi}\left(2^{j+l}x, u_2, \ldots, u_n\right) \\
&\le \sum_{j=0}^{\infty} \frac{1}{8^{j+l}} \widetilde{\varphi}\left(2^{j+l}x, u_2, \ldots, u_n\right) = \sum_{j=l}^{\infty} \frac{1}{8^j} \widetilde{\varphi}\left(2^{j}x, u_2, \ldots, u_n\right)
\end{aligned} \tag{2.64}$$

for all $x, u_2, \ldots, u_n \in X$, and $l \in \mathbb{N}$. By (2.52), we see that the right-hand side of the above inequality tends to 0 as $l \to \infty$. Therefore, $\|C(x) - S(x), u_2, \ldots, u_n\|_Y = 0$ for all $u_2, \ldots, u_n \in X$. By Lemma 1.6, we can conclude that $C(x) = S(x)$ for all $x \in X$. So $C = S$. This proves the uniqueness of C. □

Theorem 2.6. *Let X be a linear space and Y an n-Banach space. Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : X^{n+1} \to [0, \infty)$ such that*

$$\begin{gathered}
\sum_{j=1}^{\infty} 8^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, u_2, \ldots, u_n\right) < \infty, \\
\left\| Df(x,y), u_2, \ldots, u_n \right\|_Y \le \varphi(x, y, u_2, \ldots, u_n)
\end{gathered} \tag{2.65}$$

for all $x, y, u_2, \ldots, u_n \in X$. Then, there is a unique cubic mapping $C : X \to Y$ such that

$$\left\| f(2x) - 2f(x) - C(x), u_2, \ldots, u_n \right\|_Y \le \sum_{j=1}^{\infty} 8^{j-1} \widetilde{\varphi}\left(\frac{x}{2^j}, u_2, \ldots, u_n\right) \tag{2.66}$$

for all $x, u_2, \ldots, u_n \in X$, where $\widetilde{\varphi}(x, u_2, \ldots, u_n)$ is defined as in Theorem 2.1.

Proof. The proof is similar to the proof of Theorem 2.5. □

Corollary 2.7. *Let X be a normed space and Y an n-Banach space. Let $\theta \in [0, \infty), p, r_2, \ldots, r_n \in (0, \infty)$ such that $p \ne 3$, and let $f : X \to Y$ be a mapping with $f(0) = 0$ such that*

$$\left\| Df(x,y), u_2, \ldots, u_n \right\|_Y \le \theta\left(\|x\|_X^p + \|y\|_X^p\right) \|u_2\|_X^{r_2} \cdots \|u_n\|_X^{r_n} \tag{2.67}$$

for all $x, y, u_2, \ldots, u_n \in X$. Then, there exists a unique cubic mapping $C : X \to Y$ such that

$$\left\| f(2x) - 2f(x) - C(x), u_2, \ldots, u_n \right\|_Y \leq \frac{\theta\epsilon \|x\|_X^p \|u_2\|_X^{r_2} \cdots \|u_n\|_X^{r_n}}{|(8-2^p)(k^3-k)|} \tag{2.68}$$

for all $x, u_2, \ldots, u_n \in X$, where ϵ is defined as in Corollary 2.3.

Proof. Define $\varphi(x, y) = \theta(\|x\|_X^p + \|y\|_X^p)\|u_2\|_X^{r_2} \cdots \|u_n\|_X^{r_n}$ for all $x, y, u_2, \ldots, u_n \in X$, and apply Theorems 2.5 and 2.6. □

The following example shows that the the generalized Hyers-Ulam stability problem for the case of $p = 3$ was excluded in Corollary 2.7.

Example 2.8. Let $X = \mathbb{C}$ be a linear space over $\mathbb{R}$, and let $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ be defined as in Example 2.4. Then, $(X, \|\cdot,\cdot\|)$ is a 2-normed linear space.

Let $\phi : \mathbb{C} \to \mathbb{C}$ be defined by

$$\phi(x) = \begin{cases} x^3, & \text{for } |x| < 1, \\ 1, & \text{for } |x| \geq 1. \end{cases} \tag{2.69}$$

Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(x) = \sum_{m=0}^{\infty} \alpha^{-3m} \phi(\alpha^m x) \tag{2.70}$$

for all $x \in \mathbb{C}$, where $\alpha > |k|$. Then, f satisfies the functional inequality

$$\|Df(x, y), u\| \leq \frac{4\alpha^6(|k|+1)}{\alpha^3 - 1}\left(|x|^3 + |y|^3\right)|u| \tag{2.71}$$

for all $x, y, u \in \mathbb{C}$, but there do not exist a cubic mapping $C : \mathbb{C} \to \mathbb{C}$ and a constant $d > 0$ such that $\|f(x) - C(x), u\| \leq d\,|x|^3|u|$ for all $x, u \in \mathbb{C}$.

It is clear that $|f(x)| \leq \alpha^3/(\alpha^3 - 1)$ for all $x \in \mathbb{C}$. If $|x|^3 + |y|^3 = 0$ or $|x|^3 + |y|^3 \geq 1/\alpha^3$ for all $x, y \in \mathbb{C}$, then inequality (2.71) holds. Now suppose that $0 < |x|^3 + |y|^3 < 1/\alpha^3$. Then, there exists an integer $n \geq 1$ such that

$$\frac{1}{\alpha^{3(n+1)}} \leq |x|^3 + |y|^3 < \frac{1}{\alpha^{3n}}. \tag{2.72}$$

Hence, $\alpha^m|kx \pm y| < 1, \alpha^m|x \pm y| < 1, \alpha^m|x| < 1$ for all $m = 0, 1, \ldots, n-1$. From the definition of f and (2.72), we obtain that

$$\begin{aligned}\|Df(x,y),u\| &= \Bigg\|\sum_{m=n}^{\infty}\alpha^{-3m}\phi(\alpha^m(kx+y)) + \sum_{m=n}^{\infty}\alpha^{-3m}\phi(\alpha^m(kx-y)) - k\sum_{m=n}^{\infty}\alpha^{-3m}\phi(\alpha^m(x+y)) \\ &\quad -k\sum_{m=n}^{\infty}\alpha^{-3m}\phi(\alpha^m(x-y)) - 2\sum_{m=n}^{\infty}\alpha^{-3m}\phi(\alpha^m kx) + 2k\sum_{m=n}^{\infty}\alpha^{-3m}\phi(\alpha^m x), u\Bigg\| \\ &\leq \frac{4\alpha^6(|k|+1)}{\alpha^3-1}\left(|x|^3 + |y|^3\right)|u|.\end{aligned} \tag{2.73}$$

Therefore, f satisfies (2.71). Now, we claim that the functional equation (1.1) is not stable for $p = 3$ in Corollary 2.7. Suppose on the contrary that there exist a cubic mapping $C : \mathbb{C} \to \mathbb{C}$ and a constant $d > 0$ such that $\|f(x) - C(x), u\| \leq d\ |x|^3|u|$ for all $x, u \in \mathbb{C}$. Then, there exists a constant $\beta \in \mathbb{C}$ such that $C(x) = \beta x^3$ for all rational numbers x. So, we obtain that

$$\|f(x),u\| \leq (d + |\beta|)|x|^3|u| \tag{2.74}$$

for all rational numbers x and all $u \in \mathbb{C}$. Let $s \in \mathbb{N}$ with $s+1 > d+|\beta|$. If x is a rational number in $(0, \alpha^{-s})$ and $u = bi$ $(b \in \mathbb{R})$, then $\alpha^m x \in (0,1)$ for all $m = 0, 1, \ldots, s$, and we get

$$\begin{aligned}\|f(x),u\| &= \left\|\sum_{m=0}^{\infty}\frac{\phi(\alpha^m x)}{\alpha^{3m}}, u\right\| \geq \sum_{m=0}^{s}\frac{\phi(\alpha^m x)}{\alpha^{3m}}|b| \\ &= (s+1)x^3|b| > (d+|\beta|)x^3|b| = (d+|\beta|)|x|^3|u|,\end{aligned} \tag{2.75}$$

which contradicts (2.74).

Theorem 2.9. *Let X be a linear space and Y an n-Banach space. Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : X^{n+1} \to [0,\infty)$ such that*

$$\sum_{j=0}^{\infty}\frac{1}{2^j}\varphi\left(2^j x, 2^j y, u_2, \ldots, u_n\right) < \infty, \tag{2.76}$$

$$\|Df(x,y), u_2, \ldots, u_n\|_Y \leq \varphi(x, y, u_2, \ldots, u_n) \tag{2.77}$$

for all $x, y, u_2, \ldots, u_n \in X$. Then, there exist a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that

$$\|f(x) - A(x) - C(x), u_2, \ldots, u_n\|_Y \leq \frac{1}{6}\sum_{j=0}^{\infty}\left(\frac{1}{2^{j+1}} + \frac{1}{8^{j+1}}\right)\tilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right) \tag{2.78}$$

for all $x, u_2, \ldots, u_n \in X$, where $\tilde{\varphi}(x, u_2, \ldots, u_n)$ is defined as in Theorem 2.1.

Proof. By Theorems 2.1 and 2.5, there exist an additive mapping $A' : X \to Y$ and a cubic mapping $C' : X \to Y$ such that

$$\begin{aligned} \|f(2x) - 8f(x) - A'(x), u_2, \ldots, u_n\|_Y &\le \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} \widetilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right), \\ \|f(2x) - 2f(x) - C'(x), u_2, \ldots, u_n\|_Y &\le \sum_{j=0}^{\infty} \frac{1}{8^{j+1}} \widetilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right) \end{aligned} \tag{2.79}$$

for all $x, u_2, \ldots, u_n \in X$. Hence,

$$\left\| f(x) + \frac{1}{6}A'(x) - \frac{1}{6}C'(x), u_2, \ldots, u_n \right\|_Y \le \frac{1}{6} \sum_{j=0}^{\infty} \left(\frac{1}{2^{j+1}} + \frac{1}{8^{j+1}} \right) \widetilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right) \tag{2.80}$$

for all $x \in X$. So, we obtain (2.78) by letting $A(x) = -(1/6)A'(x)$ and $C(x) = (1/6)C'(x)$ for all $x \in X$.

To prove the uniqueness of A and C, let $A'', C'' : X \to Y$ be another additive and cubic mapping satisfying (2.78). Fix $x \in X$. Let $A_1 = A - A''$ and $C_1 = C - C''$. So,

$$\begin{aligned} &\|A_1(x) + C_1(x), u_2, \ldots, u_n\|_Y \\ &\quad \le \|f(x) - A(x) - C(x), u_2, \ldots, u_n\|_Y + \|f(x) - A''(x) - C''(x), u_2, \ldots, u_n\|_Y \\ &\quad \le \frac{1}{3} \sum_{j=0}^{\infty} \left(\frac{1}{2^{j+1}} + \frac{1}{8^{j+1}} \right) \widetilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right) \end{aligned} \tag{2.81}$$

for all $x, u_2, \ldots, u_n \in X$. Then (2.76) implies that

$$\lim_{n \to \infty} \frac{1}{8^n} \|A_1(2^n x) + C_1(2^n x), u_2, \ldots, u_n\|_Y = 0 \tag{2.82}$$

for all $x, u_2, \ldots, u_n \in X$. Thus, $C_1 = 0$. So, it follows from (2.81) that

$$\|A_1(x), u_2, \ldots, u_n\|_Y \le \frac{1}{3} \sum_{j=0}^{\infty} \left(\frac{1}{2^{j+1}} + \frac{1}{8^{j+1}} \right) \widetilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right) \tag{2.83}$$

for all $u_2, \ldots, u_n \in X$. Therefore, $A_1 = 0$. □

Similarly to Theorem 2.9, one can prove the following result.

Theorem 2.10. *Let X be a linear space and Y an n-Banach space. Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : X^{n+1} \to [0, \infty)$ such that*

$$\sum_{j=0}^{\infty} 8^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, u_2, \ldots, u_n\right) < \infty,$$
$$\|Df(x,y), u_2, \ldots, u_n\|_Y \leq \varphi(x, y, u_2, \ldots, u_n) \tag{2.84}$$

for all $x, y, u_2, \ldots, u_n \in X$. Then, there exist a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that

$$\|f(x) - A(x) - C(x), u_2, \ldots, u_n\|_Y \leq \frac{1}{6}\sum_{j=1}^{\infty}\left(2^{j-1} + 8^{j-1}\right)\tilde{\varphi}\left(\frac{x}{2^j}, u_2, \ldots, u_n\right) \tag{2.85}$$

for all $x, u_2, \ldots, u_n \in X$, where $\tilde{\varphi}(x, u_2, \ldots, u_n)$ is defined as in Theorem 2.1.

Proof. The proof is similar to the proof of Theorem 2.9 and the result follows from Theorems 2.2 and 2.6. □

Theorem 2.11. *Let X be a linear space and Y an n-Banach space. Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : X^{n+1} \to [0, \infty)$ such that*

$$\sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, u_2, \ldots, u_n\right) < \infty, \quad \sum_{j=0}^{\infty} \frac{1}{8^j}\varphi\left(2^j x, 2^j y, u_2, \ldots, u_n\right) < \infty,$$
$$\|Df(x,y), u_2, \ldots, u_n\|_Y \leq \varphi(x, y, u_2, \ldots, u_n) \tag{2.86}$$

for all $x, y, u_2, \ldots, u_n \in X$. Then, there exist a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that

$$\begin{aligned}&\|f(x) - A(x) - C(x), u_2, \ldots, u_n\|_Y \\ &\quad \leq \frac{1}{6}\left[\sum_{j=1}^{\infty} 2^{j-1}\tilde{\varphi}\left(\frac{x}{2^j}, u_2, \ldots, u_n\right) + \sum_{j=0}^{\infty}\frac{1}{8^{j+1}}\tilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right)\right]\end{aligned} \tag{2.87}$$

for all $x, u_2, \ldots, u_n \in X$, where $\tilde{\varphi}(x, u_2, \ldots, u_n)$ is defined as in Theorem 2.1.

Proof. The proof is similar to the proof of Theorem 2.9 and the result follows from Theorems 2.2 and 2.5. □

Corollary 2.12. *Let X be a normed space and Y an n-Banach space. Let $\theta \in [0, \infty), r_2, \ldots, r_n \in (0, \infty)$, $p \in (0,1) \cup (1,3) \cup (3, \infty)$, and let $f : X \to Y$ be a mapping with $f(0) = 0$ such that*

$$\|Df(x,y), u_2, \ldots, u_n\|_Y \leq \theta\left(\|x\|_X^p + \|y\|_X^p\right)\|u_2\|_X^{r_2} \cdots \|u_n\|_X^{r_n} \tag{2.88}$$

for all $x, y, u_2, \ldots, u_n \in X$. Then, there exist a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that

$$\left\| f(x) - A(x) - C(x), u_2, \ldots, u_n \right\|_Y \leq \frac{1}{6|k^3 - k|}\left(\frac{1}{|2 - 2^p|} + \frac{1}{|8 - 2^p|}\right)\theta\epsilon\|x\|_X^p\|u_2\|_X^{r_2}\cdots\|u_n\|_X^{r_n} \tag{2.89}$$

for all $x, u_2, \ldots, u_n \in X$, where ϵ is defined as in Corollary 2.3.

Proof. Define $\varphi(x, y) = \theta(\|x\|_X^p + \|y\|_X^p)\|u_2\|_X^{r_2}\cdots\|u_n\|_X^{r_n}$ for all $x, y, u_2, \ldots, u_n \in X$, and apply Theorems 2.9–2.11. □

Remark 2.13. The generalized Hyers-Ulam stability problem for the cases of $p = 1$ and $p = 3$ was excluded in Corollary 2.12 (see Examples 2.4 and 2.8).

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