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Research Article

Limit-Point/Limit-Circle Results for Equations with Damping

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The authors study the nonlinear limit-point and limit-circle properties for the second order nonlinear damped differential equation $(a(t)|y'|^{p-1}y')' + b(t)|y'|^{q-1}y' + r(t)|y|^{\lambda-1}y = 0$, where $0 < \lambda \le p \le q$, $a(t) > 0$, and $r(t) > 0$. Some examples are given to illustrate the main results.

1. Introduction

In this paper, we study the nonlinear equation

$$\left(a(t)\left|y'\right|^{p-1}y'\right)' + b(t)\left|y'\right|^{q-1}y' + r(t)\left|y\right|^{\lambda-1}y = 0 \tag{1.1}$$

and its special case

$$\left(a(t)\left|y'\right|^{p-1}y'\right)' + b(t)\left|y'\right|^{p-1}y' + r(t)\left|y\right|^{\lambda-1}y = 0. \tag{1.2}$$

We set $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{R} = (-\infty,\infty)$ and assume throughout that $0 < \lambda \le p \le q$, $a \in C^1(\mathbb{R}_+)$, $b \in C^0(\mathbb{R}_+)$, $a^{1/p}r \in C^1(\mathbb{R}_+)$, $a(t) > 0$, and $r(t) > 0$.

We will only consider solutions defined on their maximal interval of existence to the right.

Remark 1.1. The functions a, b, and r are smooth enough so that all nontrivial solutions of (1.1) defined on $\mathbb{R}_+$ are nontrivial in any neighborhood of ∞ (see Theorem 13(i) in [1]).

Moreover, if either $q = p$ or $b(t) \geq 0$ on $\mathbb{R}_+$, then all nontrivial solutions of (1.1) are defined on $\mathbb{R}_+$.

We can write (1.1) as the equivalent system

$$\begin{aligned} y_1' &= a^{-1/p}(t)|y_2|^{1/p} \operatorname{sgn} y_2, \\ y_2' &= -b(t)a^{-q/p}(t)|y_2|^{q/p} \operatorname{sgn} y_2 - r(t)|y_1|^{\lambda} \operatorname{sgn} y_1, \end{aligned} \tag{1.3}$$

where the relationship between a solution y of (1.1) and a solution (y_1, y_2) of the system (1.3) is given by

$$y_1(t) = y(t), \qquad y_2(t) = a(t)|y'(t)|^{p-1} y'(t). \tag{1.4}$$

We are interested in what is known as the nonlinear limit-point and nonlinear limit-circle properties of solutions as given in the following definition (see the monograph [2] as well as the papers [3–14]).

Definition 1.2. A solution y of (1.1) defined on $\mathbb{R}_+$ is said to be of the nonlinear limit-circle type if

$$\int_0^\infty |y(t)|^{\lambda+1} dt < \infty, \tag{NLC}$$

and it is said to be of the nonlinear limit-point type otherwise, that is, if

$$\int_0^\infty |y(t)|^{\lambda+1} dt = \infty. \tag{NLP}$$

Equation (1.1) will be said to be of the nonlinear limit-circle type if every solution y of (1.1) defined on $\mathbb{R}_+$ satisfies (NLC) and to be of the nonlinear limit-point type if there is at least one solution y for which (NLP) holds.

The properties defined above are nonlinear generalizations of the well-known linear limit-point/limit-circle properties introduced by Weyl [15] more than 100 years ago. For the history and a survey of what is known about the linear and nonlinear problems as well as their relationships to other properties of solutions such as boundedness, oscillation, and convergence to zero, we refer the reader to the monograph by Bartušek et al. [2] as well as the recent papers of Bartušek and Graef [4, 6, 9–11].

Here, we are also interested in what we call the strong nonlinear limit-point and strong nonlinear limit-circle properties of solutions of (1.1) as given in the following definitions. These notions were first introduced in [7] and [8], respectively, and further studied, for example, in [4, 6]. We define the function $R : \mathbb{R}_+ \to \mathbb{R}$ by

$$R(t) = a^{1/p}(t)r(t) \tag{1.5}$$

and let constant δ be given by

$$\delta = \frac{p+1}{p}. \tag{1.6}$$

Definition 1.3. A solution y of (1.1) defined on $\mathbb{R}_+$ is said to be of the strong nonlinear limit-point type if

$$\int_0^\infty |y(t)|^{\lambda+1} dt = \infty, \qquad \int_0^\infty \frac{|y_2(t)|^\delta}{R(t)} dt = \infty. \tag{SNLP}$$

Equation (1.1) is said to be of the strong nonlinear limit-point type if every nontrivial solution defined on $\mathbb{R}_+$ is of the strong nonlinear limit-point type and there is at least one nontrivial solution defined on $\mathbb{R}_+$.

Definition 1.4. A solution y of (1.1) defined on $\mathbb{R}_+$ is said to be of the strong nonlinear limit-circle type if

$$\int_0^\infty |y(t)|^{\lambda+1} dt < \infty, \qquad \int_0^\infty \frac{|y_2(t)|^\delta}{R(t)} dt < \infty. \tag{SNLC}$$

Equation (1.1) is said to be of the strong nonlinear limit-circle type if every solution defined on $\mathbb{R}_+$ is of the strong nonlinear limit-circle type.

From the above definitions we see that for an equation to be of the nonlinear limit-circle type, every solution must satisfy (NLC); whereas for an equation to be of the nonlinear limit-point type, there needs to be only one solution satisfying (NLP). For an equation to be of the strong nonlinear limit-point type, every solution defined on $\mathbb{R}_+$ must satisfy (SNLP) and there must be at least one such nontrivial solution.

If $b(t) \equiv 0$, (1.1) becomes

$$\left(a(t)|y'|^{p-1}y'\right)' + r(t)|y|^{\lambda-1}y = 0, \tag{1.7}$$

and moreover, if $\lambda = p$, then (1.7) reduces to the well-known half-linear equation,

$$\left(a(t)|y'|^{p-1}y'\right)' + r(t)|y|^{p-1}y = 0, \tag{1.8}$$

a general discussion of which can be found in the monograph by Došlý and Řehák [16]. Using terminology introduced by the authors in [5–7], if $\lambda > p$, we say that (1.1) is of the *super-half-linear* type, and if $\lambda < p$, we will say that it is of the *sub-half-linear* type. Since in this paper we are assuming that $\lambda \le p$, we are in the half-linear and sub-half-linear cases.

The limit-point/limit-circle problem for the damped equation

$$(a(t)y')' + b(t)y' + r(t)y^\lambda = 0 \tag{1.9}$$

with $b(t) \geq 0$ was considered in the papers [17, 18], where $\lambda \leq 1$ is the ratio of odd positive integers and $\lambda \geq 1$ is an odd integer, respectively. The results in both of these papers tend to be modifications of results in [12–14] to accommodate the damping term.

It will be convenient to define the following constants:

$$\begin{gathered}\alpha = \frac{p+1}{(\lambda+2)p+1}, \qquad \beta = \frac{(\lambda+1)p}{(\lambda+2)p+1}, \qquad \gamma = \frac{p+1}{p(\lambda+1)}, \\ \beta_1 = \frac{p}{(\lambda+2)p+1}, \qquad \omega_2 = \frac{1}{\lambda+1} + \frac{vp}{p+1}, \qquad \alpha_1 = \alpha\, \gamma^{-1/(\lambda+1)}, \\ \beta_2 = \frac{(\lambda+1)(p+1)}{p-\lambda+(v-1)p(\lambda+1)} \quad \text{for either } p > \lambda \text{ or } v > 1, \\ \omega_1 = \frac{vp+1}{p+1}, \qquad v_1 = \beta_1(v-1), \qquad \omega = \frac{1}{\lambda+1} + \frac{p}{p+1}, \qquad v = \frac{q}{p} \geq 1.\end{gathered} \tag{1.10}$$

Notice that $\alpha = 1-\beta$, $\omega_2 - 1 = 1/\beta_2$, $\omega_2 \geq \omega_1$, and $\omega \geq 1$. We define the function $g : \mathbb{R}_+ \to \mathbb{R}$ by

$$g(t) = -\frac{a^{1/p}(t) R'(t)}{R^{\alpha+1}(t)}, \tag{1.11}$$

and in the reminder of this paper we will make use of the assumption that

$$\lim_{t\to\infty} g(t) = 0, \qquad \int_0^\infty |g'(s)|\,ds < \infty. \tag{H}$$

If (H) holds, we define the constants

$$\gamma_1 = \alpha\, \gamma^{-1/(\lambda+1)} \sup_{s\in\mathbb{R}_+} |g(s)|, \qquad \gamma_2 = \delta + \gamma_1. \tag{1.12}$$

For any solution $y : \mathbb{R}_+ \to \mathbb{R}$ of (1.1), we let

$$\begin{aligned} F(t) &= R^{\beta}(t)\left[\frac{a(t)}{r(t)}|y'(t)|^{p+1} + \gamma |y(t)|^{\lambda+1}\right] \\ &= R^{\beta}(t)\left(\frac{|y_2(t)|^{\delta}}{R(t)} + \gamma |y(t)|^{\lambda+1}\right). \end{aligned} \tag{1.13}$$

Note that $F \geq 0$ on $\mathbb{R}_+$ for every solution of (1.1).

For any continuous function $h : \mathbb{R}_+ \to \mathbb{R}$, we let $h_+(t) = \max\{h(t), 0\}$ and $h_-(t) = \max\{-h(t), 0\}$ so that $h(t) = h_+(t) - h_-(t)$.

2. Lemmas

In this section we present a number of lemmas that will facilitate proving our main results.

Lemma 2.1. *For every nontrivial solution y of* (1.1) *defined on $\mathbb{R}_+$, $F(t) > 0$ for $t \geq 0$.*

Proof. Suppose, to the contrary, that (1.1) has a nontrivial solution y such that $F(t_0) = 0$ for a number $t_0 \in \mathbb{R}_+$. Then (1.13) implies $y(t_0) = y'(t_0) = 0$ and so (1.1) has the solution $\overline{y}$ defined by

$$\overline{y}(t) = y(t) \quad \text{for } t \in [0, t_0], \qquad \overline{y}(t) = 0 \quad \text{for } t \geq t_0. \tag{2.1}$$

But this contradicts Remark 1.1 and proves the lemma. □

Lemma 2.2. *Let y be a solution of* (1.1). *Then:*

(i) *for $t \in \mathbb{R}_+$, we have*

$$|y(t)| \leq \gamma^{-1/(\lambda+1)} R^{-\beta_1}(t) F^{1/(\lambda+1)}(t), \qquad |y_2(t)| \leq R^{\beta_1}(t) F^{p/(p+1)}(t). \tag{2.2}$$

(ii) *for $0 \leq \tau < t$, we have*

$$\begin{aligned} F(t) = {} & F(\tau) - \alpha g(\tau) y(\tau) y_2(\tau) + \alpha g(t) y(t) y_2(t) \\ & - \alpha \int_\tau^t g'(s) y(s) y_2(s) ds \\ & - \int_\tau^t \Big[\delta R^{-\alpha}(s) |y_2(s)|^{1/p} \operatorname{sgn} y_2(s) \\ & \qquad - \alpha g(s) y(s) \Big] \frac{b(s)}{a^v(s)} |y_2(s)|^{v-1} y_2(s) ds, \end{aligned} \tag{2.3}$$

$$\begin{aligned} & \left| \int_\tau^t \Big[\delta R^{-\alpha}(s) |y_2(s)|^{1/p} \operatorname{sgn} y_2(s) - \alpha g(s) y(s) \Big] \frac{b(s)}{a^v(s)} |y_2(s)|^{v-1} y_2(s) ds \right| \\ & \qquad \leq \delta \int_\tau^t \frac{|b(s)|}{a^v(s)} R^{v_1}(s) F^{\omega_1}(s) ds + \gamma_1 \int_\tau^t \frac{|b(s)|}{a^v(s)} R^{v_1}(s) F^{\omega_2}(s) ds. \end{aligned} \tag{2.4}$$

Proof. Let y be a solution of (1.1). Then it is a solution of the equation

$$\left(a(t) |z'|^{p-1} z' \right)' + r(t) |z|^{\lambda-1} z = e(t) \tag{2.5}$$

with $e(t) = -b(t)|y'(t)|^{q-1} y'(t) = -(b(t)/a^v(t))|y_2(t)|^{v-1} y_2(t)$. Then (2.2) and (2.3) follow from Lemma 1.2 in [5] applied to (2.5). Relation (2.4) follows from (2.2). □

The following two lemmas give us sufficient conditions for the boundedness of F from above and from bellow by positive constants.

Lemma 2.3. *Let* (H) *hold and assume that*

$$\int_0^\infty \frac{|b(s)|}{a^v(s)} R^{v_1}(s) ds < \infty. \tag{2.6}$$

Then for any nontrivial solution y *of* (1.1) *defined on* $\mathbb{R}_+$, *the function* F *is bounded from bellow on* $\mathbb{R}_+$ *by a positive constant depending on* y.

Proof. Suppose, to the contrary, that there is a nontrivial solution of (1.1) such that

$$\liminf_{t \to \infty} F(t) = 0. \tag{2.7}$$

By Lemma 2.1, $F(t) > 0$ on $\mathbb{R}_+$. Let $\bar{t} \in \mathbb{R}_+$ be such that

$$2\alpha_1 \sup_{s \in [\bar{t}, \infty)} |g(s)| + \alpha_1 \int_{\bar{t}}^\infty |g'(s)| ds + (\gamma_1 + \delta) \int_{\bar{t}}^\infty \frac{|b(s)|}{a^v(s)} R^{v_1}(s) ds \le \frac{1}{2}, \tag{2.8}$$

the existence of such a $\bar{t}$ follows from (H) and (2.6). Then, for any $t_0 \ge \bar{t}$ such that $F(t_0) \le 1$, there exist τ and σ such that $t_0 \le \sigma < \tau$ and

$$2F(\tau) = F(\sigma) = F(t_0) > 0, \qquad F(\tau) \le F(t) \le F(\sigma) \tag{2.9}$$

for $\sigma \le t \le \tau$. Then (2.2) implies

$$|y(t) y_2(t)| \le \gamma^{-1/(\lambda+1)} F^\omega(t) \tag{2.10}$$

on $\mathbb{R}_+$. From this, (2.3) (with $\tau = \sigma$ and $t = \tau$), (2.4), (2.9), and the fact that $F(\sigma) \le 1$, we have

$$\begin{aligned} \frac{F(\sigma)}{2} = F(\sigma) - F(\tau) \le& \left[\alpha_1 |g(\tau)| + \alpha_1 |g(\sigma)| + \alpha_1 \int_{\bar{t}}^\infty |g'(s)| ds\right] F^\omega(\sigma) \\ &+ \gamma_1 \int_{\bar{t}}^\infty \frac{|b(s)|}{a^v(s)} R^{v_1}(s) ds\, F^{\omega_2}(\sigma) + \delta \int_{\bar{t}}^\infty \frac{|b(s)|}{a^v(s)} R^{v_1}(s) ds\, F^{\omega_1}(\sigma). \end{aligned} \tag{2.11}$$

Hence, using (2.8) and the facts that $\omega_2 \ge \omega_1 \ge 1$, $\omega \ge 1$, and $F(\sigma) \le 1$, we obtain

$$F(\sigma) \le \frac{1}{2} F(\sigma). \tag{2.12}$$

This contradiction to $F(\sigma) > 0$ proves the lemma. □

Lemma 2.4. *Assume that* $b \ge 0$ *for large* t, (H) *holds,*

$$\int_0^\infty \frac{b(t)}{a^v(t)} R^{v_1}(t) dt < \infty, \tag{2.13}$$

and either (i) $\lambda = p = q$, or (ii) $q > \lambda$ and

$$\liminf_{t\to\infty} R^{\beta}(t)\left(\int_t^{\infty}\left[\left|g'(s)\right| + \frac{b(s)}{a^{v}(s)}R^{v_1}(s)\right]ds\right)^{\beta_2}\exp\left\{\int_0^t \frac{R'_{-}(s)}{R(s)}ds\right\} = 0. \tag{2.14}$$

Then for every solution of (1.1) *the function F is bounded on $\mathbb{R}_+$.*

Proof. Let y be a nontrivial solution of (1.1). Then according to Remark 1.1 and Lemma 2.1, y is defined on $\mathbb{R}_+$ and $F(t) > 0$ on $\mathbb{R}_+$. In view of (H) and (2.13), we can choose $\bar{t} \in \mathbb{R}_+$ such that

$$\int_{\bar{t}}^{\infty}\left[\left|g'(s)\right| + \frac{b(s)}{a^{v}(s)}R^{v_1}(s)\right]ds \le \frac{1}{2}\left[3\alpha_1 + 2^{\omega_1}\delta + 2^{\omega_2}\gamma_1\right]^{-1}. \tag{2.15}$$

Suppose that F is not bounded, that is,

$$\limsup_{t\to\infty} F(t) = \infty. \tag{2.16}$$

Then, for any $t_0 \ge \bar{t}$ with $F(t_0) \ge 1$, there exist σ and τ such that $t_0 \le \sigma < \tau$, $(1/2)F(\tau) = F(\sigma) = F(t_0)$, and

$$1 \le F(\sigma) \le F(t) \le F(\tau) \quad \text{for } \sigma \le t \le \tau. \tag{2.17}$$

Since g is of bounded variation and $\lim_{t\to\infty} g(t) = 0$, we see that

$$\left|g(\sigma)\right| = \left|g(\sigma) - g(\infty)\right| \le \int_{\sigma}^{\infty}\left|g'(s)\right|ds. \tag{2.18}$$

Setting $\tau = \sigma$ and $t = \tau$ in (2.2)–(2.4), we have (2.10) and

$$\begin{aligned} F(\sigma) = F(\tau) - F(\sigma) \le \Bigg[&\alpha_1\left|g(\sigma)\right| + \alpha_1\left|g(\tau)\right| + \alpha_1\int_{\sigma}^{\tau}\left|g'(s)\right|ds \\ &+\gamma_1\int_{\sigma}^{\tau}\frac{b(s)}{a^{v}(s)}R^{v_1}(s)ds\Bigg]F^{\omega_2}(\tau) + \delta\int_{\sigma}^{\infty}\frac{b(s)}{a^{v}(s)}R^{v_1}(s)dsF^{\omega_1}(\tau). \end{aligned} \tag{2.19}$$

From this, (2.8), (2.15), and (2.18), we obtain

$$\begin{aligned} F(\sigma) &\le \left[3\alpha_1\int_{\sigma}^{\infty}\left|g'(s)\right|ds + \left(2^{\omega_2}\gamma_1 + 2^{\omega_1}\delta\right)\int_{\sigma}^{\infty}\frac{b(s)}{a^{v}(s)}R^{v_1}(s)ds\right]F^{\omega_2}(\sigma) \\ &\le K\int_{\sigma}^{\infty}\left[\left|g'(s)\right| + \frac{b(s)}{a^{v}(s)}R^{v_1}(s)\right]ds\ F^{\omega_2}(\sigma) \le \frac{1}{2}F^{\omega_2}(\sigma), \end{aligned} \tag{2.20}$$

where $K = 3\alpha_1 + 2^{\omega_1}\delta + 2^{\omega_2}\gamma_1$.

If $\lambda = p = q$, then $\omega_2 = 1$ and (2.20) gives us a contradiction.

Now let $q > \lambda$ and (2.14) hold. Then $\omega_2 > 1$ and (2.20) implies

$$F(t_0) = F(\sigma) \geq K^{-\beta_2}\left(\int_\sigma^\infty \left[|g'(s)| + \frac{b(s)}{a^v(s)} R^{v_1}(s)\right] ds\right)^{-\beta_2}. \tag{2.21}$$

Hence,

$$F(t) \geq K_1\left(\int_t^\infty \left[|g'(s)| + \frac{b(s)}{a^v(s)} R^{v_1}(s)\right] ds\right)^{-\beta_2} \tag{2.22}$$

for all $t \geq \bar{t}$ such that $F(t) \geq 1$, where $K_1 = K^{-\beta_2}$. At the same time, (2.20) implies $F(t) \geq 2^{\beta_2} > 1$ for these values of t. Thus, (2.22) holds for all $t \geq \bar{t}$. On the other hand, if $z(t) = F(t)R^{-\beta}(t)$, then (1.13) implies

$$\begin{aligned} z'(t) &= \left(R^{-1}(t)\right)' |y_2(t)|^{\delta} - \delta r^{-1}(t) y'(t) b(t) |y'(t)|^{q} \operatorname{sgn} y'(t) \\ &\leq \left(R^{-1}(t)\right)' |y_2(t)|^{\delta} \leq \frac{R'_-(t)}{R(t)} R^{-\beta}(t) F(t) = \frac{R'_-(t)}{R(t)}\ z(t) \end{aligned} \tag{2.23}$$

for $t \geq \bar{t}$. So,

$$z(t) \leq z\left(\bar{t}\right) \exp \int_0^t \frac{R'_-(s)}{R(s)} ds. \tag{2.24}$$

From this and (2.22),

$$K_1\left\{\int_t^\infty \left[|g'(s)| + \frac{b(s)}{a^v(s)} R^{v_1}(s)\right] ds\right\}^{-\beta_2} \leq F(t) = R^{\beta}(t) z(t) \leq z\left(\bar{t}\right) R^{\beta}(t) \exp \int_0^t \frac{R'_-(s)}{R(s)} ds, \tag{2.25}$$

which contradicts (2.14). Hence, F is bounded from above on $\mathbb{R}_+$. Since $F > 0$ on $\mathbb{R}_+$, the conclusion follows. □

Lemma 2.5. *Let* (H) *and* (2.6) *hold. Then there exists a solution* y *of* (1.1) *defined on* $\mathbb{R}_+$*, a constant* $c_0 > 0$*, and* $t_0 \in \mathbb{R}_+$ *such that*

$$0 < \frac{3}{4} c_0 \leq F(t) \leq \frac{3}{2} c_0 \quad \text{for } t \geq t_0. \tag{2.26}$$

Moreover, c_0 *can be chosen arbitrary small.*

Proof. Condition (H) implies that g is bounded, so we can choose $M > 0$, $t_0 \in \mathbb{R}_+$, and c_0 such that

$$|g(t)| \leq M \quad \text{for } t \geq t_0, \qquad \int_{t_0}^{\infty} |g'(s)| ds \leq M, \qquad \int_{t_0}^{\infty} \frac{|b(s)|}{a^v(s)} R^{v_1}(s) ds \leq M,$$
$$M \leq \frac{1}{4}\left(\frac{3}{2}\right)^{-\omega_2} \left[3\alpha\gamma^{-1/(\lambda+1)} + \delta + \gamma_1\right]^{-1}, \qquad 0 < c_0 \leq \frac{2}{3}. \tag{2.27}$$

Consider a solution y of (1.1) such that $F(t_0) = c_0$. First, we will show that

$$F(t) \leq \frac{3}{2}c_0 \leq 1 \quad \text{for } t \geq t_0. \tag{2.28}$$

Suppose (2.28) does not hold. Then there exist $t_2 > t_1 \geq t_0$ such that

$$F(t_2) = \frac{3}{2}c_0, \qquad F(t_1) = c_0, \qquad c_0 < F(t) < \frac{3}{2}c_0 \tag{2.29}$$

for $t \in (t_1, t_2)$. Lemma 2.2 (with $\tau = t_1$ and $t = t_2$), and the facts that $\omega \leq \omega_2$, $\omega_1 \leq \omega_2$, and $c_0 < 1$ imply

$$\begin{aligned}\frac{c_0}{2} = \frac{F(t_1)}{2} &= F(t_2) - F(t_1) \leq 3\alpha\gamma^{-1/(\lambda+1)} M\left(\frac{3}{2}c_0\right)^{\omega} \\ &\quad + M(\delta F^{\omega_1}(t_2)) + \gamma_1 M F^{\omega_2}(t_2) \\ &\leq M\left(3\alpha\gamma^{-1/(\lambda+1)} + \delta + \gamma_1\right)\left(\frac{3}{2}\right)^{\omega_2} c_0^{\omega} \leq \frac{1}{4}c_0^{\omega}.\end{aligned} \tag{2.30}$$

Hence, $c_0^{\omega-1} \geq 2$ which contradicts the choice of c_0, and so (2.28) holds.

Now, Lemma 2.2 (with $t = t,\ \tau = t_0$) similarly implies

$$\begin{aligned}|F(t) - c_0| &\leq 3\alpha\gamma^{-1/(\lambda+1)} M\left(\frac{3}{2}c_0\right)^{\omega} + M\left[\left(\frac{3}{2}\right)^{\omega_1} \delta c_0^{\omega_1} + \left(\frac{3}{2}\right)^{\omega_2} \gamma_1 c_0^{\omega_2}\right] \\ &\leq M c_0 \left[3\left(\frac{3}{2}\right)^{\omega} \alpha\gamma^{-1/(\lambda+1)} + \left(\frac{3}{2}\right)^{\omega_1} \delta + \left(\frac{3}{2}\right)^{\omega_2} \gamma_1\right] \leq \frac{c_0}{4},\end{aligned} \tag{2.31}$$

and the statement of the lemma is proved. □

Lemma 2.6. *Suppose that* (H) *and* (2.6) *hold and*

$$\int_0^{\infty} R^{-\beta}(t) dt = \infty. \tag{2.32}$$

In addition, assume that either

$$\int_0^\infty \left(\left| \left(\frac{1}{r(s)} \right)' \right| + \frac{|b(s)|}{a^v(s)r(s)} R^{\beta_1(v-1)}(s) \right) ds < \infty \tag{2.33}$$

or

$$\int_0^\infty \left(\frac{|a'(s)|}{a(s)r(s)} + \frac{|b(s)|}{a^v(s)r(s)} R^{v_1}(s) \right) ds < \infty \tag{2.34}$$

holds. If y is a solution of (1.1) *with*

$$c_1 \le F(t) \le c_2 \tag{2.35}$$

on $\mathbb{R}_+$ for some positive constants c_1 and c_2, then

$$\int_0^\infty |y(t)|^{\lambda+1} dt = \infty. \tag{2.36}$$

Moreover, if r does not tend to zero as $t \longrightarrow \infty$, then

$$\int_0^\infty \frac{|y_2(t)|^\delta}{R(t)} dt = \infty. \tag{2.37}$$

Proof. Let y be a nontrivial solution of (1.1) satisfying (2.35). Then in view of (1.13), (2.32), and (2.35)

$$\gamma \int_0^t |y(s)|^{\lambda+1} ds + \int_0^t \frac{|y_2(s)|^\delta}{R(s)} ds = \int_0^t \frac{F(s)}{R^\beta(s)} ds \longrightarrow \infty \tag{2.38}$$

as $t \longrightarrow \infty$. Now, (2.2) and (2.35) imply

$$|y(t)y_2(t)| \le \gamma^{-1/(\lambda+1)} c_2^\omega \stackrel{\text{def}}{=} M_1 \tag{2.39}$$

for $t \ge 0$ so there exists $t_0 \ge 0$ such that

$$|g(t)| \le \frac{c_1}{2M_1 \max(1, \gamma)} \tag{2.40}$$

for $t \ge t_0$. It follows from (1.1) that

$$\begin{aligned} \int_0^t |y(s)|^{\lambda+1} ds &= -\int_0^t \frac{y(s)y_2'(s)}{r(s)} ds - \int_0^t \frac{b(s)}{a^v(s)r(s)} y(s) |y_2(s)|^{v-1} y_2(s) ds \\ &= -\frac{y(t)y_2(t)}{r(t)} + D + \int_0^t \frac{|y_2(s)|^\delta}{R(s)} ds + J(t), \end{aligned} \tag{2.41}$$

where $D = y(0)y_2(0)r^{-1}(0)$ and

$$J(t) = \int_0^t \left[\left(\frac{1}{r(s)}\right)' - \frac{b(s)}{a^v(s)r(s)}\left|y_2(s)\right|^{v-1}\right] y(s)y_2(s)ds. \tag{2.42}$$

Hence, from (2.39),

$$|J(t)| \leq M_1 D_1 \int_0^t \left[\left|\left(\frac{1}{r(s)}\right)'\right| + \frac{|b(s)|}{a^v(s)r(s)} R^{\beta_1(v-1)}(s)\right] ds \tag{2.43}$$

with $D_1 = 1 + c_2^{(q-p)/(p+1)}$ for $t \geq t_0$. Moreover, (1.13), (2.35), and (2.38) imply

$$c_3 \int_{t_0}^t R^{-\beta}(s)ds \leq \int_{t_0}^t \left|y(s)\right|^{\lambda+1} ds + \int_{t_0}^t \frac{\left|y_2(s)\right|^{\delta}}{R(s)} ds \leq c_4 \int_{t_0}^t R^{-\beta}(s)ds \tag{2.44}$$

for $t \geq t_0$ with $c_3 = c_1/\max(1,\gamma)$ and $c_4 = c_2/\min(1,\gamma)$.

If (2.33) holds, then (2.43) implies J is bounded on $\mathbb{R}_+$, and in view of (2.41), we have

$$\left|\int_{t_0}^t \left|y(s)\right|^{\lambda+1} ds - \int_{t_0}^t \frac{\left|y_2(s)\right|^{\delta}}{R(s)} ds + \frac{y(t)y_2(t)}{r(t)}\right| \leq J_1(t) + m\int_{t_0}^t R^{-\beta}(s)ds \tag{2.45}$$

for $t \geq t_0$ with $J_1(t) = |J(t)| + (|y(0)y_2(0)|/r(0))$ and $m = 0$. Note, that J_1 is bounded on $\mathbb{R}_+$.

Now let (2.34) hold. Then, using (2.2), (2.35), and (2.40) and setting $c_5 = \max(1, c_2^{(q-p)/(p+1)})$, we have

$$\begin{aligned}
&\int_{t_0}^t \left[\left|\left(\frac{1}{r(s)}\right)'\right| + \frac{|b(s)|}{a^v(s)r(s)}\left|y_2(s)\right|^{v-1}\right] ds \\
&\quad\leq \int_{t_0}^t \left(\left|\left(\frac{a^{1/p}}{R(s)}\right)'\right| + \frac{|b(s)|}{a^v(s)r(s)} R^{\beta_1(v-1)}(s) c_2^{(q-p)/(p+1)}\right) ds \\
&\quad\leq c_5 \int_0^t \left(\frac{|a'(s)|}{pa(s)r(s)} + \frac{|b(s)|}{a^v(s)r(s)} R^{\beta_1(v-1)}(s)\right) ds + \int_{t_0}^t \frac{\left|g(s)\right|}{R^{\beta}(s)} ds \\
&\quad\leq M_2 + \frac{c_3}{2M_1}\int_{t_0}^t R^{-\beta}(s)ds
\end{aligned} \tag{2.46}$$

for $t \geq t_0$, where $M_2 = c_5 \int_0^{\infty}((|a'(s)|/pa(s)r(s)) + ((|b(s)|/a^v(s)r(s))R^{\beta_1(v-1)}(s)))ds < \infty$ by (2.34). From (2.46) together with (2.39), (2.41), and (2.42), inequality (2.45) holds with

$m = c_3/2$ and $J_1(t) = M_1M_2$. Thus, (2.45) holds with $m = c_3/2$ if either (2.33) or (2.34) holds, and J_1 is bounded for $t \geq t_0$. Moreover,

$$-J_1(t) - \frac{c_3}{2}\int_{t_0}^{t} R^{-\beta}(s)ds \leq \int_{t_0}^{t} |y(s)|^{\lambda+1}ds - \int_{t_0}^{t} \frac{|y_2(s)|^{\delta}}{R(s)}ds + \frac{y(t)\ y_2(t)}{r(t)} \leq J_1(t) + \frac{c_3}{2}\int_{t_0}^{t} R^{-\beta}(s)ds \tag{2.47}$$

for $t \geq t_0$. Adding the left hand inequalities in (2.44) and (2.47) gives

$$2\int_{t_0}^{t} |y(s)|^{\lambda+1}ds + \frac{y(t)y_2(t)}{r(t)} \geq \frac{c_3}{2}\int_{t_0}^{t} R^{-\beta}(s)ds - J_1(t) \longrightarrow \infty \tag{2.48}$$

as $t \to \infty$, and subtracting gives

$$2\int_{t_0}^{t} \frac{|y_2(s)|^{\delta}}{R(s)}ds - \frac{y(t)y_2(t)}{r(t)} \geq \frac{c_3}{2}\int_{t_0}^{t} R^{-\beta}(s)ds - J_1(t) \longrightarrow \infty \tag{2.49}$$

as $t \to \infty$.

If y' is oscillatory, let $\{t_k\}_{k=1}^{\infty} \to \infty$ be a sequence of zeros of y'. Then letting $t = t_k$ in (2.48) and (2.49), it is clear that the conclusion of the lemma holds.

Let y' be nonoscillatory. Then either

$$y(t)y_2(t) > 0 \quad \text{for large } t \tag{2.50}$$

or

$$y(t)y_2(t) < 0 \quad \text{for large } t. \tag{2.51}$$

We first prove (2.36). It clearly holds if (2.50) does. So suppose (2.51) holds. Then $y(t)\ y_2(t)\ r^{-1}(t) < 0$ for large t and (2.48) gives us the contradiction. Hence, (2.36) holds.

Finally, we prove (2.37). From (2.49), (2.37) holds if (2.50) does. Let (2.51) hold and assume that (2.37) does not. Then (2.49) implies

$$\lim_{t\to\infty} \frac{y(t)\ y_2(t)}{r(t)} = -\infty. \tag{2.52}$$

In view of (2.39) and (2.51), $y(t)\ y_2(t) \geq -M_1$, so $\lim_{t\to\infty} r(t) = 0$. This contradicts the assumptions of the lemma and completes the proof. □

3. LP/LC Problem for (1.1)

In this section we present our main results for (1.1) and give some examples to illustrate them.

Theorem 3.1. *Let $b \geq 0$ for large t and assume that* (*H*) *and* (2.13) *hold. In addition, if $q > \lambda$, assume that* (2.14) *also holds. Then* (1.1) *is of the strong nonlinear limit-circle type if and only if*

$$\int_0^\infty R^{-\beta}(t)dt < \infty. \tag{3.1}$$

Proof. Let y be a nontrivial solution of (1.1). By Remark 1.1, y is defined on $\mathbb{R}_+$. The hypotheses of Lemmas 2.3 and 2.4 are satisfied, so there are constants c and c_1 such that

$$0 < c \leq F(t) \leq c_1 \tag{3.2}$$

on $\mathbb{R}_+$. Hence, from this and (1.13),

$$\begin{aligned} c\int_0^\infty R^{-\beta}(t)dt &\leq \gamma \int_0^\infty |y(t)|^{\lambda+1}dt + \int_0^\infty \frac{|y_2(t)^\delta|}{R(t)}dt = \int_0^\infty F(t)R^{-\beta}(t)dt \\ &\leq c_1 \int_0^\infty R^{-\beta}(t)dt. \end{aligned} \tag{3.3}$$

The conclusion of the theorem then follows from (3.1). □

In case $b(t) \equiv 0$, the results in this paper reduce to previously known results by the present authors except that the necessary part of Theorem 3.1 is new.

Theorem 3.2. *Let* (*H*), (2.6), *and either* (2.33) *or* (2.34) *hold. If*

$$\int_0^\infty R^{-\beta}(t)dt = \infty, \tag{3.4}$$

then (1.1) *is of the nonlinear limit-point type.*

Proof. The hypotheses of Lemmas 2.5 and 2.6 are satisfied, so if y is a solution given by Lemma 2.5, then (2.36) holds, and the conclusion follows. □

Theorem 3.3. *Let $b \geq 0$ for large t and let conditions* (*H*), (2.13), *and either* (2.33) *or* (2.34) *hold. In addition, if $q > \lambda$, assume that* (2.14) *holds. If*

$$\int_0^\infty R^{-\beta}(t)dt = \infty, \tag{3.5}$$

then every nontrivial solution of (1.1) *is of the nonlinear limit-point type. If, moreover, r does not tend to zero as $t \to \infty$, then* (1.1) *is of the strong nonlinear limit-point type.*

Proof. Note that the hypotheses of Lemmas 2.3, 2.4, and 2.6 are satisfied. Let y be a nontrivial solution of (1.1). Then Remark 1.1 implies y is defined on $\mathbb{R}_+$, and by Lemmas 2.3 and 2.4, there are positive constants C_1 and C_2 such that

$$0 < C_1 \leq F(t) \leq C_2 \quad \text{on } \mathbb{R}_+. \tag{3.6}$$

Thus, by Lemma 2.6, (2.36) holds, and if r does not tend to zero as $t \to \infty$, then (2.37) holds. This proves the theorem. □

Remark 3.4. Note that Lemmas 2.1, 2.2, and 2.6 are valid without the assumption that $\lambda \leq p$.

4. LP/LC Problem for (1.2)

One of the main assumptions in Section 3 is (2.6), which takes the form

$$\int_0^\infty \frac{|b(t)|}{a(t)} dt < \infty \tag{4.1}$$

for (1.2). It is possible to remove this condition when studying (1.2). The technique to accomplish this is contained in the following lemma; a direct computation proves the lemma.

Lemma 4.1. *Equation* (1.2) *and the equation*

$$\left(\overline{a}(t)\left|y'\right|^{p-1}y'\right)' + \overline{r}(t)\left|y\right|^{\lambda-1}y = 0 \tag{4.2}$$

are equivalent where

$$\overline{a}(t) = a(t)\exp\left\{\int_0^t \frac{b(s)}{a(s)}ds\right\}, \qquad \overline{r}(t) = r(t)\exp\left\{\int_0^t \frac{b(s)}{a(s)}ds\right\}. \tag{4.3}$$

That is, every solution of (1.2) *is a solution of* (4.2) *and vice versa.*

Based on this lemma, results for (1.2) can be obtained by combining Lemma 4.1 and known results for (4.2), such as those that can be found, for example, in [3, 5, 7, 9, 10]. Here we only state a sample of the many such possible results.

Define

$$\overline{R}(t)=\overline{a}^{1/p}(t)\overline{r}(t)=R(t)\exp\left\{\delta\int_0^t\frac{b(s)}{a(s)}ds\right\},$$
$$\overline{g}(t)=-\frac{\overline{a}^{1/p}(t)\overline{R}'(t)}{\overline{R}^{\alpha+1}(t)}=-\frac{a^{1/p}(t)}{R^{\alpha+1}(t)}\left[R'(t)+\delta\frac{b(t)}{a(t)}R(t)\right]$$
$$\times\exp\left\{\frac{\lambda-p}{(\lambda+2)p+1}\int_0^t\frac{b(s)}{a(s)}ds\right\},\tag{4.4}$$
$$\omega_3=\frac{(\lambda+1)(p+1)}{p-\lambda}\quad\text{for } p>\lambda,\qquad \omega_3=\infty\quad\text{for } p=\lambda.$$

Theorem 4.2. *Assume that*

$$\lim_{t\to\infty}\overline{g}(t)=0,\qquad\int_0^\infty\left|\overline{g}'(s)\right|ds<\infty,\tag{4.5}$$

and either (i) $\lambda=p$ or (ii) $\lambda<p$ and

$$\liminf_{t\to\infty}\overline{R}^{\beta}(t)\left(\int_t^\infty\left|\overline{g}'(s)\right|ds\right)^{\omega_3}\exp\left\{\int_0^t\left(\overline{R}^{-1}(\sigma)\right)'_+\overline{R}(\sigma)d\sigma\right\}=0.\tag{4.6}$$

Then (1.2) *is of the strong nonlinear limit-circle type if and only if*

$$\int_0^\infty\overline{R}^{-\beta}(\sigma)d\sigma<\infty.\tag{4.7}$$

Proof. The conclusion follows from Theorem 2.11 in [5] applied to (4.2) and Lemma 4.1. □

Our next result follows from Theorem 3.2 being applied to (4.2) and Lemma 4.1.

Theorem 4.3. *Let* (4.5) *and either*

$$\int_0^\infty\left|\left(\frac{1}{\overline{r}(t)}\right)'\right|dt<\infty\qquad\text{or}\quad\int_0^\infty\frac{\left|\overline{a}'(r)\right|}{\overline{a}(t)\overline{r}(t)}dt<\infty\tag{4.8}$$

hold. If

$$\int_0^\infty\overline{R}^{-\beta}(\sigma)d\sigma=\infty,\tag{4.9}$$

then (1.2) *is of the nonlinear limit-point type.*

Our final theorem is a strong nonlinear limit-point result for (1.2).

Theorem 4.4. *Assume that* (4.5) *holds and*

$$\lim_{t\to\infty}\frac{a'(t)+b(t)}{a^{1-\beta/p}(t)r(t)}\exp\left\{\frac{1}{\omega_3}\int_0^t\frac{b(s)}{a(s)}ds\right\}=0. \tag{4.10}$$

If $\int_0^\infty \overline{R}^{-\beta}(\sigma)d\sigma=\infty$, *then* (1.1) *is of the nonlinear limit-point type. If, in addition,* r *does not tend to zero as* $t\to\infty$, *then* (1.2) *is of the strong nonlinear limit-point type.*

Proof. This result follows from Theorem 2.16 in [5] and Lemma 4.1 above. Note that Theorem 2.16 in [5] is proved for $r(t)\ge r_0>0$ for $t\in\mathbb{R}_+$, but it is easy to see from (2.34) in [5] and the end of its proof that Theorem 2.16 holds as long as r does not tend to zero as $t\to\infty$. □

We conclude the paper with some examples to illustrate our main results.

Example 4.5. Consider the equation

$$\left(|y'|^{p-1}y'\right)'+b(t)|y'|^{q-1}y'+t^{\sigma}|y|^{\lambda-1}y=0,\quad t\ge 1. \tag{4.11}$$

Assume that $\sigma\alpha>-1$ and $\beta\sigma\le 1$. If b satisfies either

$$\sigma\ge 0,\qquad \int_1^\infty t^{\sigma v_1}|b(t)|dt<\infty, \tag{4.12}$$

or

$$\sigma<0,\qquad \int_1^\infty t^{\sigma(v_1-1)}|b(t)|dt<\infty, \tag{4.13}$$

then the conditions of Theorem 3.2 are satisfied, so (4.11) is of the nonlinear limit-point type.

Example 4.6. Consider the special case of (4.11) with $p=1$, namely,

$$y''+b(t)|y'|^{q-1}y'+t^{\sigma}|y|^{\lambda-1}y=0,\quad t\ge 1. \tag{4.14}$$

Then we have $0<\lambda\le 1\le q$, so if

$$-\frac{\lambda+3}{2}<\sigma\le\frac{\lambda+3}{\lambda+1} \tag{4.15}$$

and either

$$\sigma\ge 0,\qquad \int_1^\infty t^{\sigma(q-1)/(\lambda+3)}|b(t)|dt<\infty \tag{4.16}$$

or

$$\sigma < 0, \qquad \int_1^\infty t^{((q-\lambda-4)/(\lambda+3))\sigma} |b(t)| dt < \infty \tag{4.17}$$

holds, (4.14) is of the nonlinear limit-point type.

Example 4.7. Consider the equation

$$y'' + b(t)y' + t^\sigma y = 0, \quad t \geq 1, \tag{4.18}$$

with $b(t) \geq 0$. Note that here $p = q = \lambda = 1$. Assume that

$$\int_1^\infty b(t) dt < \infty. \tag{4.19}$$

Then by Theorem 3.1, (4.18) is of the strong nonlinear limit-circle type if and only if $\sigma > 2$. By Theorem 3.3, (4.18) is of the strong nonlinear limit-point type if $0 < \sigma \leq 2$. It is worth noting that this agrees with the well-known limit-circle criteria

$$\int_0^\infty r^{-1/2}(t) dt < \infty \tag{4.20}$$

of Dunford and Schwartz [19, page 1414] (also see the discussion in [2]).

For our next example we consider the case where $p = q = \lambda$. It may be convenient to refer to this case as the *fully half-linear* equation.

Example 4.8. Consider the equation

$$\left(t^a |y'|^{\epsilon-1} y'\right)' + t^b |y'|^{\epsilon-1} y' + t^\sigma |y|^{\epsilon-1} y = 0, \quad t \geq 1, \tag{4.21}$$

where $\epsilon > 0$. If $\sigma + \epsilon + 1 > a > \max\{b+1, \epsilon(1-\sigma)+1\}$, then (4.21) is of the strong nonlinear limit-circle type by Theorem 3.1. On the other hand, if $\sigma > 0$ and $b+1 < a < \min\{\sigma+\epsilon+1, \epsilon(1-\sigma)+1\}$, then (4.21) is of the strong nonlinear limit-point type by Theorem 3.3.

Our final example will illustrate several of our theorems as well allow us to compare our results to those in [17, 18].

Example 4.9. Consider the equation

$$y'' + t^s y' + t^\sigma |y|^{\lambda-1} y = 0, \quad t \geq 1, \tag{4.22}$$

with $s \in \mathbb{R}$, $\sigma \geq 0$, and $0 < \lambda \leq 1$. Calculations show the following.

(i) Equation (4.22) is of the nonlinear limit-circle type if

(a) $\lambda = 1$ and $s > -1$ (by Theorem 4.2);
(b) $\lambda < 1$ and $-1 < s < \sigma/2$ (by Theorem 4.2);
(c) $s = -1$ and $\sigma > (1-\lambda)/(\lambda+1)$ (by Theorem 4.2);
(d) $\lambda = 1$, $s < -1$, and $\sigma > (\lambda+3)/(\lambda+1)$ (by Theorem 3.1);
(e) $\lambda < 1$, $\sigma > (\lambda+3)/(\lambda+1)$, and $s < -1 - \sigma((1-\lambda)/(\lambda+3))$ (by Theorem 3.1).

(ii) Equation (4.22) is of the nonlinear limit-point type if

(f) $s = -1$ and $\sigma \leq (1-\lambda)/(\lambda+1)$ (by Theorem 4.3);
(g) $s < -1$ and $\sigma \leq (\lambda+3)/(\lambda+1)$ (by Theorem 3.2).

Now by [17, Corollary 2.3], (4.22) is of the nonlinear limit-circle type if $s \leq -\sigma(1-\lambda)/2(\lambda+3)$ and $\sigma > (\lambda+3)/(\lambda+1)$. The nonlinear limit-point result [17, Theorem 2.6] does not apply to (4.22). This shows that our results substantially extend the ones in [17] in the case of nonlinear limit-circle type results and are new in the case of nonlinear limit-point results. The results in [17] follow from ours if $s \geq -1$ and for $s < -1$ and $\lambda = 1$. There are errors in the proofs of the results in [18].

More specifically, for (4.22) with $\lambda = 1$, that is,

$$y'' + t^s y' + t^\sigma y = 0, \quad t \geq 1, \tag{4.23}$$

the results in [17, 18] show that (4.23) is of the nonlinear limit-circle type if $s \leq 0$ and $\sigma > 2$. and by results in the present paper (4.23) is of the nonlinear limit-point type if and only if

(h) $s < -1$ and $\sigma > 2$;
(i) $s = -1$ and $\sigma > 0$;
(j) $s > -1$ and σ is arbitrary.

Hence, the results in [17, 18] follow from ours, and our results are substantially better; note that we obtain necessary and sufficient condition for (4.23) to be of the nonlinear limit-circle type.

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