Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 245012, 12 pages
http://dx.doi.org/10.1155/2013/245012

Research Article

Unbounded Positive Solutions and Mann Iterative Schemes of a Second-Order Nonlinear Neutral Delay Difference Equation

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Received 12 May 2012; Accepted 12 December 2012

Academic Editor: Alexander I. Domoshnitsky

This paper is concerned with solvability of the second-order nonlinear neutral delay difference equation $\Delta^2(x_n + a_n x_{n-\tau}) + \Delta h(n, x_{h_{1n}}, x_{h_{2n}}, \ldots, x_{h_{kn}}) + f(n, x_{f_{1n}}, x_{f_{2n}}, \ldots, x_{f_{kn}}) = b_n, \forall n \geq n_0$. Utilizing the Banach fixed point theorem and some new techniques, we show the existence of uncountably many unbounded positive solutions for the difference equation, suggest several Mann-type iterative schemes with errors, and discuss the error estimates between the unbounded positive solutions and the sequences generated by the Mann iterative schemes. Four nontrivial examples are given to illustrate the results presented in this paper.

1. Introduction and Preliminaries

Recently, the oscillation, nonoscillation, asymptotic behavior, and existence of solutions of different classes of linear and nonlinear second-order difference equations have been studied by many authors; see, for example, [1–26] and the references cited therein. Using the Banach fixed point theorem, Jinfa [5] discussed the existence of a bounded nonoscillatory solution for the second-order neutral delay difference equation with positive and negative coefficients:

$$\Delta^2 (x_n + p x_{n-m}) + p_n x_{n-k} - q_n x_{n-l} = 0, \quad \forall n \geq n_0 \tag{1}$$

under the condition $p \neq -1$. Luo and Bainov [13] and M. Migda and J. Migda [16] considered the asymptotic behaviors of nonoscillatory solutions for the second-order neutral difference equation with maxima:

$$\Delta^2 (x_n + p_n x_{n-k}) + q_n \max \{x_s : n-l \leq s \leq n\} = 0, \quad \forall n \geq 1 \tag{2}$$

and the second-order neutral difference equation:

$$\Delta^2 (x_n + p x_{n-k}) + f (n, x_n) = 0, \quad \forall n \geq 1, \tag{3}$$

respectively. Meng and Yan [15] studied the existence of bounded nonoscillatory solutions for the second-order nonlinear nonautonomous neutral delay difference equation:

$$\Delta^2 (x_n - p x_{n-\tau}) = \sum_{n=1}^{m} q_i x_{n-\sigma_i} + f \left(n, x_{n-\eta_{1n}}, \ldots, x_{n-\eta_{ln}}\right), \quad \forall n \geq n_0. \tag{4}$$

Applying the cone compression and expansion theorem in Fréchet spaces, Tian and Ge [21] established the existence of multiple positive solutions of the second-order discrete equation on the half-line:

$$\Delta^2 x_{n-1} - p \Delta x_{n-1} - q x_{n-1} + f (n, x_n) = 0, \quad \forall n \geq 1 \tag{5}$$

with certain boundary value conditions. But to the best of our knowledge, results on multiplicity of unbounded solutions for neutral delay difference equations are very scarce in the literature. Nothing has been done with the existence of uncountably many unbounded positive solutions for (1)~(5) and any other second-order neutral delay difference equations:

Inspired and motivated by the results in [1–26], in this paper we introduce and study the second-order nonlinear neutral delay difference equation:

$$\Delta^2 \left(x_n + a_n x_{n-\tau}\right) + \Delta h\left(n, x_{h_{1n}}, x_{h_{2n}}, \dots, x_{h_{kn}}\right) + f\left(n, x_{f_{1n}}, x_{f_{2n}}, \dots, x_{f_{kn}}\right) = b_n, \quad \forall n \geq n_0, \tag{6}$$

where $\tau, k, n_0 \in \mathbb{N}$, $\{a_n\}_{n\in\mathbb{N}_{n_0}}, \{b_n\}_{n\in\mathbb{N}_{n_0}} \subset \mathbb{R}$, $h, f \in C(\mathbb{N}_{n_0} \times \mathbb{R}^k, \mathbb{R})$, $\{h_{ln}\}_{n\in\mathbb{N}_{n_0}}, \{f_{ln}\}_{n\in\mathbb{N}_{n_0}} \subseteq \mathbb{N}$, and

$$\lim_{n\to\infty} h_{ln} = \lim_{n\to\infty} f_{ln} = +\infty, \quad l \in \{1, 2, \dots, k\}. \tag{7}$$

By means of the Banach fixed point theorem and some new techniques, we establish sufficient conditions for the existence of uncountably many unbounded positive solutions of (6), suggest a few Mann iterative schemes with errors for approximating these unbounded positive solutions, and prove their convergence and the error estimates. The results obtained in this paper extend the result in [5]. Four nontrivial examples are interested in the text to illustrate the importance of our results.

Throughout this paper, we assume that Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{Z}$, $\mathbb{N}_0$, and $\mathbb{N}$ denote the sets of all integers, nonnegative integers, and positive integers, respectively,

$$\begin{gathered} \mathbb{N}_t = \{n : n \in \mathbb{N} \text{ with } n \geq t\}, \quad \forall t \in \mathbb{N}, \\ \beta = \min\left\{n_0 - \tau, \inf\left\{h_{ln}, f_{ln} : 1 \leq l \leq k, n \in \mathbb{N}_{n_0}\right\}\right\} \in \mathbb{N}, \\ H_n = \max\{h_{ln} : l \in \{1, 2, \dots, k\}\}, \\ F_n = \max\{f_{ln} : l \in \{1, 2, \dots, k\}\}, \quad \forall n \in \mathbb{N}_{n_0}, \end{gathered} \tag{8}$$

l_β^∞ represents the Banach space of all real sequences on $\mathbb{N}_\beta$ with norm

$$\begin{gathered} \|x\| = \sup_{n\in\mathbb{N}_\beta} \left|\frac{x_n}{n}\right| < +\infty \quad \text{for each } x = \{x_n\}_{n\in\mathbb{N}_\beta} \in l_\beta^\infty, \\ A(N, M) = \left\{x = \{x_n\}_{n\in\mathbb{N}_\beta} \in l_\beta^\infty : N \leq \frac{x_n}{n} \leq M, \ n \in \mathbb{N}_\beta\right\} \\ \text{for any } M > N > 0. \end{gathered} \tag{9}$$

It is clear that $A(N, M)$ is a closed and convex subset of l_β^∞. By a solution of (6), we mean a sequence $\{x_n\}_{n\in\mathbb{N}_\beta}$ with a positive integer $T \geq n_0 + \tau + \beta$ such that (6) holds for all $n \geq T$.

The following lemmas play important roles in this paper.

Lemma 1 (see [27]). *Let $\{\alpha_n\}_{n\in\mathbb{N}_0}$, $\{\beta_n\}_{n\in\mathbb{N}_0}$, $\{\gamma_n\}_{n\in\mathbb{N}_0}$, and $\{t_n\}_{n\in\mathbb{N}_0}$ be four nonnegative sequences satisfying the inequality*

$$\alpha_{n+1} \leq (1 - t_n)\alpha_n + t_n\beta_n + \gamma_n, \quad \forall n \in \mathbb{N}_0, \tag{10}$$

where $\{t_n\}_{n\in\mathbb{N}_0} \subset [0, 1]$, $\sum_{n=0}^\infty t_n = +\infty$, $\lim_{n\to\infty}\beta_n = 0$, and $\sum_{n=0}^\infty \gamma_n < +\infty$. Then $\lim_{n\to\infty}\alpha_n = 0$.

Lemma 2 (see [11]). *Let $\tau, n_0 \in \mathbb{N}$ and $\{b_n\}_{n\in\mathbb{N}_{n_0}}$ be a nonnegative sequence. Then*

(a) $\sum_{i=0}^\infty \sum_{s=n_0+i\tau}^\infty b_s < +\infty \Leftrightarrow \sum_{s=n_0}^\infty s b_s < +\infty$;

(b) $\sum_{i=0}^\infty \sum_{s=n_0+i\tau}^\infty \sum_{t=s}^\infty b_t < +\infty \Leftrightarrow \sum_{s=n_0}^\infty \sum_{t=s}^\infty s b_t < +\infty$.

2. Existence of Uncountably Many Unbounded Positive Solutions

Using the Banach fixed point theorem and the Mann iterative schemes with errors, we next discuss the existence of uncountably many unbounded positive solutions of (6), prove that the Mann iterative schemes with errors converge to these unbounded positive solutions, and compute the error estimates between the Mann iterative schemes with errors and the unbounded positive solutions.

Theorem 3. *Assume that there exist two constants M and N with $M > N > 0$ and four nonnegative sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$, $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$, $\{R_n\}_{n\in\mathbb{N}_{n_0}}$, and $\{W_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying*

$$\begin{gathered} \left|f(n, u_1, u_2, \dots, u_k) - f(n, \overline{u}_1, \overline{u}_2, \dots, \overline{u}_k)\right| \\ \leq P_n \max\{|u_l - \overline{u}_l| : 1 \leq l \leq k\}, \\ \left|h(n, u_1, u_2, \dots, u_k) - h(n, \overline{u}_1, \overline{u}_2, \dots, \overline{u}_k)\right| \\ \leq R_n \max\{|u_l - \overline{u}_l| : 1 \leq l \leq k\}, \\ \forall (n, u_l, \overline{u}_l) \in \mathbb{N}_{n_0} \times (\mathbb{R}^+ \setminus \{0\})^2, \quad 1 \leq l \leq k, \end{gathered} \tag{11}$$

$$\begin{gathered} \left|f(n, u_1, u_2, \dots, u_k)\right| \leq Q_n, \quad \left|h(n, u_1, u_2, \dots, u_k)\right| \leq W_n, \\ \forall (n, u_l) \in \mathbb{N}_{n_0} \times (\mathbb{R}^+ \setminus \{0\}), \quad 1 \leq l \leq k, \end{gathered} \tag{12}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^\infty \sum_{s=n+i\tau}^\infty \max\{R_s H_s, W_s\} = 0, \tag{13}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^\infty \sum_{s=n+i\tau}^\infty \sum_{t=s}^\infty \max\{P_t F_t, Q_t, |b_t|\} = 0, \tag{14}$$

$$a_n = -1 \quad \textit{eventually}. \tag{15}$$

Then

(a) for any $L \in (N, M)$, *there exist* $\theta \in (0,1)$ *and* $T \geq n_0 + \tau + \beta$ *such that for each* $x_0 = \{x_{0n}\}_{n\in\mathbb{N}_\beta} \in A(N,M)$, *the Mann iterative sequence with errors* $\{x_m\}_{m\in\mathbb{N}_0} = \{x_{mn}\}_{(m,n)\in\mathbb{N}_0\times\mathbb{N}_\beta}$ *generated by the scheme*

$$x_{m+1n} = \begin{cases} (1-\alpha_m-\beta_m)\,x_{mn} \\ \quad +\alpha_m \Big\{ nL \\ \quad\quad -\displaystyle\sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\Big[h\left(s, x_{mh_{1s}}, x_{mh_{2s}}, \ldots, x_{mh_{ks}}\right) \\ \quad\quad\quad -\displaystyle\sum_{t=s}^{\infty}\left(f\left(t, x_{mf_{1t}}, x_{mf_{2t}}, \ldots, x_{mf_{kt}}\right)-b_t\right)\Big]\Big\} \\ \quad +\beta_m\gamma_{mn}, \qquad n \geq T,\ m \geq 0, \\ (1-\alpha_m-\beta_m)\,x_{mT} \\ \quad +\alpha_m \Big\{ TL \\ \quad\quad -\displaystyle\sum_{i=1}^{\infty}\sum_{s=T+i\tau}^{\infty}\Big[h\left(s, x_{mh_{1s}}, x_{mh_{2s}}, \ldots, x_{mh_{ks}}\right) \\ \quad\quad\quad -\displaystyle\sum_{t=s}^{\infty}\left(f\left(t, x_{mf_{1t}}, x_{mf_{2t}}, \ldots, x_{mf_{kt}}\right)-b_t\right)\Big]\Big\} \\ \quad +\beta_m\gamma_{mT}, \qquad \beta \leq n < T,\ m \geq 0 \end{cases} \tag{16}$$

converges to an unbounded positive solution $x \in A(N,M)$ *of* (6) *and has the following error estimate:*

$$\|x_{m+1}-x\| \leq [1-(1-\theta)\,\alpha_m]\,\|x_m - x\| + 2M\beta_m, \quad \forall m \in \mathbb{N}_0, \tag{17}$$

where $\{\gamma_m\}_{m\in\mathbb{N}_0} = \{\gamma_{mn}\}_{(m,n)\in\mathbb{N}_0\times\mathbb{N}_\beta}$ *is an arbitrary sequence in* $A(N,M)$ *and* $\{\alpha_m\}_{m\in\mathbb{N}_0}$ *and* $\{\beta_m\}_{m\in\mathbb{N}_0}$ *are any sequences in* $[0,1]$ *such that*

$$\begin{gathered} \sum_{m=0}^{\infty}\alpha_m = +\infty, \\ \sum_{m=0}^{\infty}\beta_m < +\infty \text{ or there exists a sequence } \\ \{\xi_m\}_{m\in\mathbb{N}_0} \subset [0,+\infty) \text{ satisfying} \\ \beta_m = \xi_m\alpha_m, \quad m \in \mathbb{N}_0, \quad \lim_{m\to\infty}\xi_m = 0; \end{gathered} \tag{18}$$

(b) equation (6) *possesses uncountably many unbounded positive solutions in* $A(N,M)$.

Proof. First of all we show that (a) holds. Set $L \in (N,M)$. It follows from (13), (14), and (15) that there exist $\theta \in (0,1)$ and $T \geq n_0 + \tau + \beta$ satisfying

$$\theta = \frac{1}{T}\sum_{i=1}^{\infty}\sum_{s=T+i\tau}^{\infty}\left(R_sH_s + \sum_{t=s}^{\infty}P_tF_t\right), \tag{19}$$

$$\frac{1}{T}\sum_{i=1}^{\infty}\sum_{s=T+i\tau}^{\infty}\left(W_s + \sum_{t=s}^{\infty}Q_t + \sum_{t=s}^{\infty}|b_t|\right) < \min\{M-L, L-N\}, \tag{20}$$

$$a_n = -1, \quad \forall n \geq T. \tag{21}$$

Define a mapping $S_L : A(N,M) \to l_\beta^\infty$ by

$$S_Lx_n = \begin{cases} nL \\ \quad -\displaystyle\sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\Big\{h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right) \\ \quad\quad -\displaystyle\sum_{t=s}^{\infty}\left[f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right)-b_t\right]\Big\}, \\ \qquad\qquad n \geq T, \\ S_Lx_T, \qquad \beta \leq n < T \end{cases} \tag{22}$$

for each $x = \{x_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$. In view of (11), (12), (19), (20), and (22), we deduce that for each $x = \{x_n\}_{n\in\mathbb{N}_\beta}$, $y = \{y_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$ and for all $n \geq T$

$$\begin{aligned} \left|\frac{S_Lx_n}{n} - \frac{S_Ly_n}{n}\right| &\leq \frac{1}{n}\sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\Big[\big|h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right) \\ &\quad - h\left(s, y_{h_{1s}}, y_{h_{2s}}, \ldots, y_{h_{ks}}\right)\big| \\ &\quad + \sum_{t=s}^{\infty}\big|f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right) \\ &\quad - f\left(t, y_{f_{1t}}, y_{f_{2t}}, \ldots, y_{f_{kt}}\right)\big|\Big] \\ &\leq \frac{1}{n}\sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\Big[R_s\max\left\{\left|x_{h_{ls}} - y_{h_{ls}}\right| : 1 \leq l \leq k\right\} \\ &\quad + \sum_{t=s}^{\infty}P_t\max\left\{\left|x_{f_{lt}} - y_{f_{lt}}\right| : 1 \leq l \leq k\right\}\Big] \end{aligned}$$

$$\leq \frac{\|x-y\|}{n} \sum_{i=1}^{\infty} \sum_{s=n+i\tau}^{\infty} \left[R_s \max\{h_{ls} : 1 \leq l \leq k\} + \sum_{t=s}^{\infty} P_t \max\{f_{lt} : 1 \leq l \leq k\} \right]$$
$$\leq \frac{\|x-y\|}{T} \sum_{i=1}^{\infty} \sum_{s=T+i\tau}^{\infty} \left(R_s H_s + \sum_{t=s}^{\infty} P_t F_t \right)$$
$$= \theta \|x-y\|,$$

$$\left| \frac{S_L x_n}{n} - L \right|$$
$$= \left| \frac{1}{n} \sum_{i=1}^{\infty} \sum_{s=n+i\tau}^{\infty} \left\{ h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right) - \sum_{t=s}^{\infty} \left[f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right) - b_t \right] \right\} \right|$$
$$\leq \frac{1}{n} \sum_{i=1}^{\infty} \sum_{s=n+i\tau}^{\infty} \left\{ \left| h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right) \right| + \sum_{t=s}^{\infty} \left[\left| f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right) \right| + |b_t| \right] \right\}$$
$$\leq \frac{1}{T} \sum_{i=1}^{\infty} \sum_{s=T+i\tau}^{\infty} \left[W_s + \sum_{t=s}^{\infty} (Q_t + |b_t|) \right]$$
$$< \min\{M-L, L-N\}, \tag{23}$$

which yield that

$$S_L(A(N,M)) \subseteq A(N,M), \qquad \|S_L x - S_L y\| \leq \theta \|x-y\|, \quad \forall x, y \in A(N,M), \tag{24}$$

which means that S_L is a contraction in $A(N,M)$. It follows from the Banach fixed point theorem that S_L has a unique fixed point $x = \{x_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$, that is,

$$x_n = nL - \sum_{i=1}^{\infty} \sum_{s=n+i\tau}^{\infty} \left\{ h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right) - \sum_{t=s}^{\infty} \left[f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right) - b_t \right] \right\}, \quad \forall n \geq T,$$

$$x_{n-\tau} = (n-\tau)L - \sum_{i=1}^{\infty} \sum_{s=n+(i-1)\tau}^{\infty} \left\{ h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right) - \sum_{t=s}^{\infty} \left[f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right) - b_t \right] \right\}, \quad \forall n \geq T+\tau, \tag{25}$$

which imply that

$$x_n - x_{n-\tau} = \tau L + \sum_{s=n}^{\infty} \left\{ h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right) - \sum_{t=s}^{\infty} \left[f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right) - b_t \right] \right\}, \quad \forall n \geq T+\tau, \tag{26}$$

which yields that

$$\Delta(x_n - x_{n-\tau}) = -h\left(n, x_{h_{1n}}, x_{h_{2n}}, \ldots, x_{h_{kn}}\right) + \sum_{t=n}^{\infty} \left[f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right) - b_t \right], \quad \forall n \geq T+\tau,$$
$$\Delta^2(x_n - x_{n-\tau}) = -\Delta h\left(n, x_{h_{1n}}, x_{h_{2n}}, \ldots, x_{h_{kn}}\right) - f\left(n, x_{f_{1n}}, x_{f_{2n}}, \ldots, x_{f_{kn}}\right) + b_n, \quad \forall n \geq T+\tau, \tag{27}$$

which together with (21) gives that $x = \{x_n\}_{n\in\mathbb{N}_\beta}$ is an unbounded positive solution of (6) in $A(N,M)$. It follows from (16), (19), (21), (22), and (24) that for any $m \in \mathbb{N}_0$ and $n \geq T$

$$\left| \frac{x_{m+1n}}{n} - \frac{x_n}{n} \right|$$
$$= \frac{1}{n} \left| (1 - \alpha_m - \beta_m) x_{mn} \right.$$

$$+\alpha_m\Bigg\{nL-\sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\Bigg[h\left(s,x_{mh_{1s}},x_{mh_{2s}},\ldots,x_{mh_{ks}}\right)-\sum_{t=s}^{\infty}\left(f\left(t,x_{mf_{1t}},x_{mf_{2t}},\ldots,x_{mf_{kt}}\right)-b_t\right)\Bigg]\Bigg\}+\beta_m\gamma_{mn}-x_n\Bigg|$$
$$\le(1-\alpha_m-\beta_m)\frac{|x_{mn}-x_n|}{n}+\alpha_m\frac{|S_Lx_{mn}-S_Lx_n|}{n}+\beta_m\frac{|\gamma_{mn}-x_n|}{n}$$
$$\le(1-\alpha_m-\beta_m)\|x_m-x\|+\theta\alpha_m\|x_m-x\|+2M\beta_m$$
$$\le[1-(1-\theta)\alpha_m]\|x_m-x\|+2M\beta_m, \tag{28}$$

which implies that

$$\|x_{m+1}-x\|\le[1-(1-\theta)\alpha_m]\|x_m-x\|+2M\beta_m,\quad \forall m\in\mathbb{N}_0. \tag{29}$$

That is, (17) holds. Thus, Lemma 1 and (17) and (18) guarantee that $\lim_{m\to\infty}x_m=x$.

Next we show that (b) holds. Let $L_1,L_2\in(N,M)$ and $L_1\ne L_2$. As in the proof of (a), we deduce similarly that for each $c\in\{1,2\}$ there exist constants $\theta_c\in(0,1)$, $T_c\ge n_0+\tau+\beta$, and a mapping S_{L_c} satisfying (19)~(24), where θ, L, and T are replaced by θ_c, L_c and T_c, respectively, and the mapping S_{L_c} has a fixed point $z^c=\{z_n^c\}_{n\in\mathbb{N}_\beta}\in A(N,M)$, which is an unbounded positive solution of (6) in $A(N,M)$, that is,

$$z_n^c=nL_c-\sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\Bigg\{h\left(s,z^c_{h_{1s}},z^c_{h_{2s}},\ldots,z^c_{h_{ks}}\right)-\sum_{t=s}^{\infty}\left[f\left(t,z^c_{f_{1t}},z^c_{f_{2t}},\ldots,z^c_{f_{kt}}\right)-b_t\right]\Bigg\},\quad\forall n\ge T_c, \tag{30}$$

which together with (11) and (17) implies that for $n\ge\max\{T_1,T_2\}$

$$\left|\frac{z_n^1}{n}-\frac{z_n^2}{n}\right|$$
$$\ge|L_1-L_2|-\frac{1}{n}\sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\Bigg\{\left|h\left(s,z^1_{h_{1s}},z^1_{h_{2s}},\ldots,z^1_{h_{ks}}\right)-h\left(s,z^2_{h_{1s}},z^2_{h_{2s}},\ldots,z^2_{h_{ks}}\right)\right|+\sum_{t=s}^{\infty}\left|f\left(t,z^1_{f_{1t}},z^1_{f_{2t}},\ldots,z^1_{f_{kt}}\right)-f\left(t,z^2_{f_{1t}},z^2_{f_{2t}},\ldots,z^2_{f_{kt}}\right)\right|\Bigg\}$$
$$\ge|L_1-L_2|-\frac{1}{n}\sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\Bigg[R_s\max\left\{\left|z^1_{h_{ls}}-z^2_{h_{ls}}\right|:1\le l\le k\right\}+\sum_{t=s}^{\infty}P_t\max\left\{\left|z^1_{f_{lt}}-z^2_{f_{lt}}\right|:1\le l\le k\right\}\Bigg]$$
$$\ge|L_1-L_2|-\frac{\|z^1-z^2\|}{n}\sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\left(R_sH_s+\sum_{t=s}^{\infty}P_tF_t\right)$$
$$\ge|L_1-L_2|-\frac{\|z^1-z^2\|}{\max\{T_1,T_2\}}\times\sum_{i=1}^{\infty}\sum_{s=\max\{T_1,T_2\}+i\tau}^{\infty}\left(R_sH_s+\sum_{t=s}^{\infty}P_tF_t\right)$$
$$\ge|L_1-L_2|-\max\{\theta_1,\theta_2\}\|z^1-z^2\|, \tag{31}$$

which yields that

$$\|z^1-z^2\|\ge\frac{|L_1-L_2|}{1+\max\{\theta_1,\theta_2\}}>0, \tag{32}$$

that is, $z^1\ne z^2$. This completes the proof. □

Theorem 4. *Assume that there exist two constants M and N with $M>N>0$ and four nonnegative sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$, $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$, $\{R_n\}_{n\in\mathbb{N}_{n_0}}$, and $\{W_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (11), (12),

$$\lim_{n\to\infty}\frac{1}{n}\sum_{s=n}^{\infty}\max\{R_sH_s,W_s\}=0, \tag{33}$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\max\{P_tF_t,Q_t,|b_t|\}=0, \tag{34}$$

$$a_n=1\quad\textit{eventually}. \tag{35}$$

Then

(a) for any $L \in (N, M)$, there exist $\theta \in (0,1)$ and $T \geq n_0 + \tau + \beta$ such that for each $x_0 = \{x_{0n}\}_{n\in\mathbb{N}_\beta} \in A(N, M)$, the Mann iterative sequence with errors $\{x_m\}_{m\in\mathbb{N}_0} = \{x_{mn}\}_{(m,n)\in\mathbb{N}_0\times\mathbb{N}_\beta}$ generated by the scheme

$$x_{m+1n} = \begin{cases} (1-\alpha_m-\beta_m)\, x_{mn} \\ \quad +\alpha_m \Big\{ nL \\ \qquad + \sum_{i=1}^{\infty} \sum_{s=n+(2i-1)\tau}^{n+2i\tau-1} \Big[h\left(s, x_{mh_{1s}}, x_{mh_{2s}}, \ldots, x_{mh_{ks}}\right) \\ \qquad\quad - \sum_{t=s}^{\infty} \left(f\left(t, x_{mf_{1t}}, x_{mf_{2t}}, \ldots, x_{mf_{kt}}\right) - b_t \right) \Big] \Big\} \\ \quad + \beta_m \gamma_{mn}, \qquad n \geq T,\ m \geq 0, \\ (1-\alpha_m-\beta_m)\, x_{mT} \\ \quad +\alpha_m \Big\{ TL \\ \qquad + \sum_{i=1}^{\infty} \sum_{s=T+(2i-1)\tau}^{T+2i\tau-1} \Big[h\left(s, x_{mh_{1s}}, x_{mh_{2s}}, \ldots, x_{mh_{ks}}\right) \\ \qquad\quad - \sum_{t=s}^{\infty} \left(f\left(t, x_{mf_{1t}}, x_{mf_{2t}}, \ldots, x_{mf_{kt}}\right) - b_t \right) \Big] \Big\} \\ \quad + \beta_m \gamma_{mT}, \qquad \beta \leq n < T,\ m \geq 0 \end{cases} \tag{36}$$

converges to an unbounded positive solution $x \in A(N, M)$ of (6) and has the error estimate (17), where $\{\gamma_m\}_{m\in\mathbb{N}_0} = \{\gamma_{mn}\}_{(m,n)\in\mathbb{N}_0\times\mathbb{N}_\beta}$ is an arbitrary sequence in $A(N, M)$, and $\{\alpha_m\}_{m\in\mathbb{N}_0}$ and $\{\beta_m\}_{m\in\mathbb{N}_0}$ are any sequences in $[0, 1]$ satisfying (18);

(b) equation (6) possesses uncountably many unbounded positive solutions in $A(N, M)$.

Proof. Let $L \in (N, M)$. It follows from (33)~(35) that there exist $\theta \in (0, 1)$ and $T \geq n_0 + \tau + \beta$ satisfying

$$\theta = \frac{1}{T} \sum_{s=T}^{\infty} \left(R_s H_s + \sum_{t=s}^{\infty} P_t F_t \right), \tag{37}$$

$$\frac{1}{T} \sum_{s=T}^{\infty} \left(W_s + \sum_{t=s}^{\infty} \left(Q_t + |b_t|\right) \right) < \min\{M-L, L-N\}, \tag{38}$$

$$a_n = 1, \quad \forall n \geq T. \tag{39}$$

Define a mapping $S_L : A(N, M) \to l_\beta^\infty$ by

$$S_L x_n = \begin{cases} nL \\ \quad + \sum_{i=1}^{\infty} \sum_{s=n+(2i-1)\tau}^{n+2i\tau-1} \Big\{ h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right) \\ \qquad - \sum_{t=s}^{\infty} \left[f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right) - b_t \right] \Big\}, \\ \qquad\qquad n \geq T, \\ S_L x_T, \qquad \beta \leq n < T \end{cases} \tag{40}$$

for each $x = \{x_n\}_{n\in\mathbb{N}_\beta} \in A(N, M)$. Using (11), (12), (37), (38), and (40), we get that for each $x = \{x_n\}_{n\in\mathbb{N}_\beta}$, $y = \{y_n\}_{n\in\mathbb{N}_\beta} \in A(N, M)$ and $n \geq T$

$$\begin{aligned} &\left| \frac{S_L x_n}{n} - \frac{S_L y_n}{n} \right| \\ &\quad \leq \frac{1}{n} \sum_{i=1}^{\infty} \sum_{s=n+(2i-1)\tau}^{n+2i\tau-1} \Big[\left| h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right) - h\left(s, y_{h_{1s}}, y_{h_{2s}}, \ldots, y_{h_{ks}}\right) \right| \\ &\qquad + \sum_{t=s}^{\infty} \left| f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right) - f\left(t, y_{f_{1t}}, y_{f_{2t}}, \ldots, y_{f_{kt}}\right) \right| \Big] \\ &\quad \leq \frac{\|x-y\|}{n} \sum_{i=1}^{\infty} \sum_{s=n+(2i-1)\tau}^{n+2i\tau-1} \left(R_s H_s + \sum_{t=s}^{\infty} P_t F_t \right) \\ &\quad \leq \frac{\|x-y\|}{T} \sum_{s=T}^{\infty} \left(R_s H_s + \sum_{t=s}^{\infty} P_t F_t \right) \\ &\quad = \theta \|x-y\|, \\ &\left| \frac{S_L x_n}{n} - L \right| \\ &\quad \leq \frac{1}{n} \\ &\qquad \times \sum_{i=1}^{\infty} \sum_{s=n+(2i-1)\tau}^{n+2i\tau-1} \Big\{ \left| h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right) \right| \\ &\qquad + \sum_{t=s}^{\infty} \left[\left| f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right) \right| + |b_t| \right] \Big\} \\ &\quad \leq \frac{1}{T} \sum_{i=T}^{\infty} \left[W_s + \sum_{t=s}^{\infty} \left(Q_t + |b_t|\right) \right] \\ &\quad < \min\{M-L, L-N\}, \end{aligned} \tag{41}$$

which imply (24). Consequently, (24) means that S_L is a contraction in $A(N,M)$ and has a unique fixed point $x = \{x_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$, which is also an unbounded positive solution of (6) in $A(N,M)$. The rest of the proof is similar to the proof of Theorem 3 and is omitted. This completes the proof. □

Theorem 5. *Assume that there exist three constants a, M, and N with $(1-a)M > N > 0$ and four nonnegative sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$, $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$, $\{R_n\}_{n\in\mathbb{N}_{n_0}}$, and $\{W_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (11), (12), (33), (34) *and*

$$0 \le a_n \le a < 1 \quad \text{eventually}. \tag{42}$$

Then

(a) for any $L \in (aM+N, M)$, there exist $\theta \in (0,1)$ and $T \ge n_0 + \tau + \beta$ such that for any $x_0 = \{x_{0n}\}_{n\in\mathbb{N}_\beta} \in A(N,M)$, the Mann iterative sequence with errors $\{x_m\}_{m\in\mathbb{N}_0} = \{x_{mn}\}_{(m,n)\in\mathbb{N}_0\times\mathbb{N}_\beta}$ generated by the scheme

$$x_{m+1n} = \begin{cases} (1-\alpha_m-\beta_m)\,x_{mn} + \alpha_m \Big\{ nL - a_n x_{mn-\tau} + \displaystyle\sum_{s=n}^{\infty}\Big[h\left(s, x_{mh_{1s}}, x_{mh_{2s}}, \ldots, x_{mh_{ks}}\right) - \sum_{t=s}^{\infty}\left(f\left(t, x_{mf_{1t}}, x_{mf_{2t}}, \ldots, x_{mf_{kt}}\right) - b_t\right)\Big]\Big\} + \beta_m \gamma_{mn}, & n \ge T,\ m \ge 0, \\ (1-\alpha_m-\beta_m)\,x_{mT} + \alpha_m \Big\{ TL - a_T x_{mT-\tau} + \displaystyle\sum_{s=T}^{\infty}\Big[h\left(s, x_{mh_{1s}}, x_{mh_{2s}}, \ldots, x_{mh_{ks}}\right) - \sum_{t=s}^{\infty}\left(f\left(t, x_{mf_{1t}}, x_{mf_{2t}}, \ldots, x_{mf_{kt}}\right) - b_t\right)\Big]\Big\} + \beta_m \gamma_{mT}, & \beta \le n < T,\ m \ge 0 \end{cases} \tag{43}$$

converges to an unbounded positive solution $x \in A(N,M)$ of (6) *and has the error estimate* (17)*, where $\{\gamma_m\}_{m\in\mathbb{N}_0} = \{\gamma_{mn}\}_{(m,n)\in\mathbb{N}_0\times\mathbb{N}_\beta}$ is an arbitrary sequence in $A(N,M)$ and $\{\alpha_m\}_{m\in\mathbb{N}_0}$ and $\{\beta_m\}_{m\in\mathbb{N}_0}$ are any sequences in $[0,1]$ satisfying* (18)*;*

(b) equation (6) *possesses uncountably many unbounded positive solutions in $A(N,M)$.*

Proof. Put $L \in (aM+N, M)$. It follows from (33), (34), and (42) that there exist $\theta \in (0,1)$ and $T \ge n_0 + \tau + \beta$ satisfying

$$\begin{gathered} \theta = a + \frac{1}{T}\sum_{s=T}^{\infty}\left(R_s H_s + \sum_{t=s}^{\infty} P_t F_t \right), \\ \frac{1}{T}\sum_{s=T}^{\infty}\left[W_s + \sum_{t=s}^{\infty}\left(Q_t + |b_t|\right)\right] < \min\{M-L, L-aM-N\}, \\ 0 \le a_n \le a < 1, \quad \forall n \ge T. \end{gathered} \tag{44}$$

Define a mapping $S_L : A(N,M) \to l_\beta^\infty$ by

$$S_L x_n = \begin{cases} nL - a_n x_{n-\tau} + \displaystyle\sum_{s=n}^{\infty}\Big\{ h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right) - \sum_{t=s}^{\infty}\left[f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right) - b_t\right]\Big\}, & n \ge T, \\ S_L x_T, & \beta \le n < T, \end{cases} \tag{45}$$

for each $x = \{x_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$. In view of (11), (12), and (44) and (45), we obtain that for each $x = \{x_n\}_{n\in\mathbb{N}_\beta}$, $y = \{y_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$ and $n \ge T$,

$$\begin{aligned} \left| \frac{S_L x_n}{n} - \frac{S_L y_n}{n} \right| &\le a_n \left| \frac{x_{n-\tau} - y_{n-\tau}}{n} \right| + \frac{1}{n}\sum_{s=n}^{\infty}\Big[\left| h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right) - h\left(s, y_{h_{1s}}, y_{h_{2s}}, \ldots, y_{h_{ks}}\right)\right| \\ &\quad + \sum_{t=s}^{\infty}\left| f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right) - f\left(t, y_{f_{1t}}, y_{f_{2t}}, \ldots, y_{f_{kt}}\right)\right| \Big] \\ &\le \left[a + \frac{1}{T}\sum_{s=T}^{\infty}\left(R_s H_s + \sum_{t=s}^{\infty} P_t F_t \right)\right] \|x-y\| \\ &= \theta \|x-y\|, \end{aligned}$$

$$
\begin{aligned}
\frac{S_L x_n}{n} &\le L \\
&\quad + \frac{1}{n}\sum_{s=n}^{\infty}\Bigg\{\left|h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right)\right| \\
&\qquad + \sum_{t=s}^{\infty}\left[\left|f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right)\right| + |b_t|\right]\Bigg\} \\
&\le L + \frac{1}{T}\sum_{s=T}^{\infty}\left[W_s + \sum_{t=s}^{\infty}\left(Q_t + |b_t|\right)\right] \\
&< L + \min\{M - L, L - aM - N\} \\
&\le M, \\
\frac{S_L x_n}{n} &\ge L - aM \\
&\quad - \frac{1}{n}\sum_{s=n}^{\infty}\Bigg\{\left|h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right)\right| \\
&\qquad + \sum_{t=s}^{\infty}\left[\left|f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right)\right| + |b_t|\right]\Bigg\} \\
&\ge L - aM - \frac{1}{T}\sum_{s=T}^{\infty}\left[W_s + \sum_{t=s}^{\infty}\left(Q_t + |b_t|\right)\right] \\
&> L - aM - \min\{M - L, L - aM - N\} \\
&\ge N,
\end{aligned}
\tag{46}
$$

which give (24), in turn, which implies that S_L is a contraction in $A(N, M)$ and possesses a unique fixed point $x = \{x_n\}_{n\in\mathbb{N}_\beta} \in A(N, M)$, which is an unbounded positive solution of (6) in $A(N, M)$. The rest of the proof is similar to that of Theorem 3 and is omitted. This completes the proof. □

Theorem 6. *Assume that there exist constants a, M, and N with $(1 + a)M > N > 0$ and four nonnegative sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$, $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$, $\{R_n\}_{n\in\mathbb{N}_{n_0}}$, and $\{W_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (11), (12), (33), (34), *and*

$$
-1 < a \le a_n \le 0 \quad \text{eventually}. \tag{47}
$$

Then

(a) for any $L \in (N, (1 + a)M)$, there exist $\theta \in (0, 1)$ and $T \ge n_0 + \tau + \beta$ such that for any $x_0 = \{x_{0n}\}_{n\in\mathbb{N}_\beta} \in A(N, M)$ and the Mann iterative sequence with errors $\{x_m\}_{m\in\mathbb{N}_0} = \{x_{mn}\}_{(m,n)\in\mathbb{N}_0\times\mathbb{N}_\beta}$ generated by (43) *converges to an unbounded positive solution $x \in A(N, M)$ of* (6) *and has the error estimate* (17), *where $\{\gamma_m\}_{m\in\mathbb{N}_0} = \{\gamma_{mn}\}_{(m,n)\in\mathbb{N}_0\times\mathbb{N}_\beta}$ is an arbitrary sequence in $A(N, M)$, $\{\alpha_m\}_{m\in\mathbb{N}_0}$ and $\{\beta_m\}_{m\in\mathbb{N}_0}$ are any sequences in $[0, 1]$ satisfying* (18)*;*

(b) equation (6) *possesses uncountably many unbounded positive solutions in $A(N, M)$.*

Proof. Put $L \in (N, (1 + a)M)$. It follows from (33), (34), and (47) that there exist $\theta \in (0, 1)$ and $T \ge n_0 + \tau + \beta$ satisfying

$$
\theta = -a + \frac{1}{T}\sum_{s=T}^{\infty}\left(R_s H_s + \sum_{t=s}^{\infty} P_t F_t\right), \tag{48}
$$

$$
\frac{1}{T}\sum_{s=T}^{\infty}\left[W_s + \sum_{t=s}^{\infty}\left(Q_t + |b_t|\right)\right] < \min\{(1 + a)M - L, L - N\}, \tag{49}
$$

$$
-1 < a \le a_n \le 0, \quad \forall n \ge T. \tag{50}
$$

Define a mapping $S_L : A(N, M) \to l_\beta^\infty$ by (45). By virtue of (12), (45), (48), and (50), we easily verify that

$$
\begin{aligned}
\frac{S_L x_n}{n} &\le L - aM \\
&\quad + \frac{1}{n}\sum_{s=n}^{\infty}\Bigg\{\left|h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right)\right| \\
&\qquad + \sum_{t=s}^{\infty}\left[\left|f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right)\right| + |b_t|\right]\Bigg\} \\
&\le L - aM + \frac{1}{T}\sum_{s=T}^{\infty}\left[W_s + \sum_{t=s}^{\infty}\left(Q_t + |b_t|\right)\right] \\
&< L + \min\{(1 + a)M - L, L - N\} \\
&\le M, \\
\frac{S_L x_n}{n} &\ge L \\
&\quad - \frac{1}{n}\sum_{s=T}^{n-1}\Bigg\{\left|h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right)\right| \\
&\qquad + \sum_{t=s}^{\infty}\left[\left|f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right)\right| + |b_t|\right]\Bigg\} \\
&\ge L - \frac{1}{T}\sum_{s=T}^{\infty}\left[W_s + \sum_{t=s}^{\infty}\left(Q_t + |b_t|\right)\right] \\
&> L - \min\{(1 + a)M - L, L - N\} \\
&\ge N,
\end{aligned}
\tag{51}
$$

which yield that $S_L(A(N, M)) \subseteq A(N, M)$. The rest of the proof is similar to that of Theorem 5 and is omitted. This completes the proof. □

Remark 7. Theorems 3~6 extend and improve Theorem 1 in [5].

3. Examples

In this section we suggest four examples to explain the results presented in Section 2. Note that Theorem 1 in [5] is useless for all these examples.

Example 8. Consider the second-order nonlinear neutral delay difference equation:

$$\Delta^2 (x_n - x_{n-\tau}) + \Delta\left(\frac{\sin^2 x_{n-2}}{n^4}\right) + \frac{1}{(n^6+n^3+2)\left(1+|x_{n^2}|^3\right)} = \frac{n^2-3n+1}{n^7+n^4+1}, \quad n \ge 3, \tag{52}$$

where $\tau \in \mathbb{N}$ is fixed. Let $n_0 = 3$, $k = 1$, $\beta = \min\{3-\tau, 1\}$, M and N two positive constants with $M > N$ and

$$\begin{gathered} a_n = -1, \qquad b_n = \frac{n^2-3n+1}{n^7+n^4+1}, \\ f(n,u) = \frac{1}{(n^6+n^3+2)\left(1+|u|^3\right)}, \\ h(n,u) = \frac{\sin^2 u}{n^4}, \qquad F_n = f_{1n} = n^2, \qquad H_n = h_{1n} = n-2, \\ P_n = \frac{3M^2}{(1+N^3)^2 n^6}, \qquad Q_n = \frac{1}{n^6}, \qquad R_n = \frac{2}{n^4}, \\ W_n = \frac{1}{n^4}, \quad \forall (n,u) \in \mathbb{N}_{n_0} \times \mathbb{R}. \end{gathered} \tag{53}$$

It is easy to see that (11), (12), and (15) are satisfied. Note that

$$\begin{aligned} \sum_{s=n}^{\infty} s \max\{R_s H_s, W_s\} &= \sum_{s=n}^{\infty} s \max\left\{\frac{2(s-2)}{s^4}, \frac{1}{s^4}\right\} \\ &= \sum_{s=n}^{\infty} \frac{2(s-2)}{s^3} < +\infty, \quad \forall n \in \mathbb{N}_{n_0}, \end{aligned} \tag{54}$$

$$\begin{aligned} \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} s \max\{P_t F_t, Q_t, |b_t|\} &= \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} s \max\left\{\frac{3M^2}{(1+N^3)^2 t^4}, \frac{1}{t^6}, \frac{|t^2-3t+1|}{t^7+t^4+1}\right\} \\ &\le \max\left\{1, \frac{3M^2}{(1+N^3)^2}\right\} \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} \frac{s}{t^4} \\ &\le \max\left\{1, \frac{3M^2}{(1+N^3)^2}\right\} \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} \frac{1}{t^3} \\ &= \max\left\{1, \frac{3M^2}{(1+N^3)^2}\right\} \sum_{t=n}^{\infty} \frac{t-n+1}{t^3} \\ &\le \max\left\{1, \frac{3M^2}{(1+N^3)^2}\right\} \sum_{t=n}^{\infty} \frac{1}{t^2} < +\infty, \quad \forall n \in \mathbb{N}_{n_0}, \end{aligned} \tag{55}$$

which together with Lemma 2 yield that (13) and (14) hold. It follows from Theorem 3 that (52) possesses uncountably many unbounded positive solutions in $A(N,M)$. On the other hand, for any $L \in (N,M)$, there exist $\theta \in (0,1)$ and $T \ge n_0 + \tau + \beta$ such that for each $x_0 = \{x_{0n}\}_{n\in\mathbb{N}_\beta} \in A(N,M)$, the Mann iterative sequence with errors $\{x_m\}_{m\in\mathbb{N}_0} = \{x_{mn}\}_{(m,n)\in\mathbb{N}_0\times\mathbb{N}_\beta}$ generated by (16) converges to an unbounded positive solution $x \in A(N,M)$ of (52) and has the error estimate (17), where $\{\gamma_m\}_{m\in\mathbb{N}_0} = \{\gamma_{mn}\}_{(m,n)\in\mathbb{N}_0\times\mathbb{N}_\beta}$ is an arbitrary sequence in $A(N,M)$ and $\{\alpha_m\}_{m\in\mathbb{N}_0}$ and $\{\beta_m\}_{m\in\mathbb{N}_0}$ are any sequences in $[0,1]$ satisfying (18).

Example 9. Consider the second-order nonlinear neutral delay difference equation:

$$\begin{aligned} &\Delta^2 (x_n + x_{n-\tau}) + \Delta\left(\frac{\sin^2 x_{3n^3+1}}{n^3(n^2+2)\left(1+x_{2n^2-3}^4\right)}\right) \\ &\quad + \frac{(-1)^n n^3 \left(x_{n^2-n-1} + x_{(n+1)(n+2)}\right)}{(n^{11}+n^5+1)\left(1+x_{n^2-n-1}^2 + x_{(n+1)(n+2)}^2\right)} \\ &= \frac{n^2-\ln n}{n^6+n^5+1}, \quad n \ge 5, \end{aligned} \tag{56}$$

where $\tau \in \mathbb{N}$ is fixed. Let $n_0 = 5$, $k = 2$, $\beta = 5-\tau$, M and N two positive constants with $M > N$ and

$$\begin{gathered} a_n = 1, \qquad b_n = \frac{n^2-\ln n}{n^6+n^5+1}, \\ f(n,u,v) = \frac{(-1)^n n^3 (u+v)}{(n^{11}+n^5+1)(1+u^2+v^2)}, \\ h(n,u,v) = \frac{\sin^2 v}{n^3(n^2+2)(1+u^4)}, \\ f_{1n} = n^2-n-1, \qquad F_n = f_{2n} = (n+1)(n+2), \\ h_{1n} = 2n^2-3, \qquad H_n = h_{2n} = 3n^3+1, \\ P_n = Q_n = \frac{4}{n^8}, \qquad R_n = W_n = \frac{10}{n^5}, \quad \forall (n,u,v) \in \mathbb{N}_{n_0} \times \mathbb{R}^2. \end{gathered} \tag{57}$$

It is clear that (11), (12), and (35) are fulfilled. Note that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{s=n}^{\infty}\max\{R_sH_s,W_s\}$$
$$=\lim_{n\to\infty}\frac{1}{n}\sum_{s=n}^{\infty}\max\left\{\frac{10\left(3s^3+1\right)}{s^5},\frac{10}{s^5}\right\}=0,$$
$$\frac{1}{n}\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\max\{P_tF_t,Q_t,|b_t|\}$$
$$=\frac{1}{n}\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\max\left\{\frac{4(t+1)(t+2)}{t^8},\frac{4}{t^8},\frac{t^2-\ln t}{t^6+t^5+1}\right\}$$
$$\le\frac{4}{n}\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\frac{1}{t^4}\le\frac{4}{n}\sum_{t=n}^{\infty}\frac{1}{t^3}\longrightarrow 0\quad\text{as } n\longrightarrow\infty, \tag{58}$$

which yields that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\max\{P_tF_t,Q_t,|b_t|\}=0. \tag{59}$$

Thus, Theorem 4 guarantees that (56) possesses uncountably unbounded positive solutions in $A(N,M)$. On the other hand, for any $L\in(N,M)$, there exist $\theta\in(0,1)$ and $T\ge\tau+n_0+\beta$ such that the Mann iterative sequence with error $\{x_m\}_{m\in\mathbb{N}_0}$ generated by (36) converges to an unbounded positive solution $x\in A(N,M)$ of (56) and has the error estimate (17), where $\{\gamma_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $A(N,M)$ and $\{\alpha_m\}_{m\in\mathbb{N}_0}$ and $\{\beta_m\}_{m\in\mathbb{N}_0}$ are any sequences in $[0,1]$ satisfying (18).

Example 10. Consider the second-order nonlinear neutral delay difference equation:

$$\Delta^2\left(x_n+\frac{3n^3-1}{4n^3+2}x_{n-\tau}\right)+\Delta\left(\frac{\sin\left(\ln\left(1+n^2\left|x_{3n^2-1}\right|\right)\right)}{n^9-\sqrt{n}-4}-\frac{n^2-(-1)^{n(n-1)/2}}{(n^7+3n^5-1)\,2^{|x_{4n^3+1}|}}\right)+\frac{(-1)^n}{n^6\left(1+x_{n-2}^2\right)}-\frac{1}{(n^5+1)\sqrt{1+|x_{n+4}|}}=\frac{(-1)^nn^3-1}{n^8\ln^3n+1},\quad n\ge 7, \tag{60}$$

where $\tau\in\mathbb{N}$ is fixed. Let $n_0=7$, $k=2$, $a=3/4$, $\beta=\min\{7-\tau,5\}$, M and N two positive constants with $M>4N$ and

$$a_n=\frac{3n^3-1}{4n^3+2},\qquad b_n=\frac{(-1)^nn^3-1}{n^8\ln^3n+1},$$
$$f(n,u,v)=\frac{(-1)^n}{n^6(1+u^2)}-\frac{1}{(n^5+1)\sqrt{1+|v|}},$$
$$h(n,u,v)=\frac{\sin\left(\ln\left(1+n^2|u|\right)\right)}{n^9-\sqrt{n}-4}-\frac{n^2-(-1)^{n(n-1)/2}}{(n^7+3n^5-1)\,2^{|v|}},$$
$$f_{1n}=n-2,$$
$$F_n=f_{2n}=n+4,\qquad h_{1n}=3n^2-1,$$
$$H_n=h_{2n}=4n^3+1,$$
$$P_n=Q_n=\frac{2}{n^4},\qquad R_n=W_n=\frac{2}{n^5},\quad\forall(n,u,v)\in\mathbb{N}_{n_0}\times\mathbb{R}^2. \tag{61}$$

It is not difficult to verify that (11), (12), and (42) are fulfilled. Note that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{s=n}^{\infty}\max\{R_sH_s,W_s\}$$
$$=\lim_{n\to\infty}\frac{1}{n}\sum_{s=n}^{\infty}\max\left\{\frac{2\left(4s^3+1\right)}{s^5},\frac{2}{s^5}\right\}$$
$$=\lim_{n\to\infty}\frac{1}{n}\sum_{s=n}^{\infty}\frac{2\left(4s^3+1\right)}{s^5}=0,$$
$$\lim_{n\to\infty}\frac{1}{n}\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\max\{P_tF_t,Q_t,|b_t|\}$$
$$=\lim_{n\to\infty}\frac{1}{n}\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\max\left\{\frac{2t+8}{t^4},\frac{2}{t^4},\frac{\left|(-1)^tt^3-1\right|}{t^8\ln^3t+1}\right\}$$
$$=\lim_{n\to\infty}\frac{1}{n}\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\frac{2t+8}{t^4}=0. \tag{62}$$

That is, (33) and (34) are satisfied. Consequently Theorem 5 implies that (60) possesses uncountably many unbounded positive solutions in $A(N,M)$. On the other hand, for any $L\in((3/4)M+N,M)$, there exist $\theta\in(0,1)$ and $T\ge n_0+\tau+\beta$ such that the Mann iterative sequence with error $\{x_m\}_{m\in\mathbb{N}_0}$ generated by (43) converges to an unbounded positive solution $x\in A(N,M)$ of (60) and has the error estimate (17), where $\{\gamma_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $A(N,M)$ and $\{\alpha_m\}_{m\in\mathbb{N}_0}$ and $\{\beta_m\}_{m\in\mathbb{N}_0}$ are any sequences in $[0,1]$ satisfying (18).

Example 11. Consider the second-order nonlinear neutral delay difference equation:

$$\Delta^2\left(x_n+\frac{1-2n^2}{2+3n^2}x_{n-\tau}\right)+\Delta\left(\frac{n^2-1}{\left(n^5+2n^3-1\right)\left(1+x_{2n-15}^2\right)}\right)+\frac{\sin^2\left(n^3x_{5n^2-2}\right)}{(n+2)^8}=\frac{(-1)^nn^4+n^3+3n^2-1}{n^9+n^7+3n^6+n^4+1},\quad n\geq 11, \tag{63}$$

where $\tau\in\mathbb{N}$ is fixed. Let $n_0=11$, $k=1$, $a=-4/5$, $\beta=\min\{11-\tau,7\}$, M and N two positive constants with $M>5N$ and

$$a_n=\frac{1-2n^2}{2+3n^2},\qquad b_n=\frac{(-1)^nn^4+n^3+3n^2-1}{n^9+n^7+3n^6+n^4+1},$$
$$f(n,u)=\frac{\sin^2\left(n^3u\right)}{(n+2)^8},$$
$$h(n,u)=\frac{n^2-1}{\left(n^5+2n^3-1\right)\left(1+u^2\right)}, \tag{64}$$
$$F_n=f_{1n}=5n^2-2,$$
$$H_n=h_{1n}=2n-15,\qquad P_n=\frac{2}{n^5},\qquad Q_n=\frac{1}{n^8},$$
$$R_n=\frac{2}{n^3},\qquad W_n=\frac{1}{n^3},\quad\forall(n,u)\in\mathbb{N}_{n_0}\times\mathbb{R}.$$

Obviously, (11), (12), and (50) are satisfied. Note that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{s=n}^{\infty}\max\left\{R_sH_s,W_s\right\}=\lim_{n\to\infty}\frac{1}{n}\sum_{s=n}^{\infty}\max\left\{\frac{4s-30}{s^3},\frac{1}{s^3}\right\}=0,$$
$$\frac{1}{n}\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\max\left\{P_tF_t,Q_t,\left|b_t\right|\right\}=\frac{1}{n}\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\max\left\{\frac{2\left(5t^2-2\right)}{t^5},\frac{1}{t^8},\frac{\left|(-1)^tt^4+t^3+3t^2-1\right|}{t^9+t^7+3t^6+t^4+1}\right\}\leq\frac{10}{n}\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\frac{1}{t^3}\longrightarrow 0\quad\text{as } n\longrightarrow\infty, \tag{65}$$

which gives that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\max\left\{P_tF_t,Q_t,\left|b_t\right|\right\}=0. \tag{66}$$

That is, (33) and (34) hold. Thus, Theorem 6 shows that (63) possesses uncountably many unbounded positive solutions in $A(N,M)$. On the other hand, for any $L\in(N,M/5)$, there exist $\theta\in(0,1)$ and $T\geq n_0+\tau+\beta$ such that the Mann iterative sequence with error $\{x_m\}_{m\in\mathbb{N}_0}$ generated by (43) converges to an unbounded positive solution $x\in A(N,M)$ of (63) and has the error estimate (17), where $\{\gamma_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $A(N,M)$ and $\{\alpha_m\}_{m\in\mathbb{N}_0}$ and $\{\beta_m\}_{m\in\mathbb{N}_0}$ are any sequences in $[0,1]$ satisfying (18).

Acknowledgments

This research was supported by the Science Research Foundation of Educational Department of Liaoning Province (L2012380) and Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2012R1A1A2002165).

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