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Research Article

Stability of Impulsive Neural Networks with Time-Varying and Distributed Delays

Qi Luo,[1] Xinjie Miao,[1] Qian Wei,[1] and Zhengxin Zhou[2]

[1] *School of Information & Control, Nanjing University of Information Science & Technology, Nanjing, Jiangsu 210044, China*
[2] *School of Mathematical Science, Yangzhou University, Yangzhou, Jiangsu 225009, China*

Correspondence should be addressed to Qi Luo; hb2-102@nuist.edu.cn

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This work is devoted to investigating the stability of impulsive cellular neural networks with time-varying and distributed delays. We use the new method of fixed point theory to obtain some new and concise sufficient conditions to ensure the existence and uniqueness of solution and the global exponential stability of trivial equilibrium. The presented algebraic criteria are easily checked and do not require the differentiability of delays.

1. Introduction

Since cellular neural networks (CNNs) were proposed by Chua and Yang in 1988 [1, 2], many researchers have put great effort into this subject due to their numerous successful applications in various fields such as optimization, linear and nonlinear programming, associative memory, pattern recognition, and computer vision.

Owing to the finite switching speed of amplifiers, there is no doubt that time delays exist in the communication and response of neurons. Moreover, as neural networks usually have a spatial extent due to the presences of a multitude of parallel pathways with a variety of axon sizes and lengths, there is a distribution of conduction velocities along these pathways and a distribution of propagation designed with discrete delays. Therefore, a more appropriate and ideal way is to incorporate continuously distributed delays with a result that a more effective model of cellular neural networks with time-varying and distributed delays proposed.

In fact, beside delay effects, stochastic and impulsive as well as diffusing effects are also likely to exist in neural networks. So far, there have been many results [3–11] on the study of dynamic behaviors of complex CNNs such as impulsive delayed reaction-diffusion CNNs and stochastic delayed reaction-diffusion CNNs. Summing up the existing researches on the stability of complex CNNs, we see that the primary method is Lyapunov theory. However, there are also lots of difficulties in the applications of corresponding theories to specific problems. It is therefore necessary to seek some new methods to deal with the stability in order to overcome those difficulties.

Recently, it is inspiring that Burton and other authors have applied the fixed point theory to investigate the stability of deterministic systems and obtained some more applicable conclusions, for example, see the monograph [12] and the work in [13–24]. In addition, more recently, there have been a few papers where the fixed point theory is employed to investigate the stability of stochastic (delayed) differential equations, for instance, see [25–31]. Precisely, in [26–28], Luo used the fixed point theory to study the exponential stability of mild solutions for stochastic partial differential equations with bounded delays and with infinite delays. In [29, 30], Sakthivel used the fixed point theory to discuss the asymptotic stability in pth moment of mild solutions to nonlinear impulsive stochastic partial differential equations with bounded delays and with infinite delays. In [31], Luo used the fixed point theory to study the exponential stability of stochastic Volterra-Levin equations. We wonder if we can obtain some new and more applicable stability criteria of complex CNNs by applying the fixed point theory.

With this motivation, in this paper, we aim to discuss the global exponential stability of impulsive CNNs with time-varying and distributed delays. It is worth noting that our research technique is based on the contraction mapping principle rather than the usual method of Lyapunov theory. We deal with, by employing the fixed point theorem, the existence and uniqueness of solution and the global exponential stability of trivial equilibrium at the same time, for which Lyapunov method feels helpless. The obtained stability criteria are easily checked and do not require the differentiability of delays.

2. Preliminaries

Let R^n denote the n-dimensional Euclidean space and $\|\cdot\|$ represent the Euclidean norm $\mathcal{N} \triangleq \{1,2,\ldots,n\}$ and $R_+ = [0,\infty)$. $C[X,Y]$ corresponds to the space of continuous mappings from the topological space X to the topological space Y.

In this paper, we consider the following impulsive cellular neural networks with time-varying and distributed delays:

$$\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = & -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j\left(x_j(t)\right) \\ & + \sum_{j=1}^{n} c_{ij} g_j\left(x_j\left(t-\tau_{ij}(t)\right)\right) \\ & + \sum_{j=1}^{n} d_{ij} \int_0^{\rho(t)} \sigma_j\left(x_j(t-\theta)\right)\mathrm{d}\theta \\ & \qquad t \geq 0,\ t \neq t_k,\end{aligned} \tag{1}$$

$$\Delta x_i(t_k) = x_i(t_k+0) - x_i(t_k) = I_{ik}\left(x_i(t_k)\right), \quad k = 1,2,\ldots, \tag{2}$$

where $i \in \mathcal{N}$ and n is the number of neurons in the neural network. $x_i(t)$ corresponds to the state of the ith neuron at time t. f_j, g_j, and σ_j denote the activation functions, respectively. The constant $a_i > 0$ represents the rate with which the ith neuron will reset its potential to the resting state when disconnected from the network and external inputs. The constants b_{ij}, c_{ij}, and d_{ij} represent the connection weights of the jth neuron to the ith neuron, respectively. $\tau_{ij}(t)$ and $\rho(t)$ correspond to the transmission delays meeting $0 \leq \tau_{ij}(t) \leq \tau$ (τ = constant) and $0 \leq \rho(t) \leq \rho$ (ρ = constant). The fixed impulsive moments t_k ($k = 1,2,\ldots$) satisfy $0 = t_0 < t_1 < t_2 < \cdots$ and $\lim_{k\to\infty} t_k = \infty$. $x_i(t_k+0)$ and $x_i(t_k-0)$ stand for the right-hand and left-hand limits of $x_i(t)$ at time t_k, respectively. $I_{ik}(x_i(t_k))$ shows the impulsive perturbation of the ith neuron at the impulsive moment t_k.

Throughout this paper, we always assume that $f_i(0) = g_i(0) = \sigma_i(0) = I_{ik}(0) = 0$ for $i \in \mathcal{N}$ and $k = 1,2,\ldots$. Thereby, problems (1) and (2) admit a trivial equilibrium $\mathbf{x} = 0$.

Denote by $\mathbf{x}(t) \triangleq \mathbf{x}(t;s,\varphi) = (x_1(t;s,\varphi_1),\ldots,x_n(t;s,\varphi_n))^T \in R^n$ the solution to (1) and (2) with the initial condition

$$x_i(s) = \varphi_i(s), \quad -m^* \leq s \leq 0,\ i \in \mathcal{N}, \tag{3}$$

where $m^* = \max\{\tau,\rho\}$, $\varphi_i(s) \in C[[-m^*,0],R]$ and $\varphi(s) = (\varphi_1(s),\ldots,\varphi_n(s))^T \in R^n$.

The solution $\mathbf{x}(t) \triangleq \mathbf{x}(t;s,\varphi) \in R^n$ to (1)–(3) is, for the time variable t, a piecewise continuous vector-valued function with the first-kind discontinuous points t_k ($k = 1,2,\ldots$), where it is left-continuous; that is, the following relations are true:

$$x_i(t_k-0) = x_i(t_k),\ x_i(t_k+0) = x_i(t_k) + I_{ik}\left(x_i(t_k)\right), \quad i \in \mathcal{N},\ k = 1,2,\ldots. \tag{4}$$

Definition 1. The trivial equilibrium $\mathbf{x} = 0$ is said to be globally exponentially stable if for any initial condition $\varphi(s) \in C[[-m^*,0],R^n]$, there exists a pair of positive constants λ and M such that

$$\left\|\mathbf{x}(t;s,\varphi)\right\| \leq M e^{-\lambda t}, \quad \forall t \geq 0. \tag{5}$$

The consideration of this paper is based on the following fixed point theorem.

Theorem 2 (see [32]). *Let Υ be a contraction operator on a complete metric space Θ, then there exists a unique point $\zeta \in \Theta$ for which $\Upsilon(\zeta) = \zeta$.*

3. Main Results

In this section, we will, for (1)–(3), use the contraction mapping principle to prove the existence and uniqueness of the solution and the global exponential stability of trivial equilibrium all at once. Before proceeding, we firstly introduce some assumptions as follows.

(A1) There exist nonnegative constants l_j such that for any $\eta, \upsilon \in R$,

$$\left|f_j(\eta) - f_j(\upsilon)\right| \leq l_j\left|\eta - \upsilon\right|, \quad j \in \mathcal{N}. \tag{6}$$

(A2) There exist nonnegative constants k_j such that for any $\eta, \upsilon \in R$,

$$\left|g_j(\eta) - g_j(\upsilon)\right| \leq k_j\left|\eta - \upsilon\right|, \quad j \in \mathcal{N}. \tag{7}$$

(A3) There exist nonnegative constants p_{jk} such that for any $\eta, \upsilon \in R$,

$$\left|I_{jk}(\eta) - I_{jk}(\upsilon)\right| \leq p_{jk}\left|\eta - \upsilon\right|, \quad j \in \mathcal{N},\ k = 1,2,\ldots. \tag{8}$$

(A4) There exist nonnegative constants ω_j such that for any $\eta, \upsilon \in R$,

$$\left|\sigma_j(\eta) - \sigma_j(\upsilon)\right| \leq \omega_j\left|\eta - \upsilon\right|, \quad j \in \mathcal{N}. \tag{9}$$

Let $\mathcal{H} = \mathcal{H}_1 \times \cdots \times \mathcal{H}_n$, and let $\mathcal{H}_i$ ($i \in \mathcal{N}$) be the space embracing functions $\phi_i(t) : [-m^*,+\infty) \to R$, wherein $\phi_i(t)$ satisfies the following:

(1) $\phi_i(t)$ is continuous on $t \neq t_k$ ($k = 1,2,\ldots$),

(2) $\lim_{t\to t_k^-} \phi_i(t)$ and $\lim_{t\to t_k^+} \phi_i(t)$ exist; moreover, $\lim_{t\to t_k^-} \phi_i(t) = \phi_i(t_k)$ for $k = 1,2,\ldots$,

(3) $\phi_i(s) = \varphi_i(s)$ on $s \in [-m^*, 0]$,

(4) $e^{\alpha t}\phi_i(t) \to 0$ as $t \to \infty$, where $\alpha = \text{const}$ and $0 < \alpha < \min_{i\in\mathcal{N}}\{a_i\}$,

where t_k and $\varphi_i(s)$ are defined as shown in Section 2. Also $\mathcal{H}$ is a complete metric space when it is equipped with a metric defined by

$$d\left(\overline{\mathbf{q}}(t), \overline{\mathbf{h}}(t)\right) = \sum_{i=1}^{n} \sup_{t\ge -m^*} \left|q_i(t) - h_i(t)\right|, \tag{10}$$

where $\overline{\mathbf{q}}(t) = (q_1(t), \ldots, q_n(t)) \in \mathcal{H}$ and $\overline{\mathbf{h}}(t) = (h_1(t), \ldots, h_n(t)) \in \mathcal{H}$.

Theorem 3. *Assume that conditions (A1)–(A4) hold provided that*

(i) *there exists a constant* μ *such that* $\inf_{k=1,2,\ldots}\{t_k - t_{k-1}\} \ge \mu$,

(ii) *there exist constants* p_i *such that* $p_{ik} \le p_i\mu$ *for* $i \in \mathcal{N}$ *and* $k = 1, 2, \ldots$,

(iii) $\sum_{i=1}^{n}\{(1/a_i)\max_{j\in\mathcal{N}}|b_{ij}l_j| + (1/a_i)\max_{j\in\mathcal{N}}|c_{ij}k_j| + (\rho/a_i)\max_{j\in\mathcal{N}}|\omega_j d_{ij}|\} + \max_{i\in\mathcal{N}}\{p_i(\mu + (1/a_i))\} \triangleq \chi < 1$,

and then the trivial equilibrium $\mathbf{x} = 0$ *is globally exponentially stable.*

Proof. Multiplying both sides of (1) with $e^{a_i t}$ gives, for $t > 0$ and $t \ne t_k$,

$$\begin{aligned} de^{a_i t}x_i(t) &= e^{a_i t}dx_i(t) + a_i x_i(t)e^{a_i t}dt \\ &= e^{a_i t}\Bigg\{\sum_{j=1}^{n} b_{ij}f_j\left(x_j(t)\right) + \sum_{j=1}^{n} c_{ij}g_j\left(x_j\left(t - \tau_{ij}(t)\right)\right) \\ &\qquad + \sum_{j=1}^{n} d_{ij}\int_0^{\rho(t)} \sigma_j\left(x_j(t-\theta)\right)d\theta\Bigg\}dt, \end{aligned} \tag{11}$$

which yields after integrating from $t_{k-1} + \varepsilon$ $(\varepsilon > 0)$ to $t \in (t_{k-1}, t_k)$ $(k = 1, 2, \ldots)$ that

$$\begin{aligned} x_i(t)e^{a_i t} &= x_i(t_{k-1} + \varepsilon)e^{a_i(t_{k-1}+\varepsilon)} \\ &+ \int_{t_{k-1}+\varepsilon}^{t} e^{a_i s}\left\{\begin{matrix}\sum_{j=1}^{n} b_{ij}f_j\left(x_j(s)\right) + \sum_{j=1}^{n} c_{ij}g_j\left(x_j\left(s - \tau_{ij}(s)\right)\right) \\ + \sum_{j=1}^{n} d_{ij}\int_0^{\rho(s)} \sigma_j\left(x_j(s-\theta)\right)d\theta\end{matrix}\right\}ds. \end{aligned} \tag{12}$$

Letting $\varepsilon \to 0$ in (12), we have, for $t \in (t_{k-1}, t_k)$ $(k = 1, 2, \ldots)$,

$$\begin{aligned} x_i(t)e^{a_i t} &= x_i(t_{k-1} + 0)e^{a_i t_{k-1}} \\ &+ \int_{t_{k-1}}^{t} e^{a_i s}\left\{\begin{matrix}\sum_{j=1}^{n} b_{ij}f_j\left(x_j(s)\right) + \sum_{j=1}^{n} c_{ij}g_j\left(x_j\left(s - \tau_{ij}(s)\right)\right) \\ + \sum_{j=1}^{n} d_{ij}\int_0^{\rho(s)} \sigma_j\left(x_j(s-\theta)\right)d\theta\end{matrix}\right\}ds. \end{aligned} \tag{13}$$

Setting $t = t_k - \varepsilon$ $(\varepsilon > 0)$ in (13), we get

$$\begin{aligned} x_i(t_k - \varepsilon)e^{a_i(t_k-\varepsilon)} &= x_i(t_{k-1} + 0)e^{a_i t_{k-1}} \\ &+ \int_{t_{k-1}}^{t_k-\varepsilon} e^{a_i s}\left\{\begin{matrix}\sum_{j=1}^{n} b_{ij}f_j\left(x_j(s)\right) + \sum_{j=1}^{n} c_{ij}g_j\left(x_j\left(s - \tau_{ij}(s)\right)\right) \\ + \sum_{j=1}^{n} d_{ij}\int_0^{\rho(s)} \sigma_j\left(x_j(s-\theta)\right)d\theta\end{matrix}\right\}ds, \end{aligned} \tag{14}$$

which generates by letting $\varepsilon \to 0$

$$\begin{aligned} x_i(t_k - 0)e^{a_i t_k} &= x_i(t_{k-1} + 0)e^{a_i t_{k-1}} \\ &+ \int_{t_{k-1}}^{t_k} e^{a_i s}\left\{\begin{matrix}\sum_{j=1}^{n} b_{ij}f_j\left(x_j(s)\right) + \sum_{j=1}^{n} c_{ij}g_j\left(x_j\left(s - \tau_{ij}(s)\right)\right) \\ + \sum_{j=1}^{n} d_{ij}\int_0^{\rho(s)} \sigma_j\left(x_j(s-\theta)\right)d\theta\end{matrix}\right\}ds. \end{aligned} \tag{15}$$

Noting $x_i(t_k - 0) = x_i(t_k)$, (15) can be rearranged as

$$\begin{aligned} x_i(t_k)e^{a_i t_k} &= x_i(t_{k-1} + 0)e^{a_i t_{k-1}} \\ &+ \int_{t_{k-1}}^{t_k} e^{a_i s}\left\{\begin{matrix}\sum_{j=1}^{n} b_{ij}f_j\left(x_j(s)\right) + \sum_{j=1}^{n} c_{ij}g_j\left(x_j\left(s - \tau_{ij}(s)\right)\right) \\ + \sum_{j=1}^{n} d_{ij}\int_0^{\rho(s)} \sigma_j\left(x_j(s-\theta)\right)d\theta\end{matrix}\right\}ds. \end{aligned} \tag{16}$$

Combining (13) and (16), we derive that

$$\begin{aligned} x_i(t)e^{a_i t} &= x_i(t_{k-1} + 0)e^{a_i t_{k-1}} \\ &+ \int_{t_{k-1}}^{t} e^{a_i s}\left\{\begin{matrix}\sum_{j=1}^{n} b_{ij}f_j\left(x_j(s)\right) + \sum_{j=1}^{n} c_{ij}g_j\left(x_j\left(s - \tau_{ij}(s)\right)\right) \\ + \sum_{j=1}^{n} d_{ij}\int_0^{\rho(s)} \sigma_j\left(x_j(s-\theta)\right)d\theta\end{matrix}\right\}ds \end{aligned} \tag{17}$$

is true for $t \in (t_{k-1}, t_k]$ $(k = 1, 2, \ldots)$. Hence, we get, for $t \in (t_{k-1}, t_k]$ $(k = 1, 2, \ldots)$,

$$\begin{aligned}
&x_i(t)\,\mathrm{e}^{a_i t} \\
&= \left\{x_i(t_{k-1}) + I_{i(k-1)}(x_i(t_{k-1}))\right\}\mathrm{e}^{a_i t_{k-1}} \\
&\quad + \int_{t_{k-1}}^{t} \mathrm{e}^{a_i s}\left\{\begin{array}{c}\sum_{j=1}^{n} b_{ij} f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij} g_j(x_j(s-\tau_{ij}(s))) \\ + \sum_{j=1}^{n} d_{ij}\int_0^{\rho(s)} \sigma_j(x_j(s-\theta))\,\mathrm{d}\theta\end{array}\right\}\mathrm{d}s \\
&= x_i(t_{k-1})\,\mathrm{e}^{a_i t_{k-1}} \\
&\quad + \int_{t_{k-1}}^{t} \mathrm{e}^{a_i s}\left\{\begin{array}{c}\sum_{j=1}^{n} b_{ij} f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij} g_j(x_j(s-\tau_{ij}(s))) \\ + \sum_{j=1}^{n} d_{ij}\int_0^{\rho(s)} \sigma_j(x_j(s-\theta))\,\mathrm{d}\theta\end{array}\right\}\mathrm{d}s \\
&\quad + I_{i(k-1)}(x_i(t_{k-1}))\,\mathrm{e}^{a_i t_{k-1}},
\end{aligned} \tag{18}$$

which results in

$$\begin{aligned}
&x_i(t_{k-1})\,\mathrm{e}^{a_i t_{k-1}} \\
&= x_i(t_{k-2})\,\mathrm{e}^{a_i t_{k-2}} \\
&\quad + \int_{t_{k-2}}^{t_{k-1}} \mathrm{e}^{a_i s}\left\{\begin{array}{c}\sum_{j=1}^{n} b_{ij} f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij} g_j(x_j(s-\tau_{ij}(s))) \\ + \sum_{j=1}^{n} d_{ij}\int_0^{\rho(s)} \sigma_j(x_j(s-\theta))\,\mathrm{d}\theta\end{array}\right\}\mathrm{d}s \\
&\quad + I_{i(k-2)}(x_i(t_{k-2}))\,\mathrm{e}^{a_i t_{k-2}} \\
&\qquad\qquad\vdots \\
&x_i(t_2)\,\mathrm{e}^{a_i t_2} \\
&= x_i(t_1)\,\mathrm{e}^{a_i t_1} \\
&\quad + \int_{t_1}^{t_2} \mathrm{e}^{a_i s}\left\{\begin{array}{c}\sum_{j=1}^{n} b_{ij} f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij} g_j(x_j(s-\tau_{ij}(s))) \\ + \sum_{j=1}^{n} d_{ij}\int_0^{\rho(s)} \sigma_j(x_j(s-\theta))\,\mathrm{d}\theta\end{array}\right\}\mathrm{d}s \\
&\quad + I_{i1}(x_i(t_1))\,\mathrm{e}^{a_i t_1}, \\
&x_i(t_1)\,\mathrm{e}^{a_i t_1} \\
&= \varphi_i(0) \\
&\quad + \int_{0}^{t_1} \mathrm{e}^{a_i s}\left\{\begin{array}{c}\sum_{j=1}^{n} b_{ij} f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij} g_j(x_j(s-\tau_{ij}(s))) \\ + \sum_{j=1}^{n} d_{ij}\int_0^{\rho(s)} \sigma_j(x_j(s-\theta))\,\mathrm{d}\theta\end{array}\right\}\mathrm{d}s.
\end{aligned} \tag{19}$$

We therefore conclude, for $t > 0$,

$$\begin{aligned}
&x_i(t) \\
&= \varphi_i(0)\,\mathrm{e}^{-a_i t} \\
&\quad + \mathrm{e}^{-a_i t}\int_{0}^{t} \mathrm{e}^{a_i s}\left\{\begin{array}{c}\sum_{j=1}^{n} b_{ij} f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij} g_j(x_j(s-\tau_{ij}(s))) \\ + \sum_{j=1}^{n} d_{ij}\int_0^{\rho(s)} \sigma_j(x_j(s-\theta))\,\mathrm{d}\theta\end{array}\right\}\mathrm{d}s \\
&\quad + \mathrm{e}^{-a_i t}\sum_{0<t_k<t}\left\{I_{ik}(x_i(t_k))\,\mathrm{e}^{a_i t_k}\right\}.
\end{aligned} \tag{20}$$

Note that $x_i(0) = \varphi_i(0)$ in (20). We then define the following operator π acting on $\mathscr{H}$, for $\overline{\mathbf{y}}(t) = (y_1(t), \ldots, y_n(t)) \in \mathscr{H}$:

$$\pi(\overline{\mathbf{y}})(t) = (\pi(y_1)(t), \ldots, \pi(y_n)(t)), \tag{21}$$

where $\pi(y_i)(t) : [-m^*, +\infty) \to R$ $(i \in \mathscr{N})$ obeys the rule as follows:

$$\begin{aligned}
&\pi(y_i)(t) \\
&= \varphi_i(0)\,\mathrm{e}^{-a_i t} \\
&\quad + \mathrm{e}^{-a_i t}\int_{0}^{t} \mathrm{e}^{a_i s}\left\{\begin{array}{c}\sum_{j=1}^{n} b_{ij} f_j(y_j(s)) + \sum_{j=1}^{n} c_{ij} g_j(y_j(s-\tau_{ij}(s))) \\ + \sum_{j=1}^{n} d_{ij}\int_0^{\rho(s)} \sigma_j(y_j(s-\theta))\,\mathrm{d}\theta\end{array}\right\}\mathrm{d}s \\
&\quad + \mathrm{e}^{-a_i t}\sum_{0<t_k<t}\left\{I_{ik}(y_i(t_k))\,\mathrm{e}^{a_i t_k}\right\}
\end{aligned} \tag{22}$$

on $t \geq 0$ and $\pi(y_i)(s) = \varphi_i(s)$ on $s \in [-m^*, 0]$.

In what follows, we will apply the contraction mapping principle to prove the existence and uniqueness of solution and the global exponential stability of trivial equilibrium at the same time. The subsequent proof can be divided into two steps.

Step 1. We need to prove that $\pi(\mathscr{H}) \subset \mathscr{H}$. For $y_i(t) \in \mathscr{H}_i$ $(i \in \mathscr{N})$, it is necessary to show that $\pi(y_i)(t) \subset \mathscr{H}_i$. As defined above, we see that $\pi(y_i)(s) = \varphi_i(s)$ on $s \in [-m^*, 0]$. Owing to the continuity of $\varphi_i(s)$ on $s \in [-m^*, 0]$, we immediately know that $\pi(y_i)(t)$ is continuous on $t \in [-m^*, 0]$.

Choose a fixed time $t > 0$, and it is then derived from (22) that

$$\pi(y_i)(t+r) - \pi(y_i)(t) = Q_1 + Q_2 + Q_3 + Q_4 + Q_5, \quad t > 0, \tag{23}$$

where,

$$\begin{aligned}
Q_1 &= \varphi_i(0)\,\mathrm{e}^{-a_i(t+r)} - \varphi_i(0)\;\mathrm{e}^{-a_i t}, \\
Q_2 &= \mathrm{e}^{-a_i(t+r)}\int_0^{t+r} \mathrm{e}^{a_i s}\sum_{j=1}^{n} b_{ij} f_j(y_j(s))\,\mathrm{d}s \\
&\quad - \mathrm{e}^{-a_i t}\int_0^{t} \mathrm{e}^{a_i s}\sum_{j=1}^{n} b_{ij} f_j(y_j(s))\,\mathrm{d}s,
\end{aligned}$$

$$
\begin{aligned}
Q_3 &= \mathrm{e}^{-a_i(t+r)}\int_0^{t+r}\mathrm{e}^{a_is}\sum_{j=1}^{n}c_{ij}g_j\left(y_j\left(s-\tau_{ij}(s)\right)\right)\mathrm{d}s\\
&\quad-\mathrm{e}^{-a_it}\int_0^{t}\mathrm{e}^{a_is}\sum_{j=1}^{n}c_{ij}g_j\left(y_j\left(s-\tau_{ij}(s)\right)\right)\mathrm{d}s,\\
Q_4 &= \mathrm{e}^{-a_i(t+r)}\int_0^{t+r}\mathrm{e}^{a_is}\sum_{j=1}^{n}d_{ij}\int_0^{\rho(s)}\sigma_j\left(y_j(s-\theta)\right)\mathrm{d}\theta\mathrm{d}s\\
&\quad-\mathrm{e}^{-a_it}\int_0^{t}\mathrm{e}^{a_is}\sum_{j=1}^{n}d_{ij}\int_0^{\rho(s)}\sigma_j\left(y_j(s-\theta)\right)\mathrm{d}\theta\mathrm{d}s,\\
Q_5 &= \mathrm{e}^{-a_i(t+r)}\sum_{0<t_k<(t+r)}\left\{I_{ik}\left(y_i(t_k)\right)\mathrm{e}^{a_it_k}\right\}\\
&\quad-\mathrm{e}^{-a_it}\sum_{0<t_k<t}\left\{I_{ik}\left(y_i(t_k)\right)\mathrm{e}^{a_it_k}\right\}.
\end{aligned}
\tag{24}
$$

Since $y_i(t)\in\mathscr{H}_i$, we know that $y_i(t)$ is continuous on $t\neq t_k$ $(k=1,2,\ldots)$; moreover, $\lim_{t\to t_k^-}y_i(t)$ and $\lim_{t\to t_k^+}y_i(t)$ exist, in addition, $\lim_{t\to t_k^-}y_i(t)=y_i(t_k)$.

Letting $t\neq t_k$ $(k=1,2,\ldots)$ in (23), it is easy to see that $Q_i\to0$ as $r\to0$ for $i=1,\ldots,5$. Thus, $\pi(y_i)(t+r)-\pi(y_i)(t)\to0$ as $r\to0$ holds on $t>0$ and $t\neq t_k$ $(k=1,2,\ldots)$.

Letting $t=t_k$ $(k=1,2,\ldots)$ in (23), it is not difficult to find that $Q_i\to0$ as $r\to0$ for $i=1,\ldots,4$. Letting $r<0$ be small enough, we compute

$$
\begin{aligned}
Q_5 &= \mathrm{e}^{-a_i(t_k+r)}\sum_{0<t_m<(t_k+r)}I_{im}\left(y_i(t_m)\right)\mathrm{e}^{a_it_m}\\
&\quad-\mathrm{e}^{-a_it_k}\sum_{0<t_m<t_k}I_{im}\left(y_i(t_m)\right)\mathrm{e}^{a_it_m}\\
&= \left\{\mathrm{e}^{-a_i(t_k+r)}-\mathrm{e}^{-a_it_k}\right\}\sum_{0<t_m<t_k}\left\{I_{im}\left(y_i(t_m)\right)\mathrm{e}^{a_it_m}\right\}
\end{aligned}
\tag{25}
$$

which implies $\lim_{r\to0^-}Q_5=0$. Letting $r>0$ be small enough, we have

$$
\begin{aligned}
Q_5 &= \mathrm{e}^{-a_i(t_k+r)}\sum_{0<t_m<(t_k+r)}I_{im}\left(y_i(t_m)\right)\mathrm{e}^{a_it_m}\\
&\quad-\mathrm{e}^{-a_it_k}\sum_{0<t_m<t_k}I_{im}\left(y_i(t_m)\right)\mathrm{e}^{a_it_m}\\
&= \mathrm{e}^{-a_i(t_k+r)}\left\{\sum_{0<t_m<t_k}\left\{I_{im}\left(y_i(t_m)\right)\mathrm{e}^{a_it_m}\right\}+I_{ik}\left(y_i(t_k)\right)\mathrm{e}^{a_it_k}\right\}\\
&\quad-\mathrm{e}^{-a_it_k}\sum_{0<t_m<t_k}\left\{I_{im}\left(y_i(t_m)\right)\mathrm{e}^{a_it_m}\right\}\\
&= \left\{\mathrm{e}^{-a_i(t_k+r)}-\mathrm{e}^{-a_it_k}\right\}\sum_{0<t_m<t_k}\left\{I_{im}\left(y_i(t_m)\right)\mathrm{e}^{a_it_m}\right\}\\
&\quad+\mathrm{e}^{-a_i(t_k+r)}I_{ik}\left(y_i(t_k)\right)\mathrm{e}^{a_it_k},
\end{aligned}
\tag{26}
$$

which implies $\lim_{r\to0^+}Q_5=\mathrm{e}^{-a_it_k}I_{ik}(y_i(t_k))\mathrm{e}^{a_it_k}$.

According to the above discussion, we see that $\pi(y_i)(t):[-m^*,+\infty)\to R$ is continuous on $t\neq t_k$ $(k=1,2,\ldots)$, while for $t=t_k$ $(k=1,2,\ldots)$, $\lim_{t\to t_k^-}\pi(y_i)(t)$ and $\lim_{t\to t_k^+}\pi(y_i)(t)$ exist; moreover, $\lim_{t\to t_k^-}\pi(y_i)(t)=\pi(y_i)(t_k)\neq\lim_{t\to t_k^+}\pi(y_i)(t)$.

Next, we will prove that $\mathrm{e}^{\alpha t}\pi(y_i)(t)\to0$ as $t\to\infty$ for $i\in\mathscr{N}$. To begin with, we give the expression of $\mathrm{e}^{\alpha t}\pi(y_i)(t)$ as follows:

$$
\mathrm{e}^{\alpha t}\pi(y_i)(t)=W_1+W_2+W_3+W_4+W_5,\quad t>0,
\tag{27}
$$

where

$W_1=\varphi_i(0)\mathrm{e}^{-(a_i-\alpha)t}$,

$W_2=\mathrm{e}^{\alpha t}\mathrm{e}^{-a_it}\int_0^t\mathrm{e}^{a_is}\sum_{j=1}^{n}b_{ij}f_j(y_j(s))\mathrm{d}s$,

$W_5=\mathrm{e}^{\alpha t}\mathrm{e}^{-a_it}\sum_{0<t_k<t}\{I_{ik}(y_i(t_k))\mathrm{e}^{a_it_k}\}$,

$W_3=\mathrm{e}^{\alpha t}\mathrm{e}^{-a_it}\int_0^t\mathrm{e}^{a_is}\sum_{j=1}^{n}c_{ij}g_j(y_j(s-\tau_{ij}(s)))\mathrm{d}s$, and

$W_4=\mathrm{e}^{\alpha t}\mathrm{e}^{-a_it}\int_0^t\mathrm{e}^{a_is}\sum_{j=1}^{n}d_{ij}\int_0^{\rho(s)}\sigma_j(y_j(s-\theta))\mathrm{d}\theta\mathrm{d}s$.

First, it is obvious that $\lim_{t\to\infty}W_1=0$ as $a_i-\alpha>0$. Furthermore, for $y_j(t)\in\mathscr{H}_j$ $(j\in\mathscr{N})$, we see $\lim_{t\to\infty}\mathrm{e}^{\alpha t}y_j(t)=0$. Then, for any $\varepsilon>0$, there exists a $T_j>0$ such that $s\geq T_j$ implies $|\mathrm{e}^{\alpha s}y_j(s)|<\varepsilon$. Choose $T^*=\max_{j\in\mathscr{N}}\{T_j\}$. It is derived form (A1) that

$$
\begin{aligned}
W_2 &\leq \mathrm{e}^{\alpha t}\mathrm{e}^{-a_it}\int_0^t\mathrm{e}^{a_is}\sum_{j=1}^{n}\left\{\left|b_{ij}l_j\right|\left|y_j(s)\right|\right\}\mathrm{d}s\\
&= \mathrm{e}^{\alpha t}\mathrm{e}^{-a_it}\int_0^t\mathrm{e}^{a_is}\mathrm{e}^{-\alpha s}\sum_{j=1}^{n}\left\{\left|b_{ij}l_j\right|\mathrm{e}^{\alpha s}\left|y_j(s)\right|\right\}\mathrm{d}s\\
&= \mathrm{e}^{-(a_i-\alpha)t}\int_0^{T^*}\mathrm{e}^{(a_i-\alpha)s}\sum_{j=1}^{n}\left\{\left|b_{ij}l_j\right|\mathrm{e}^{\alpha s}\left|y_j(s)\right|\right\}\mathrm{d}s\\
&\quad+\mathrm{e}^{-(a_i-\alpha)t}\int_{T^*}^{t}\mathrm{e}^{(a_i-\alpha)s}\sum_{j=1}^{n}\left\{\left|b_{ij}l_j\right|\mathrm{e}^{\alpha s}\left|y_j(s)\right|\right\}\mathrm{d}s\\
&\leq \mathrm{e}^{-(a_i-\alpha)t}\sum_{j=1}^{n}\left\{\left|b_{ij}l_j\right|\sup_{s\in[0,T^*]}\left|\mathrm{e}^{\alpha s}y_j(s)\right|\right\}\left\{\int_0^{T^*}\mathrm{e}^{(a_i-\alpha)s}\mathrm{d}s\right\}\\
&\quad+\varepsilon\sum_{j=1}^{n}\left\{\left|b_{ij}l_j\right|\right\}\mathrm{e}^{-(a_i-\alpha)t}\int_{T^*}^{t}\mathrm{e}^{(a_i-\alpha)s}\mathrm{d}s\\
&\leq \mathrm{e}^{-(a_i-\alpha)t}\sum_{j=1}^{n}\left\{\left|b_{ij}l_j\right|\sup_{s\in[0,T^*]}\left|\mathrm{e}^{\alpha s}y_j(s)\right|\right\}\left\{\int_0^{T^*}\mathrm{e}^{(a_i-\alpha)s}\mathrm{d}s\right\}\\
&\quad+\frac{\varepsilon}{a_i-\alpha}\sum_{j=1}^{n}\left\{\left|b_{ij}l_j\right|\right\},
\end{aligned}
\tag{28}
$$

which leads to $W_2\to0$ as $t\to\infty$.

Similarly, for the given $\varepsilon > 0$ above, there also exists a $T_j' > 0$ such that $s \geq T_j' - \tau$ implies $|e^{\alpha s} y_j(s)| < \varepsilon$. Select $\widehat{T} = \max_{j\in\mathcal{N}}\{T_j'\}$. It follows from (A2) that

$$\begin{aligned}
W_3 &\leq e^{\alpha t} e^{-a_i t} \int_0^t e^{a_i s} \sum_{j=1}^n \left\{ \left| c_{ij} k_j \right| \left| y_j \left(s - \tau_{ij}(s) \right) \right| \right\} ds \\
&\leq e^{-(a_i-\alpha)t} \\
&\quad \times \int_0^t e^{a_i s} e^{-\alpha\{s-\tau\}} \\
&\qquad \times \sum_{j=1}^n \left\{ \left| c_{ij} k_j \right| e^{\alpha[s-\tau_{ij}(s)]} \left| y_j \left(s - \tau_{ij}(s) \right) \right| \right\} ds \\
&= e^{\alpha\tau} e^{-(a_i-\alpha)t} \\
&\quad \times \int_0^{\widehat{T}} e^{(a_i-\alpha)s} \sum_{j=1}^n \left\{ \left| c_{ij} k_j \right| e^{\alpha[s-\tau_{ij}(s)]} \left| y_j \left(s - \tau_{ij}(s) \right) \right| \right\} ds \\
&\quad + e^{\alpha\tau} e^{-(a_i-\alpha)t} \\
&\quad \times \int_{\widehat{T}}^t e^{(a_i-\alpha)s} \sum_{j=1}^n \left\{ \left| c_{ij} k_j \right| e^{\alpha[s-\tau_{ij}(s)]} \left| y_j \left(s - \tau_{ij}(s) \right) \right| \right\} ds \\
&\leq e^{\alpha\tau} \sum_{j=1}^n \left\{ \left| c_{ij} k_j \right| \sup_{s\in[-\tau,\widehat{T}]} \left| e^{\alpha s} y_j(s) \right| \right\} \\
&\quad \times e^{-(a_i-\alpha)t} \int_0^{\widehat{T}} e^{(a_i-\alpha)s} ds \\
&\quad + e^{\alpha\tau} \varepsilon \sum_{j=1}^n \left\{ \left| c_{ij} k_j \right| \right\} e^{-(a_i-\alpha)t} \int_{\widehat{T}}^t e^{(a_i-\alpha)s} ds \\
&\leq e^{\alpha\tau} \sum_{j=1}^n \left\{ \left| c_{ij} k_j \right| \sup_{s\in[-\tau,\widehat{T}]} \left| e^{\alpha s} y_j(s) \right| \right\} \\
&\quad \times e^{-(a_i-\alpha)t} \int_0^{\widehat{T}} e^{(a_i-\alpha)s} ds + \frac{e^{\alpha\tau}\varepsilon}{a_i - \alpha} \sum_{j=1}^n \left\{ \left| c_{ij} k_j \right| \right\},
\end{aligned} \tag{29}$$

which results in $W_3 \to 0$ as $t \to \infty$. In addition, it is derived from (A4) that

$$\begin{aligned}
W_4 &\leq e^{\alpha t} e^{-a_i t} \int_0^t e^{a_i s} \sum_{j=1}^n \left\{ d_{ij} \int_0^\rho \omega_j \left| y_j(s-\theta) \right| d\theta \right\} ds \\
&= e^{\alpha t} e^{-a_i t} \\
&\quad \times \int_0^t e^{a_i s} e^{-\alpha s} \sum_{j=1}^n \left\{ d_{ij} \int_0^\rho e^{\alpha\theta} \omega_j e^{\alpha(s-\theta)} \left| y_j(s-\theta) \right| d\theta \right\} ds \\
&\leq e^{\alpha t} e^{-a_i t} \\
&\quad \times \int_0^t e^{a_i s} e^{-\alpha s} \\
&\qquad \times \sum_{j=1}^n \left\{ d_{ij} \sup_{\zeta\in[s-\rho,s]} \left\{ e^{\alpha\zeta} \left| y_j(\zeta) \right| \right\} \int_0^\rho e^{\alpha\theta} \omega_j d\theta \right\} ds.
\end{aligned} \tag{30}$$

Since $e^{\alpha\zeta}|y_j(\zeta)| \to 0$ as $\zeta \to \infty$, we know that, for any $\varepsilon > 0$, there exists a $T_j'' > 0$ such that $\zeta > T_j'' - \rho$ implies $e^{\alpha\zeta}|y_j(\zeta)| < \varepsilon$. Selecting $\overline{T} = \max_{j\in\mathcal{N}}\{T_j''\}$, it follows from (30) that

$$\begin{aligned}
W_4 &\leq e^{(\alpha-a_i)t} \\
&\quad \times \int_0^{\overline{T}} e^{(a_i-\alpha)s} \sum_{j=1}^n \left\{ d_{ij} \sup_{\zeta\in[s-\rho,s]} \left\{ e^{\alpha\zeta} \left| y_j(\zeta) \right| \right\} \right. \\
&\qquad\qquad \left. \times \int_0^\rho e^{\alpha\theta} \omega_j d\theta \right\} ds \\
&\quad + e^{(\alpha-a_i)t} \\
&\quad \times \int_{\overline{T}}^t e^{(a_i-\alpha)s} \sum_{j=1}^n \left\{ d_{ij} \sup_{\zeta\in[s-\rho,s]} \left\{ e^{\alpha\zeta} \left| y_j(\zeta) \right| \right\} \right. \\
&\qquad\qquad \left. \times \int_0^\rho e^{\alpha\theta} \omega_j d\theta \right\} ds \\
&\leq \frac{e^{\alpha\rho}}{\alpha} \left\{ \sum_{j=1}^n d_{ij} \omega_j \sup_{\zeta\in[-\rho,\overline{T}]} \left\{ e^{\alpha\zeta} \left| y_j(\zeta) \right| \right\} \right\} \\
&\quad \times e^{(\alpha-a_i)t} \int_0^{\overline{T}} e^{(a_i-\alpha)s} ds \\
&\quad + \frac{e^{\alpha\rho}}{\alpha} \sum_{j=1}^n \left\{ d_{ij} \omega_j \sup_{\zeta\in[\overline{T}-\rho,t]} \left\{ e^{\alpha\zeta} \left| y_j(\zeta) \right| \right\} \right\} \\
&\quad \times e^{(\alpha-a_i)t} \int_{\overline{T}}^t e^{(a_i-\alpha)s} ds \\
&\leq e^{(\alpha-a_i)t} \frac{e^{\alpha\rho}}{\alpha} \left\{ \sum_{j=1}^n d_{ij} \omega_j \sup_{\zeta\in[-\rho,\overline{T}]} \left\{ e^{\alpha\zeta} \left| y_j(\zeta) \right| \right\} \right\} \\
&\quad \times \int_0^{\overline{T}} e^{(a_i-\alpha)s} ds + \varepsilon \sum_{j=1}^n \left\{ d_{ij} \omega_j \right\} \frac{e^{\alpha\rho}}{\alpha(a_i-\alpha)},
\end{aligned} \tag{31}$$

which yields $W_4 \to 0$ as $t \to \infty$.

Furthermore, from (A3), we see that $|I_{ik}(x_i(t_k))| \leq p_{ik}|y_i(t_k)|$. So,

$$W_5 \leq e^{\alpha t} e^{-a_i t} \sum_{0<t_k<t} \left\{ p_{ik} \left| y_i(t_k) \right| e^{a_i t_k} \right\}. \tag{32}$$

As $y_i(t) \in \mathcal{H}_i$, we have $\lim_{t\to\infty} e^{\alpha t} y_i(t) = 0$. Then, for any $\varepsilon > 0$, there exists a nonimpulsive point $T_i > 0$ such that

$s \geq T_i$ implies $|e^{\alpha s} y_i(s)| < \varepsilon$. It then follows from conditions (i) and (ii) that

$$\begin{aligned}
W_5 &\leq e^{\alpha t} e^{-a_i t} \left\{ \sum_{0<t_k<T_i} \left\{ p_{ik} \left| y_i(t_k) \right| e^{a_i t_k} \right\} + \sum_{T_i<t_k<t} \left\{ p_{ik} \left| y_i(t_k) \right| e^{\alpha t_k} e^{(a_i-\alpha) t_k} \right\} \right\} \\
&\leq e^{\alpha t} e^{-a_i t} \sum_{0<t_k<T_i} \left\{ p_{ik} \left| y_i(t_k) \right| e^{a_i t_k} \right\} + e^{\alpha t} e^{-a_i t} p_i \varepsilon \sum_{T_i<t_k<t} \left\{ \mu e^{(a_i-\alpha) t_k} \right\} \\
&\leq e^{-(a_i-\alpha) t} \sum_{0<t_k<T_i} \left\{ p_{ik} \left| y_i(t_k) \right| e^{a_i t_k} \right\} + e^{-(a_i-\alpha) t} p_i \varepsilon \left\{ \sum_{T_i<t_r<t_k} \left\{ e^{(a_i-\alpha) t_r} (t_{r+1} - t_r) \right\} + \mu e^{(a_i-\alpha) t_k} \right\} \\
&\leq e^{-(a_i-\alpha) t} \sum_{0<t_k<T_i} \left\{ p_{ik} \left| y_i(t_k) \right| e^{a_i t_k} \right\} + e^{-(a_i-\alpha) t} p_i \varepsilon \int_{T_i}^{t} e^{(a_i-\alpha) s} ds + e^{-(a_i-\alpha) t} p_i \varepsilon \mu e^{(a_i-\alpha) t} \\
&\leq e^{-(a_i-\alpha) t} \sum_{0<t_k<T_i} \left\{ p_{ik} \left| y_i(t_k) \right| e^{a_i t_k} \right\} + \frac{p_i \varepsilon}{a_i - \alpha} + p_i \varepsilon \mu,
\end{aligned} \tag{33}$$

which means that $W_5 \to 0$ as $t \to \infty$.

Now, we can derive from (27) that $e^{\alpha t} \pi(y_i)(t) \to 0$ as $t \to \infty$ for $i \in \mathcal{N}$. It is therefore concluded that $\pi(y_i)(t) \subset \mathcal{H}_i$ which results in $\pi(\mathcal{H}) \subset \mathcal{H}$.

Step 2. We need to prove that π is contractive. For $\bar{\mathbf{z}} = (z_1(t), \ldots, z_n(t)) \in \mathcal{H}$ and $\bar{\mathbf{y}} = (y_1(t), \ldots, y_n(t)) \in \mathcal{H}$, we estimate

$$\left| \pi(y_i)(t) - \pi(z_i)(t) \right| \leq J_1 + J_2 + J_3 + J_4, \tag{34}$$

where

$$\begin{aligned}
J_1 &= e^{-a_i t} \int_0^t e^{a_i s} \sum_{j=1}^{n} \left[\left| b_{ij} \right| \left| f_j(y_j(s)) - f_j(z_j(s)) \right| \right] ds, \\
J_2 &= e^{-a_i t} \int_0^t e^{a_i s} \sum_{j=1}^{n} \left[\left| c_{ij} \right| \left| g_j(y_j(s - \tau_{ij}(s))) - g_j(z_j(s - \tau_{ij}(s))) \right| \right] ds, \\
J_3 &= e^{-a_i t} \int_0^t e^{a_i s} \sum_{j=1}^{n} \left| d_{ij} \right| \int_0^{\rho(s)} \left| \sigma_j(y_j(s-\theta)) - \sigma_j(z_j(s-\theta)) \right| d\theta ds, \\
J_4 &= e^{-a_i t} \sum_{0<t_k<t} \left\{ e^{a_i t_k} \left| I_{ik}(y_i(t_k)) - I_{ik}(z_i(t_k)) \right| \right\}.
\end{aligned} \tag{35}$$

Note that

$$\begin{aligned}
J_1 &\leq e^{-a_i t} \int_0^t e^{a_i s} \sum_{j=1}^{n} \left[\left| b_{ij} l_j \right| \left| y_j(s) - z_j(s) \right| \right] ds \\
&\leq \max_{j \in \mathcal{N}} \left| b_{ij} l_j \right| \sum_{j=1}^{n} \left\{ \sup_{s \in [0,t]} \left| y_j(s) - z_j(s) \right| \right\} \times e^{-a_i t} \int_0^t e^{a_i s} ds \\
&\leq \frac{1}{a_i} \max_{j \in \mathcal{N}} \left| b_{ij} l_j \right| \sum_{j=1}^{n} \left\{ \sup_{s \in [0,t]} \left| y_j(s) - z_j(s) \right| \right\}, \\
J_2 &\leq e^{-a_i t} \int_0^t e^{a_i s} \sum_{j=1}^{n} \left[\left| c_{ij} k_j \right| \left| y_j(s - \tau_{ij}(s)) - z_j(s - \tau_{ij}(s)) \right| \right] ds \\
&\leq \max_{j \in \mathcal{N}} \left| c_{ij} k_j \right| \sum_{j=1}^{n} \left\{ \sup_{\xi \in [-\tau,t]} \left| y_j(\xi) - z_j(\xi) \right| \right\} \times e^{-a_i t} \int_0^t e^{a_i s} ds \\
&\leq \frac{1}{a_i} \max_{j \in \mathcal{N}} \left| c_{ij} k_j \right| \sum_{j=1}^{n} \left\{ \sup_{\xi \in [-\tau,t]} \left| y_j(\xi) - z_j(\xi) \right| \right\}, \\
J_3 &\leq e^{-a_i t} \int_0^t e^{a_i s} \sum_{j=1}^{n} \left\{ \left| d_{ij} \right| \int_0^{\rho(s)} \omega_j \left| y_j(s-\theta) - z_j(s-\theta) \right| d\theta \right\} ds \\
&\leq e^{-a_i t} \int_0^t e^{a_i s} \sum_{j=1}^{n} \left\{ \left| d_{ij} \right| \sup_{\xi \in [s-\rho,s]} \left| y_j(\xi) - z_j(\xi) \right| \times \int_0^{\rho(s)} \omega_j d\theta \right\} ds \\
&\leq e^{-a_i t} \int_0^t e^{a_i s} \sum_{j=1}^{n} \left\{ \omega_j \left| d_{ij} \right| \sup_{\xi \in [-\rho,t]} \left| y_j(\xi) - z_j(\xi) \right| \rho(s) \right\} ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \max_{j\in\mathcal{N}}\left\{\omega_j\left|d_{ij}\right|\right\}\sum_{j=1}^{n}\left\{\sup_{\xi\in[-\rho,t]}\left|y_j(\xi)-z_j(\xi)\right|\right\}\\
&\quad\times \mathrm{e}^{-a_it}\int_0^t \mathrm{e}^{a_is}\rho(s)\,\mathrm{d}s\\
&\leq \frac{\rho}{a_i}\max_{j\in\mathcal{N}}\left\{\omega_j\left|d_{ij}\right|\right\}\sum_{j=1}^{n}\left\{\sup_{\xi\in[-\rho,t]}\left|y_j(\xi)-z_j(\xi)\right|\right\},\\
J_4 &\leq \mathrm{e}^{-a_it}\sum_{0<t_k<t}\left\{\mathrm{e}^{a_it_k}p_{ik}\left|y_i(t_k)-z_i(t_k)\right|\right\}\\
&\leq p_i\mathrm{e}^{-a_it}\sup_{s\in[0,t]}\left|y_i(s)-z_i(s)\right|\sum_{0<t_k<t}\left\{\mathrm{e}^{a_it_k}\mu\right\}\\
&\leq p_i\mathrm{e}^{-a_it}\sup_{s\in[0,t]}\left|y_i(s)-z_i(s)\right|\\
&\quad\times\left\{\sum_{0<t_r<t_k}\left\{\mathrm{e}^{a_it_r}\left(t_{r+1}-t_r\right)\right\}+\mathrm{e}^{a_it_k}\mu\right\}\\
&\leq p_i\sup_{s\in[0,t]}\left|y_i(s)-z_i(s)\right|\mathrm{e}^{-a_it}\left\{\int_0^t \mathrm{e}^{a_is}\mathrm{d}s+\mathrm{e}^{a_it}\mu\right\}\\
&\leq p_i\left(\mu+\frac{1}{a_i}\right)\sup_{s\in[0,t]}\left|y_i(s)-z_i(s)\right|.
\end{aligned}
\tag{36}
$$

It is then derived from (36) that

$$
\begin{aligned}
&\sup_{t\in[-m^*,T]}\left|\pi(y_i)(t)-\pi(z_i)(t)\right|\\
&\leq \frac{1}{a_i}\max_{j\in\mathcal{N}}\left|b_{ij}l_j\right|\sum_{j=1}^{n}\left\{\sup_{s\in[-m^*,T]}\left|y_j(s)-z_j(s)\right|\right\}\\
&\quad+\frac{1}{a_i}\max_{j\in\mathcal{N}}\left|c_{ij}k_j\right|\sum_{j=1}^{n}\left\{\sup_{\xi\in[-m^*,T]}\left|y_j(\xi)-z_j(\xi)\right|\right\}\\
&\quad+\frac{\rho}{a_i}\max_{j\in\mathcal{N}}\left\{\omega_j\left|d_{ij}\right|\right\}\sum_{j=1}^{n}\left\{\sup_{\xi\in[-m^*,T]}\left|y_j(\xi)-z_j(\xi)\right|\right\}\\
&\quad+p_i\left(\mu+\frac{1}{a_i}\right)\sup_{s\in[-m^*,T]}\left|y_i(s)-z_i(s)\right|,
\end{aligned}
\tag{37}
$$

which means that

$$
\begin{aligned}
&\sum_{i=1}^{n}\sup_{t\in[-m^*,T]}\left|\pi(y_i)(t)-\pi(z_i)(t)\right|\\
&\qquad\leq \chi\sum_{j=1}^{n}\left\{\sup_{s\in[-m^*,T]}\left|y_j(s)-z_j(s)\right|\right\},
\end{aligned}
\tag{38}
$$

where

$$
\begin{aligned}
\chi \triangleq \sum_{i=1}^{n}&\left\{\frac{1}{a_i}\max_{j\in\mathcal{N}}\left|b_{ij}l_j\right|+\frac{1}{a_i}\max_{j\in\mathcal{N}}\left|c_{ij}k_j\right|+\frac{\rho}{a_i}\max_{j\in\mathcal{N}}\left|\omega_jd_{ij}\right|\right\}\\
&+\max_{i\in\mathcal{N}}\left\{p_i\left(\mu+\frac{1}{a_i}\right)\right\}.
\end{aligned}
\tag{39}
$$

In view of condition (iii), we know that π is a contraction mapping, and hence, there exists a unique fixed point $\overline{\mathbf{y}}(\cdot)$ of π in $\mathscr{H}$ which means that $\overline{\mathbf{y}}^{\mathrm{T}}(\cdot)$ is the solution to (1)–(3) and $\mathrm{e}^{\alpha t}\|\overline{\mathbf{y}}^{\mathrm{T}}(\cdot)\| \to 0$ as $t \to \infty$. This completes the proof. □

Lemma 4. *Assume conditions (A1)–(A4) hold. Provided that*

(i) $\inf_{k=1,2,\ldots}\{t_k-t_{k-1}\}\geq 1$,

(ii) *there exist constants* p_i *such that* $p_{ik}\leq p_i$ *for* $i\in\mathcal{N}$ *and* $k=1,2,\ldots$,

(iii) $\sum_{i=1}^{n}\{(1/a_i)\max_{j\in\mathcal{N}}|b_{ij}l_j| + (1/a_i)\max_{j\in\mathcal{N}}|c_{ij}k_j| + (\rho/a_i)\max_{j\in\mathcal{N}}|\omega_jd_{ij}|\}+\max_{i\in\mathcal{N}}\{p_i(1+(1/a_i))\}\triangleq\chi<1$,

then the trivial equilibrium $\mathbf{x}=0$ *is globally exponentially stable.*

Proof. Lemma 4 is a direct conclusion by letting $\mu = 1$ in Theorem 3. □

Remark 5. In Theorem 3, we use the fixed point theorem to prove the existence and uniqueness of solution and the global exponential stability of trivial equilibrium all at once, while Lyapunov method fails to do this.

Remark 6. The presented sufficient conditions in Theorem 3 and Lemma 4 do not require even the differentiability of delays, let alone the monotone decreasing behavior of delays which is necessary in some relevant works.

4. Example

Consider the following two-dimensional impulsive cellular neural network with time-varying and distributed delays.

$$
\begin{aligned}
\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = &-a_ix_i(t)+\sum_{j=1}^{2}b_{ij}f_j\left(x_j(t)\right)\\
&+\sum_{j=1}^{2}c_{ij}g_j\left(x_j\left(t-\tau_{ij}(t)\right)\right)\\
&+\sum_{j=1}^{2}d_{ij}\int_0^{\rho(t)}\sigma_j\left(x_j(t-\theta)\right)\mathrm{d}\theta,\quad t\geq 0,\ t\neq t_k,\\
\Delta x_i(t_k) = x_i&(t_k+0)-x_i(t_k)=\arctan\left(0.4x_i(t_k)\right),\\
&k=1,2,\ldots,
\end{aligned}
\tag{40}
$$

with the initial conditions $x_1(s) = \cos(s)$, $x_2(s) = \sin(s)$ on $-m^* \leq s \leq 0$, where $\tau_{ij}(t) = 0.8 + 0.4\cos(t)$, $\rho(t) = 0.5 + 0.3\sin(t)$, m^* is defined as shown in (3), $a_1 = a_2 = 7$, $b_{ij} = 0$, $c_{11} = 0$, $c_{12} = 1/7$, $c_{21} = -1/7$, $c_{22} = -1/7$, $d_{11} = 3/7$, $d_{12} = 2/7$, $d_{21} = 0$, $d_{22} = 1/7$, $f_j(s) = g_j(s) = \sigma_j(s) = (|s+1| - |s-1|)/2$, and $t_k = t_{k-1} + 0.5k$.

It is easily to find that $\mu = 0.5$, $l_j = k_j = \omega_j = 1$, and $p_{ik} = 0.4$. Let $p_i = 0.8$ and compute

$$\sum_{i=1}^{2} \left\{ \frac{1}{a_i} \max_{j\in\mathcal{N}} \left| b_{ij} l_j \right| + \frac{1}{a_i} \max_{j\in\mathcal{N}} \left| c_{ij} k_j \right| + \frac{\rho}{a_i} \max_{j\in\mathcal{N}} \left| \omega_j d_{ij} \right| \right\} + \max_{i\in\mathcal{N}} \left\{ p_i \left(\mu + \frac{1}{a_i} \right) \right\} < 1. \tag{41}$$

From Theorem 3, we conclude that the trivial equilibrium $\mathbf{x} = 0$ of this two-dimensional impulsive cellular neural network with time-varying and distributed delays is globally exponentially stable.

5. Conclusions

This article is a new attempt of applying the fixed point theory to the stability analysis of impulsive neural networks with time-varying and distributed delays, which is different from the existing relevant publications where Lyapunov theory is the main technique. From what have been discussed above, we see that the contraction mapping principle is effective for not only the investigation of the existence and uniqueness of solution but also for the stability analysis of trivial equilibrium. In the future, we will continue to explore the application of other kinds of fixed point theorems to the stability research of complex neural networks.

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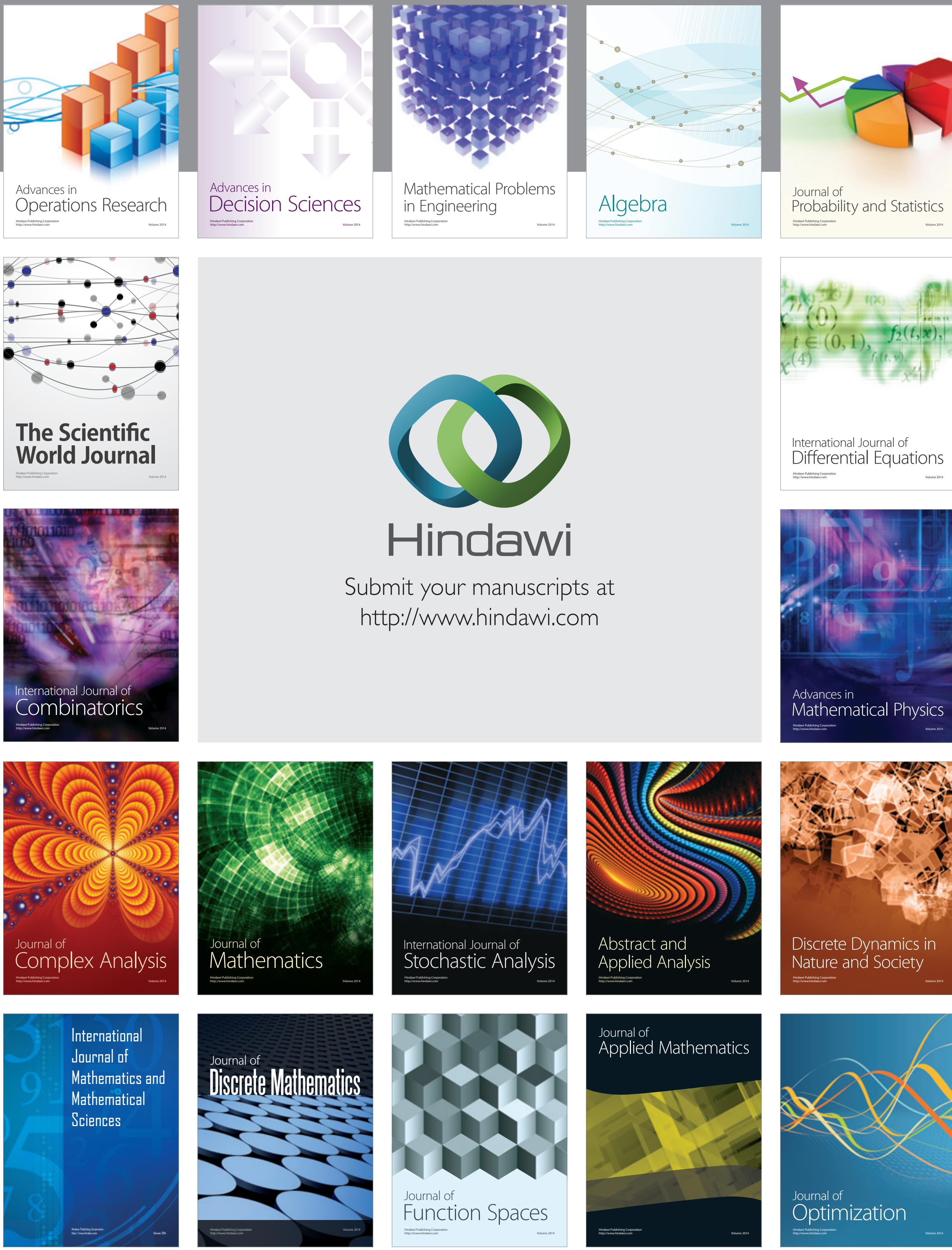
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