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Research Article

The First Integral Method to the Nonlinear Schrodinger Equations in Higher Dimensions

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The first integral method introduced by Feng is adopted for solving some important nonlinear partial differential equations, including the $(2+1)$-dimensional hyperbolic nonlinear Schrodinger (HNLS) equation, the generalized nonlinear Schrodinger (GNLS) equation with a source, and the higher-order nonlinear Schrodinger equation in nonlinear optical fibers. This method provides polynomial first integrals for autonomous planar systems. Through the established first integrals, exact traveling wave solutions are formally derived in a concise manner.

1. Introduction

It is well known that nonlinear complex physical phenomena are related to nonlinear partial differential equations (NLPDEs) which are involved in many fields from physics to biology, chemistry, mechanics, etc. As mathematical models of the phenomena, the investigation of exact solutions of NLPDEs will help us to understand these phenomena better. Many effective methods for obtaining exact solutions of NLPDEs have been established and developed, such as the Lie point symmetries method [1], the exp-function method [2, 3], the sine-cosine method [4, 5], the extended tanh-coth method [6, 7], the projective Riccati equation method [8, 9], and so on.

The first integral method was first proposed by Feng in [10] in solving Burgers-KdV equation which is based on the ring theory of commutative algebra. Recently, this useful method has been widely used by many such as in [11–21] and by the references therein. In Section 2, we have described this method for finding exact travelling wave solutions of nonlinear evolution equations. In Section 3, we have illustrated this method in detail with the (2+1)-dimensional hyperbolic nonlinear Schrodinger (HNLS) equation, the generalized nonlinear Schrodinger (GNLS) equation with a source, and the higher-order nonlinear Schrodinger equation in nonlinear optical fibers. In Section 4, we have given some conclusions.

2. The First Integral Method

Consider a general nonlinear PDE in the form

$$P\left(u, u_t, u_x, u_{xx}, u_{tt}, u_{xt}, u_{xxx}, \ldots\right) = 0, \tag{1}$$

where P is a polynomial in its arguments.

Raslan in [22] has summarized the first integral method in the following steps.

Step 1. Using a wave variable $\xi = x - ct + \varepsilon$, where ε is an arbitrary constant, (1) can be written in the following nonlinear ordinary differential equation (ODE):

$$Q\left(U, U', U'', U''', \ldots\right) = 0, \tag{2}$$

where the prime denotes the derivation with respect to ξ.

Step 2. Assume that the solution of ODE (2) can be written as

$$u(x,t) = u(\xi). \tag{3}$$

Step 3. We introduced new independent variables

$$X(\xi) = u(\xi), \qquad Y = u_{\xi}(\xi), \tag{4}$$

which leads a system of nonlinear ODEs

$$X_{\xi}(\xi) = Y(\xi), \tag{5a}$$

$$Y_{\xi}(\xi) = F(X(\xi), Y(\xi)). \tag{5b}$$

Step 4. According to the qualitative theory of ODEs [23], if we can find the integrals to (5a) and (5b) under the same conditions, then the general solution to (5a) and (5b) can be found directly. However, in general, it is really difficult to realize this even for one first integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integrals, nor is there a logical way for telling us what these first integrals are.

We will apply the division theorem to obtain one first integral to (5a) and (5b) which reduces (2) to a first-order integrable ODE.

An exact solution to (1) is then obtained by solving this equation.

Let us now recall the division theorem for two variables in the complex domain $C(w, z)$.

Theorem 1 (division theorem). *Suppose that $P(w, z)$ and $Q(w, z)$ are polynomials in $C(w, z)$, and $P(w, z)$ is irreducible in $C(w, z)$. If $Q(w, z)$ vanishes at all zero points of $P(w, z)$, then there exists a polynomial $G(w, z)$ in $C(w, z)$ such that*

$$Q(w, z) = P(w, z) G(w, z). \tag{6}$$

The division theorem follows immediately from the Hilbert-Nullstellensatz Theorem [24].

Theorem 2 (Hilbert-Nullstellensatz theorem). *Let k be a field and L an algebraic closure of k.*

(1) *Every ideal γ of $k[X_1, \ldots, X_n]$ not containing 1 admits at least one zero in L^n.*

(2) *Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ be two elements of L^n; for the set of polynomials of $k[X_1, \ldots, X_n]$ zero at x to be identical with the set of polynomials of $k[X_1, \ldots, X_n]$ zero at y it is necessary and sufficient that there exists a k-automorphism s of L such that $y_i = s(x_i)$ for $1 \le i \le n$.*

(3) *For an ideal α of $k[X_1, \ldots, X_n]$ to be maximal, it is necessary and sufficient that there exists an x in L^n such that α is the set of polynomials of $k[X_1, \ldots, X_n]$ to be zero at x.*

(4) *For a polynomial Q of $k[X_1, \ldots, X_n]$ to be zero on the set of zeros in L^n of an ideal γ of $k[X_1, \ldots, X_n]$, it is necessary and sufficient that there exists an integer $m > 0$ such that $Q^m \in \gamma$.*

3. Applications

In this section, we have investigated three NPDEs using the first integral method for the first time.

3.1. The (2+1)-Dimensional Hyperbolic Nonlinear Schrodinger Equation. Let us consider the (2+1)-dimensional hyperbolic nonlinear Schrodinger (HNLS) equation [25] which reads

$$iu_t + \frac{1}{2}u_{xx} - \frac{1}{2}u_{yy} + |u|^2 u = 0, \tag{7}$$

where x is dimensionless variable, y is the propagation coordinate, $i = \sqrt{-1}$, and t is the time. The above equation can be derived from optics [26] and large-scale Rossby waves [27]. Various types of NLS or HNLS equations describing time and space evolutions of slowly varying envelopes have wide applications in various branches of physics [28, 29].

By considering the transformations $u(x, y, t) = \varphi(\xi)\exp(i\eta)$, and the wave variable $\xi = x + ay - ct + \varsigma$, $\eta = mx + ny + wt + \varepsilon$, where, ς, and ε are arbitrary constants, (7) changes into a system of ordinary differential equations as follows:

$$\left(c^2 - 1\right)\varphi''(\xi) = 2\varphi^3(\xi) + \left(w^2 - 2n - (a + cw)^2\right)\varphi(\xi), \tag{8}$$

where prime denotes the derivative with respect to the same variable ξ.

Using (4) and (5a) and (5b), we can get

$$X'(\xi) = Y(\xi), \tag{9a}$$

$$Y'(\xi) = \left(\frac{2}{c^2 - 1}\right)X^3(\xi) + \left(\frac{w^2 - 2n - (a + cw)^2}{c^2 - 1}\right)X(\xi). \tag{9b}$$

According to the first integral method, we suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (9a) and (9b) and $P(X, Y) = \sum_{i=0}^{m} a_i(X)Y^i$ is an irreducible polynomial in the complex domain $C[X, Y]$ such that

$$P[X(\xi), Y(\xi)] = \sum_{i=0}^{m} a_i(X(\xi))\, Y^i(\xi) = 0, \tag{10}$$

where $a_i(X)$, $(i = 0, 1, 2, \ldots, m)$ are polynomials of X and $a_m(X) \neq 0$.

Equation (10) is called the first integral to (9a) and (9b). Due to the division theorem, there exists a polynomial $h(X) + g(X)Y$ in the complex domain $C[X, Y]$ such that

$$\begin{aligned}\frac{dP}{d\xi} &= \frac{\partial P}{\partial X}\frac{dX}{d\xi} + \frac{\partial P}{\partial Y}\frac{dY}{d\xi} \\ &= [h(X) + g(X)Y]\sum_{i=0}^{m} a_i(X)\, Y^i.\end{aligned} \tag{11}$$

Here, we have considered two different cases, assuming that $m = 1$ and $m = 2$ in (10).

Case 1. Suppose that $m = 1$, by equating the coefficients of Y^i $(i = 2, 1, 0)$ on both sides of (11), we have

$$a_1'(X) = g(X)\, a_1(X), \tag{12a}$$

$$a_0'(X) = h(X)\, a_1(X) + g(X)\, a_0(X), \tag{12b}$$

$$a_1(X)\left(\left(\frac{2}{c^2-1}\right)X^3 + \left(\frac{w^2 - 2n - (a+cw)^2}{c^2-1}\right)X\right) = h(X)\, a_0(X). \tag{12c}$$

Since $a_i(X)(i = 0, 1)$ are polynomials, then from (12a) we have deduced that $a_1(X)$ is constant and $g(X) = 0$. For simplicity, we have $a_1(X) = 1$.

Balancing the degrees of $h(X)$ and $a_0(X)$, we have concluded that $\deg(h(X)) = 1$ only. Suppose that $h(X) = AX + B$ and $A \neq 0$, we find $a_0(X)$

$$a_0(X) = \frac{A}{2}X^2 + BX + D, \tag{13}$$

where D is an arbitrary integration constant.

Substituting $a_0(X)$, $a_1(X)$, and $h(X)$ for (12c) and setting all the coefficients of powers X to be zero, we have obtained a system of nonlinear algebraic equations and by solving it, we have obtained

$$D = \mp\frac{1}{2}\sqrt{\frac{1}{-1+c^2}}\,[2n + (a + (-1+c)\,w)\,(a + w + cw)],$$
$$B = 0, \qquad A = \pm\sqrt{2}\sqrt{-\frac{1}{1-c} - \frac{1}{1+c}}. \tag{14}$$

Using the conditions (14) in (10), we obtain

$$Y(\xi) = \mp\frac{\sqrt{2}}{2}\sqrt{-\frac{1}{1-c} - \frac{1}{1+c}}X^2(\xi) \pm \frac{1}{2}\sqrt{\frac{1}{-1+c^2}}\,[2n + (a + (-1+c)\,w) \times (a + w + cw)], \tag{15}$$

respectively.

Combining (15) with (9a), we have obtained the exact solutions to (9a) ad (9b). The exact traveling wave solutions to the (2+1)-dimensional hyperbolic nonlinear Schrodinger equation (7) can be written as

$$u_{1,2}(x, y, t) = -\frac{1}{\sqrt{2}}\sqrt{-2n - (a + (-1+c)\,w)\,(a + w + cw)} \times \tan\Big[\sqrt{-2n - (a + (-1+c)\,w)\,(a + w + cw)} \times \left(\pm\sqrt{-1+c}\,(x + ay - ct + \varsigma) + 2(-1+c)\sqrt{-1+c}\,\xi_0\right) \times \left(\sqrt{2}(-1+c)\sqrt{-1+c}\right)^{-1}\Big] \times \exp[i(mx + ny + wt + \varepsilon)], \tag{16}$$

respectively, where ξ_0 is an arbitrary integration constant.

Case 2. Assume that $m = 2$, by equating the coefficients of Y^i $(i = 3, 2, 1, 0)$ on both sides of (11), we have

$$a_2'(X) = g(X)\, a_2(X), \tag{17a}$$

$$a_1'(X) = h(X)\, a_2(X) + g(X)\, a_1(X), \tag{17b}$$

$$a_0'(X) + 2a_2(X)\left[\left(\frac{2}{c^2-1}\right)X^3 + \left(\frac{w^2-2n-(a+cw)^2}{c^2-1}\right)X\right] = h(X)\, a_1(X) + g(X)\, a_0(X), \tag{17c}$$

$$a_1(X)\left[\left(\frac{2}{c^2-1}\right)X^3 + \left(\frac{w^2 - 2n - (a+cw)^2}{c^2-1}\right)X\right] = h(X)\, a_0(X). \tag{17d}$$

Since, $a_i(X)$ $(i = 0, 1, 2)$ are polynomials, then from (17a) it can be deduced that $a_2(X)$ is a constant and $g(X) = 0$. For simplicity, let us suppose that $a_2(X) = 1$. Balancing the degrees of $h(X)$ and $a_0(X)$ it can be concluded that $\deg(h(X)) = 1$ only.

In this case, let us assume that $h(X) = AX + B$ and $A \neq 0$, then we find $a_1(X)$ and $a_0(X)$ as follows:

$$a_1(X) = \left(\frac{A}{2}\right)X^2 + BX + D, \tag{18a}$$

$$a_0(X) = \left(\frac{A^2}{8} - \frac{1}{c^2-1}\right)X^4 + \frac{1}{2}(AB)X^3 + \left(\frac{AD + B^2}{2} - \frac{w^2 - 2n - (a+cw)^2}{c^2-1}\right)X^2 + BDX + F, \tag{18b}$$

where A, B, D, and F are arbitrary integration constants.

Substituting $a_0(X)$, $a_1(X)$, $a_2(X)$, and $h(X)$ for (17d) and setting all the coefficients of powers X to be zero, then we have obtained a system of nonlinear algebraic equations and by solving it, we get

$$F = 0, \qquad D = 0, \qquad B = 0,$$
$$w = \frac{-ac - \sqrt{a^2 + 2n - 2c^2 n}}{-1 + c^2},$$
$$A = \mp 2\sqrt{2}\sqrt{-\frac{1}{1-c} - \frac{1}{1+c}}, \tag{19a}$$
$$F = 0, \qquad D = 0, \qquad B = 0,$$
$$w = \frac{-ac + \sqrt{a^2 + 2n - 2c^2 n}}{-1 + c^2},$$
$$A = \mp 2\sqrt{2}\sqrt{-\frac{1}{1-c} - \frac{1}{1+c}}. \tag{19b}$$

Using the conditions (19a) and (19b) in (10), we have obtained

$$Y(\xi) = \pm\frac{1}{\sqrt{-1+c^2}}X^2(\xi). \tag{20}$$

Combining (20) with (9a) we have obtained the exact solutions to (9a) and (9b) and hence the exact traveling wave solutions to the (2+1)-dimensional hyperbolic nonlinear Schrodinger equation (7) can be written as

$$u_{3,4}(x,y,t) = \frac{\sqrt{-1+c}\sqrt{1+c}}{\mp(x+ay-ct+\varsigma)-\sqrt{-1+c}\sqrt{1+c}} \times \exp\left[i\left(mx+ny+\frac{-ac-\sqrt{a^2+2n-2c^2n}}{-1+c^2}t+\varepsilon\right)\right], \tag{21}$$

$$u_{5,6}(x,y,t) = \frac{\sqrt{-1+c}\sqrt{1+c}}{\mp(x+ay-ct+\varsigma)-\sqrt{-1+c}\sqrt{1+c}} \times \exp\left[i\left(mx+ny+\frac{-ac+\sqrt{a^2+2n-2c^2n}}{-1+c^2}t+\varepsilon\right)\right]. \tag{22}$$

Comparing these results with the results obtained in [1], it can be seen that the solutions here are new.

3.2. The Generalized Nonlinear Schrodinger (GNLS) Equation with a Source. Let us Consider the generalized nonlinear Schrodinger (GNLS) equation with a source [30, 31], in the form

$$iu_t + au_{xx} + bu|u|^2 + icu_{xxx} + id\left(u|u|^2\right)_x = ke^{i[\chi(\xi)-wt]}, \tag{23}$$

where $\xi = \alpha(x-\upsilon t)$ is a real function and a, b, c, d, k, α, υ, and w are all real.

The GNLS equation (23) plays an important role in many nonlinear sciences. It arises as an asymptotic limit for a slowly varying dispersive wave envelope in a nonlinear medium. For example, its significant application in optical soliton communication plasma physics has been proved.

Furthermore, the GNLS equation enjoys many remarkable properties (e.g., bright and dark soliton solutions, Lax pair, Liouvile integrability, inverse scattering transformation, conservation laws, Backlund transformation, etc.).

We have considered a plane wave transformation in the form

$$u(x,t) = \psi(\xi)\, e^{i[\chi(\xi)-wt]}, \tag{24}$$

where $\psi(\xi)$ is a real function. For convenience, let $\chi = \beta\xi + x_0$ where β and x_0 are real constants and $\xi = \alpha(x-\upsilon t)+\varsigma$. Then by replacing (23) and its appropriate derivatives in (22) and separating the real and imaginary parts of the result, we have obtained the following two ordinary differential equations:

$$c\alpha^3\psi''' + \left(-\alpha\upsilon + 2a\beta\alpha^2 - 3c\alpha^3\beta^2\right)\psi' + 3d\alpha\psi^2\psi' = 0, \tag{25}$$

$$\left(a\alpha^2 - 3c\psi^3\beta\right)\psi'' + \left(\alpha\beta\upsilon + w - a\alpha^2\beta^2 + c\alpha^3\beta^3\right)\psi + (b-d\alpha\beta)\psi^3 - k = 0. \tag{26}$$

Integrating (25) once, with respect to ξ, we have

$$c\alpha^2\psi''(\xi) + \left(-\upsilon + 2a\alpha\beta - 3c\alpha^2\beta^2\right)\psi(\xi) - M = 0, \tag{27}$$

where M is an arbitrary integration constant. Since the same function $\psi(\xi)$ satisfies (26) and (27), we have obtained the following constraint condition:

$$\frac{a\alpha^2 - 3c\psi^3\beta}{c\alpha^2} = \frac{\alpha\beta\upsilon + w - a\alpha^2\beta^2 + c\alpha^3\beta^3}{-\upsilon + 2a\alpha\beta - 3c\alpha^2\beta^2} = \frac{b-d\alpha\beta}{d} = \frac{k}{M}. \tag{28}$$

Using (4) and (5a) and (5b), we can get

$$X'(\xi) = Y(\xi), \tag{29a}$$

$$Y'(\xi) = \left(-\frac{d}{c\alpha^2}\right)X^3(\xi) + \left(\frac{\upsilon}{c\alpha^2} - \frac{2a\beta}{c\alpha} + 3\beta^2\right)X(\xi) + \frac{M}{c\alpha^2}. \tag{29b}$$

According to the first integral method, we suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (29a) and (29b) and $P(X,Y) = \sum_{i=0}^{m} a_i(X)Y^i$ is an irreducible polynomial in the complex domain $C[X,Y]$ such that

$$P\left[X(\xi), Y(\xi)\right] = \sum_{i=0}^{m} a_i(X(\xi))\, Y(\xi)^i = 0, \tag{30}$$

where $a_i(X)$, $(i = 0, 1, 2, \ldots, m)$ are polynomials of X and $a_m(X) \neq 0$.

Equation (30) is called the first integral to (29a) and (29b). Due to the division theorem, there exists a polynomial $h(X) + g(X)Y$ in the complex domain $C[X,Y]$ such that

$$\frac{dP}{d\xi} = \frac{\partial P}{\partial X}\frac{dX}{d\xi} + \frac{\partial P}{\partial Y}\frac{dP}{d\xi}, = \left[h(X) + g(X)Y\right]\sum_{i=0}^{m} a_i(X)\, Y^i. \tag{31}$$

In this example, we have taken two different cases, assuming that $m = 1$ and $m = 2$ in (30).

Case 3. Suppose that $m = 1$, by equating the coefficients of Y^i $(i = 2, 1, 0)$ on both sides of (31), we have

$$a_1'(X) = g(X)\, a_1(X), \tag{32a}$$

$$a_0'(X) = h(X)\, a_1(X) + g(X)\, a_0(X), \tag{32b}$$

$$a_1(X)\left[\left[-\frac{d}{c\alpha^2}\right]X^3 + \left[\frac{v}{c\alpha^2} - \frac{2a\beta}{c\alpha} + 3\beta^2\right]X + \frac{M}{c\alpha^2}\right] = h(X)\, a_0(X). \tag{32c}$$

Since $a_i(X)$ $(i = 0, 1)$ are polynomials, then from (32a) it can be deduced that $a_1(X)$ is constant and $g(X) = 0$. For simplicity, it was taken $a_1(X) = 1$.

Balancing the degrees of $h(X)$ and $a_0(X)$, it can be concluded that $\deg(h(X)) = 1$ only. Suppose that $h(X) = AX + B$, and $A \neq 0$, then we find

$$a_0(X) = \frac{A}{2}X^2 + BX + D, \tag{33}$$

where D is an arbitrary integration constant.

Substituting $a_0(X)$, $a_1(X)$, and $h(X)$ in (32c) and setting all the coefficients of powers X to be zero, we have obtained a system of nonlinear algebraic equations and by solving it, we obtain

$$\begin{aligned} &v = -i\sqrt{2}\sqrt{c}\sqrt{d}D\alpha + 2a\alpha\beta - 3c\alpha^2\beta^2, \\ &M = 0, \qquad A = -\frac{i\sqrt{2}\sqrt{d}}{\sqrt{c}\alpha}, \qquad B = 0, \end{aligned} \tag{34a}$$

$$\begin{aligned} &v = i\sqrt{2}\sqrt{c}\sqrt{d}D\alpha + 2a\alpha\beta - 3c\alpha^2\beta^2, \\ &M = 0, \qquad A = \frac{i\sqrt{2}\sqrt{d}}{\sqrt{c}\alpha}, \qquad B = 0. \end{aligned} \tag{34b}$$

Using the conditions (34a) and (34b) in (30), we obtain

$$Y(\xi) = \left(\pm\frac{i\sqrt{2}\sqrt{d}}{\sqrt{c}\alpha}\right)X^2(\xi) - D, \tag{35}$$

respectively.

Combining (35) with (29a), the exact solutions to (29a) and (29b) were obtained and then the exact traveling wave solutions to the generalized nonlinear Schrodinger (GNLS) equation with a source (23) can be written as

$$\begin{aligned} u_1(x,t) = {} & i(-2)^{1/4}c^{1/4}\sqrt{D}\sqrt{\alpha} \\ & \times \tanh\left[(1+i)\, d^{1/4}\sqrt{D}\,(\alpha x - \alpha v t + \varsigma - 2\sqrt{c}\alpha\xi_0) \right. \\ & \left. \times \left(2^{3/4}c^{1/4}\sqrt{\alpha}\right)^{-1}\right] \times \left(d^{1/4}\right)^{-1} \\ & \times \exp\left[i\left(\beta\{\alpha x - \alpha v t + \varsigma\} - wt\right)\right], \\ & v = -i\sqrt{2}\sqrt{c}\sqrt{d}D\alpha + 2a\alpha\beta - 3c\alpha^2\beta^2, \end{aligned} \tag{36}$$

$$\begin{aligned} u_2(x,t) = {} & i(-2)^{1/4}c^{1/4}\sqrt{D}\sqrt{\alpha} \\ & \times \tan\left[(1+i)\, d^{1/4}\sqrt{D}\,(\alpha x - \alpha v t + \varsigma - 2\sqrt{c}\alpha\xi_0) \right. \\ & \left. \times \left(2^{3/4}c^{1/4}\sqrt{\alpha}\right)^{-1}\right] \times \left(d^{1/4}\right)^{-1} \\ & \times \exp\left[i\left(\beta\{\alpha x - \alpha v t + \varsigma\} - wt\right)\right], \\ & v = i\sqrt{2}\sqrt{c}\sqrt{d}D\alpha + 2a\alpha\beta - 3c\alpha^2\beta^2, \end{aligned} \tag{37}$$

where ξ_0 is an arbitrary integration constant.

Case 4. Suppose that $m = 2$, by equating the coefficients of Y^i $(i = 3, 2, 1, 0)$ on both sides of (31), we have

$$a_2'(X) = g(X)\, a_2(X), \tag{38a}$$

$$a_1'(X) = h(X)\, a_2(X) + g(X)\, a_1(X), \tag{38b}$$

$$a_0'(X) + 2a_2(X)\left[\left[-\frac{d}{c\alpha^2}\right]X^3 + \left[\frac{v}{c\alpha^2} - \frac{2a\beta}{c\alpha} + 3\beta^2\right]X + \frac{M}{c\alpha^2}\right] = h(X)\, a_1(X) + g(X)\, a_0(X), \tag{38c}$$

$$a_1(X)\left[\left[-\frac{d}{c\alpha^2}\right]X^3 + \left[\frac{v}{c\alpha^2} - \frac{2a\beta}{c\alpha} + 3\beta^2\right]X + \frac{M}{c\alpha^2}\right] = h(X)\, a_0(X). \tag{38d}$$

Since $a_i(X)$ $(i = 0, 1, 2)$ are polynomials, then from (38a) it can be deduced that $a_2(X)$ is a constant and $g(X) = 0$. For simplicity, we have taken $a_2(X) = 1$. Balancing the degrees of $h(X)$ and $a_0(X)$ we have concluded that $\deg(h(X)) = 1$ only.

In this case, it was assumed that $h(X) = AX + B$ and $A \neq 0$; then we find $a_1(X)$ and $a_0(X)$ as follows:

$$a_1(X) = \left(\frac{A}{2}\right)X^2 + BX + D, \tag{39a}$$

$$\begin{aligned} a_0(X) = {} & \left(\frac{A^2}{8} + \frac{d}{2c\alpha^2}\right)X^4 + \frac{1}{2}(AB)X^3 \\ & + \left(\frac{AD + B^2}{2} - c\alpha^2 + \frac{2a\beta}{c\alpha} - 3\beta^2\right)X^2 \\ & + \left(BD - \frac{2M}{c\alpha^2}\right)X + F, \end{aligned} \tag{39b}$$

where A, B, D, and F are arbitrary integration constants.

Substituting $a_0(X)$, $a_1(X)$, $a_2(X)$, and $h(X)$ for (38d) and setting all the coefficients of powers X to be zero, a system of

nonlinear algebraic equations was obtained and by solving it, we got

$$M = 0, \qquad v = \frac{1}{2}\left[-i\sqrt{2}\sqrt{c}\sqrt{d}D\alpha + 4a\alpha\beta - 6c\alpha^2\beta^2\right],$$
$$F = \frac{D^2}{4}, \qquad A = -\frac{2i\sqrt{2}\sqrt{d}}{\sqrt{c}\alpha}, \qquad B = 0, \tag{40a}$$

$$M = 0, \qquad v = \frac{1}{2}\left[i\sqrt{2}\sqrt{c}\sqrt{d}D\alpha + 4a\alpha\beta - 6c\alpha^2\beta^2\right],$$
$$F = \frac{D^2}{4}, \qquad A = \frac{2i\sqrt{2}\sqrt{d}}{\sqrt{c}\alpha}, \qquad B = 0. \tag{40b}$$

Using the conditions (40a) and (40b) in (30), we obtain

$$Y(\xi) = \frac{\pm i\sqrt{2}\sqrt{d}X^2(\xi) - \sqrt{c}D\alpha}{2\sqrt{c}\alpha}, \tag{41}$$

respectively. Combining (41) with (29a) we have obtained the exact solutions to (29a) and (29b) and thus the exact traveling wave solutions to the generalized nonlinear Schrodinger (GNLS) equation with a source (23) can be written as

$$\begin{aligned} u_3(x,t) = {} & (-1)^{3/4}c^{1/4}\sqrt{D}\sqrt{\alpha} \\ & \times \tanh\left[\left(\frac{1}{2}+\frac{i}{2}\right)d^{1/4}\sqrt{D}\right. \\ & \quad \times (\alpha x - \alpha v t + \varsigma - 2\sqrt{c}\alpha\xi_0) \\ & \quad \left.\times\left(2^{1/4}c^{1/4}\sqrt{\alpha}\right)^{-1}\right] \times \left(2^{1/4}d^{1/4}\right)^{-1} \\ & \times \exp\left[i\left(\beta\{\alpha x - \alpha v t + \varsigma\} - wt\right)\right], \\ v = {} & -\frac{1}{2}i\sqrt{2}\sqrt{c}\sqrt{d}D\alpha + 2a\alpha\beta - 3c\alpha^2\beta^2, \end{aligned} \tag{42}$$

$$\begin{aligned} u_4(x,t) = {} & -(-1)^{3/4}c^{1/4}\sqrt{D}\sqrt{\alpha} \\ & \times \tan\left[(-1)^{1/4}d^{1/4}\right. \\ & \quad \times \sqrt{D}(-\alpha x + \alpha v t + \varsigma + 2\sqrt{c}\alpha\xi_0) \\ & \quad \left.\times\left(2^{3/4}c^{1/4}\sqrt{\alpha}\right)^{-1}\right] \times \left(2^{1/4}d^{1/4}\right)^{-1} \\ & \times \exp\left[i\left(\beta\{\alpha x - \alpha v t + \varsigma\} - wt\right)\right] \\ v = {} & \frac{1}{2}i\sqrt{2}\sqrt{c}\sqrt{d}D\alpha + 2a\alpha\beta - 3c\alpha^2\beta^2, \end{aligned} \tag{43}$$

respectively, where ξ_0 is an arbitrary integration constant.

Equations (36)-(37) and (42)-(43) are new types of exact traveling wave solutions to the generalized nonlinear Schrodinger (GNLS) equation with a source (23). It could not be obtained by the methods presented in [32].

3.3. The Higher-Order Nonlinear Schrodinger Equation in Nonlinear Optical Fibers. The higher-order nonlinear Schrodinger equation describing propagation of ultrashort pulses in nonlinear optical fibers [33–39] reads

$$\begin{aligned} \psi_z = {} & i\alpha_1\psi_{tt} + i\alpha_2\psi|\psi|^2 + \alpha_3\psi_{ttt} \\ & + \alpha_4\left(\psi|\psi|^2\right)_t + \alpha_5\psi\left(|\psi|^2\right)_t, \end{aligned} \tag{44}$$

where ψ is slowly varying envelope of the electric field, the subscripts z and t are the spatial and temporal partial derivative in retard time coordinates, and α_1, α_2, α_3, α_4, α_5 are the real parameters related to the group velocity dispersion (GVD), self-phase modulation (SPM), third-order dispersion (TOD), and self-steepening and self-frequency shift arising from simulated Raman scattering, respectively. Some properties of the equation, as well as many versions of it have been studied [33–39]. Up to now, the bright, dark and the combined bright and dark solitary waves and periodic waves were found of (43) and its special case.

To seek traveling wave solutions of (44), we make the gauge transformation

$$\begin{aligned} \psi(z,t) &= \varphi(\xi)\exp\left[i(kz - \omega t)\right], \\ \xi &= \beta t - \lambda z + \varepsilon, \end{aligned} \tag{45}$$

where β, k, ω, λ, ε are constants. Substituting (45) into (44) yields a complex ODE of $\varphi(\xi)$, the real and imaginary parts of which, respectively,

$$\begin{aligned} & \left(\beta^2\alpha_1 - 3\beta^2\alpha_3\omega\right)\varphi'' + \left(\alpha_3\omega^3 - \alpha_1\omega^2 - k\right)\varphi \\ & \quad + \left(\alpha_2 - \alpha_4\omega\right)\varphi^3 = 0, \\ & \beta^3\alpha_3\varphi''' + \left(2\beta\alpha_1\omega - 3\beta\alpha_3\omega^2 + \lambda\right)\varphi' \\ & \quad + \left(3\beta\alpha_4 + 2\beta\alpha_5\right)\varphi^2\varphi' = 0. \end{aligned} \tag{46}$$

It is easy to see that (46) becomes an equation

$$\varphi'' + \frac{2\beta\alpha_1\omega - 3\beta\alpha_3\omega^2 + \lambda}{\beta^3\alpha_3}\varphi + \frac{3\alpha_4 + 2\alpha_5}{3\beta^2\alpha_3}\varphi^3 = 0, \tag{47}$$

under the constraint conditions

$$\begin{aligned} \omega &= \frac{3\alpha_1\alpha_4 + 2\alpha_1\alpha_5 - 3\alpha_2\alpha_3}{6\alpha_3\left(\alpha_4 + \alpha_5\right)}, \\ k &= \frac{1}{\alpha_3}\left[\frac{1}{\beta}\left(3\alpha_3\omega - \alpha_1\right)\lambda - 2\omega\left(\alpha_1 - 2\alpha_3\omega\right)^2\right] \\ &= \frac{\left(3\alpha_2\alpha_3 - 3\alpha_1\alpha_4 - 2\alpha_1\alpha_5\right)\left(3\alpha_2\alpha_3 + \alpha_1\alpha_5\right)^2}{27\alpha_3^2\left(\alpha_4 + \alpha_5\right)^3} \\ &\quad + \frac{\left(\alpha_1\alpha_4 - 3\alpha_2\alpha_3\right)\lambda}{2\beta\alpha_3\left(\alpha_4 + \alpha_5\right)}. \end{aligned} \tag{48}$$

Using (4) and (5a) and (5b), we can get

$$X' = Y, \tag{49a}$$

$$Y' = -\left(\frac{3\alpha_4 + 2\alpha_5}{3\beta^2\alpha_3}\right)X^3 - \left(\frac{2\beta\alpha_1\omega - 3\beta\alpha_3\omega^2 + \lambda}{\beta^3\alpha_3}\right)X. \tag{49b}$$

According to the first integral method, we suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (49a) and (49b) and $P(X,Y) = \sum_{i=0}^{m} a_i(X)Y^i$ is an irreducible polynomial in the complex domain $C[X,Y]$ such that

$$P\left[X(\xi), Y(\xi)\right] = \sum_{i=0}^{m} a_i\left(X(\xi)\right)Y(\xi)^i = 0, \tag{50}$$

where $a_i(X)$, $(i = 0, 1, 2, \ldots, m)$ are polynomials of X and $a_m(X) \neq 0$.

Equation (50) is called the first integral to (49a) and (49b) due to the division theorem, there exists a polynomial $h(X) + g(X)Y$ in the complex domain $C[X,Y]$ such that

$$\frac{dP}{d\xi} = \frac{\partial P}{\partial X}\frac{dX}{d\xi} + \frac{\partial P}{\partial Y}\frac{dP}{d\xi} = \left[h(X) + g(X)Y\right]\sum_{i=0}^{m} a_i(X)Y^i. \tag{51}$$

In this example, we take two different cases, assuming that $m = 1$ and $m = 2$ in (50).

Case 5. Suppose that $m = 1$, by equating the coefficients of Y^i $(i = 2, 1, 0)$ on both sides of (51), we have

$$a_1'(X) = g(X)a_1(X), \tag{52a}$$

$$a_0'(X) = h(X)a_1(X) + g(X)a_0(X), \tag{52b}$$

$$a_1(X)\left[-\left(\frac{3\alpha_4 + 2\alpha_5}{3\beta^2\alpha_3}\right)X^3 - \left(\frac{2\beta\alpha_1\omega - 3\beta\alpha_3\omega^2 + \lambda}{\beta^3\alpha_3}\right)X\right] = h(X)a_0(X). \tag{52c}$$

Since $a_i(X)$ $(i = 0, 1)$ are polynomials, then from (52a) it was deduced that $a_1(X)$ is constant and $g(X) = 0$. For simplicity, take $a_1(X) = 1$.

Balancing the degrees of $h(X)$ and $a_0(X)$, it was concluded that $\deg(h(X)) = 1$ only. Suppose that $h(X) = AX + B$ and $A \neq 0$, then we find

$$a_0(X) = \frac{A}{2}X^2 + BX + D, \tag{53}$$

where D is an arbitrary integration constant.

Substituting $a_0(X)$, $a_1(X)$, and $h(X)$ in (52c) and setting all the coefficients of powers X to be zero, then we have obtained a system of nonlinear algebraic equations and by solving it, we obtain

$$D = \pm\frac{\sqrt{3/2}\left(\lambda + \beta\omega\left(2\alpha_1 - 3\omega\alpha_3\right)\right)}{\beta^2\sqrt{\alpha_3}\sqrt{-3\alpha_4 - 2\alpha_5}}, \quad B = 0, \quad A = \mp\frac{\sqrt{2/3}\sqrt{-3\alpha_4 - 2\alpha_5}}{\beta\sqrt{\alpha_3}}. \tag{54}$$

Using the conditions (54) in (50), we obtain

$$Y(\xi) = \left(\pm\frac{\sqrt{2/3}\sqrt{-3\alpha_4 - 2\alpha_5}}{\beta\sqrt{\alpha_3}}\right)X^2(\xi) \mp \frac{\sqrt{3/2}\left(\lambda + \beta\omega\left(2\alpha_1 - 3\omega\alpha_3\right)\right)}{\beta^2\sqrt{\alpha_3}\sqrt{-3\alpha_4 - 2\alpha_5}}, \tag{55}$$

respectively.

Combining (55) with (49a), the exact solutions to (49a) and (49b) were obtained and then the exact traveling wave solutions to the higher-order nonlinear Schrodinger equation in nonlinear optical fibers can be written as

$$\psi_{1,2}(z,t) = \pm\frac{1}{\sqrt{\beta}p}\sqrt{3}q \times \tan\left[q\left(\left(\beta t - \lambda z + \varepsilon\right)r \pm 2\sqrt{3}\beta^2\xi_0\sqrt{\alpha_3}p^2\right) \times\left(2\beta^{3/2}\sqrt{\alpha_3}p\right)^{-1}\right] \times \exp\left[i\left(kz - \omega t\right)\right], \tag{56}$$

respectively, where

$$p = \sqrt{3\alpha_4 + 2\alpha_5}, \quad q = \sqrt{\lambda + \beta w\left(2\alpha_1 - 3w\alpha_3\right)}, \quad r = \sqrt{-6\alpha_4 - 4\alpha_5}, \quad p^2 = 3\alpha_4 + 2\alpha_5, \tag{57}$$

and ξ_0 is an arbitrary integration constant.

Case 6. Suppose that $m = 2$, by equating the coefficients of Y^i $(i = 3, 2, 1, 0)$ on both sides of (51), we have

$$a_2'(X) = g(X)a_2(X), \tag{58a}$$

$$a_1'(X) = h(X)a_2(X) + g(X)a_1(X), \tag{58b}$$

$$a_0'(X) + 2a_2(X)\left[\left[-\frac{d}{c\alpha^2}\right]X^3 + \left[\frac{\upsilon}{\alpha^2} - \frac{2a\beta}{c\alpha} + 3\beta^2\right]X + \frac{M}{c\alpha^2}\right] = h(X)a_1(X) + g(X)a_0(X), \tag{58c}$$

$$a_1(X)\left[-\left(\frac{2\beta\alpha_1\omega - 3\beta\alpha_3\omega^2 + \lambda}{\beta^3\alpha_3}\right)X - \left(\frac{3\alpha_4 + 2\alpha_5}{3\beta^2\alpha_3}\right)X^3\right] = h(X)a_0(X). \tag{58d}$$

Since $a_i(X)\,(i = 0, 1, 2)$ are polynomials, then from (58a) it was deduced that $a_2(X)$ is a constant and $g(X) = 0$. For simplicity, we have taken $a_2(X) = 1$. Balancing the degrees of $h(X)$ and $a_0(X)$ we conclude that $\deg(h(X)) = 1$ only.

In this case, it was assumed that $h(X) = AX+B$ and $A \neq 0$; then we find $a_1(X)$ and $a_0(X)$ as follows:

$$a_1(X) = \left(\frac{A}{2}\right)X^2 + BX + D, \tag{59a}$$

$$\begin{aligned} a_0(X) = {} & \left(\frac{A^2}{8} + \frac{3\alpha_4 + 2\alpha_5}{6\beta^2\alpha_3}\right)X^4 + \frac{1}{2}(AB)X^3 \\ & + \left(\frac{AD + B^2}{2} + \frac{2\beta\alpha_1\omega - 3\beta\alpha_3\omega^2 + \lambda}{\beta^3\alpha_3}\right)X^2 \\ & + BDX + F, \end{aligned} \tag{59b}$$

where A, B, D, and F are arbitrary integration constants.

Substituting $a_0(X), a_1(X), a_2(X)$, and $h(X)$ into (58d) and setting all the coefficients of powers X to be zero, a system of nonlinear algebraic equations was obtained and by solving it, we get

$$\lambda = -2\beta\omega\alpha_1 + 3\beta\omega^2\alpha_3, \qquad F = 0, \qquad B = 0, \qquad D = 0,$$
$$A = \mp\frac{2\sqrt{2/3}\sqrt{-3\alpha_4 - 2\alpha_5}}{\beta\sqrt{\alpha_3}}, \tag{60a}$$

$$\lambda = -2\beta\omega\alpha_1 + 3\beta\omega^2\alpha_3 \pm \frac{D\beta^2\sqrt{\alpha_3}\sqrt{-3\alpha_4 - 2\alpha_5}}{\sqrt{6}},$$
$$F = \frac{D^2}{4}, \qquad B = 0, \qquad A = \mp\frac{2\sqrt{2/3}\sqrt{-3\alpha_4 - 2\alpha_5}}{\beta\sqrt{\alpha_3}}. \tag{60b}$$

Using the conditions (60a) in (50), we obtain

$$Y(\xi) = \pm\frac{\sqrt{-3\alpha_4 - 2\alpha_5}}{\sqrt{6}\beta\sqrt{\alpha_3}}X^2(\xi), \tag{61}$$

respectively. Combining (61) with (49a), the exact solutions to (49a) and (49b) were obtained and thus the exact traveling wave solutions to the higher-order nonlinear Schrodinger equation in nonlinear optical fibers (44) can be written as

$$\begin{aligned} & \psi_{3,4}(z,t) \\ & = 6\beta\sqrt{\alpha_3} \\ & \times \left(-6\beta c_1\sqrt{\alpha_3}\right. \\ & \qquad \left. \mp\sqrt{6}\left(\beta t - \left(-2\beta\omega\alpha_1 + 3\beta\omega^2\alpha_3\right)z + \varepsilon\right)ip\right)^{-1} \\ & \times \exp\left[i\left(kz - \omega t\right)\right], \end{aligned} \tag{62}$$

respectively.

Similarly, in the case of (60b), from (50), we get

$$Y(\xi) = -\frac{D}{2} \pm \frac{\sqrt{-3\alpha_4 - 2\alpha_5}}{\sqrt{6}\beta\sqrt{\alpha_3}}X^2(\xi), \tag{63}$$

respectively. Combining (63) with (49a), the exact solutions to (49a) and (49b) were obtained and thus the exact traveling wave solutions to the higher-order nonlinear Schrodinger equation in nonlinear optical fibers (44) can be written as

$$\begin{aligned} \psi_5(z,t) = {} & (3/2)^{1/4}\sqrt{D}\sqrt{\beta}{\alpha_3}^{1/4}p \\ & \times \tan\left[\sqrt{D}\left(\left(\beta t - \lambda z + \varepsilon\right) - 6\beta\xi_0\sqrt{\alpha_3}\right)p\right. \\ & \qquad \left. \times \left(2^{3/4}3^{1/4}\sqrt{\beta}{\alpha_3}^{1/4}(-1)^{1/4}p^{1/2}\right)^{-1}\right] \\ & \times \left((-1)^{3/4}p^{3/2}\right)^{-1} \times \exp\left[i\left(kz - \omega t\right)\right], \\ \lambda = {} & -2\beta\omega\alpha_1 + 3\beta\omega^2\alpha_3 + \frac{D\beta^2\sqrt{\alpha_3}\sqrt{-3\alpha_4 - 2\alpha_5}}{\sqrt{6}}, \end{aligned} \tag{64}$$

$$\begin{aligned} \psi_6(z,t) = {} & (3/2)^{1/4}\sqrt{D}\sqrt{\beta}{\alpha_3}^{1/4} \\ & \times p\tanh\left[\sqrt{D}\left(\left(\beta t - \lambda z + \varepsilon\right) - 6\beta\xi_0\sqrt{\alpha_3}\right)p\right. \\ & \qquad \left. \times \left(2^{3/4}3^{1/4}\sqrt{\beta}{\alpha_3}^{1/4}(-1)^{1/4}p^{1/2}\right)^{-1}\right] \\ & \times \left((-1)^{3/4}p^{3/2}\right)^{-1} \times \exp\left[i\left(kz - \omega t\right)\right], \\ \lambda = {} & -2\beta\omega\alpha_1 + 3\beta\omega^2\alpha_3 - \frac{D\beta^2\sqrt{\alpha_3}\sqrt{-3\alpha_4 - 2\alpha_5}}{\sqrt{6}}, \end{aligned} \tag{65}$$

respectively, where p is as defined in (57) and ξ_0 is an arbitrary integration constant.

Comparing these results with Liu's results [39], it can be seen that the solutions here are new.

4. Conclusion

Searching for first integrals of nonlinear ODEs is one of the most important problems since they permit us to solve a nonlinear differential equation by quadratures. Applying the first integral method, which is based on the ring theory of commutative algebra, some new exact traveling wave solutions to the (2+1)-dimensional hyperbolic nonlinear Schrodinger (HNLS) equation, generalized nonlinear Schrodinger (GNLS) equation with a source and higher-order nonlinear Schrodinger equation in nonlinear optical fibers were established.

These solutions may be important for the explanation of some practical physical problems.

The first integral method described herein is not only efficient but also has the merit of being widely applicable.

Therefore, this method can be applied to other nonlinear evolution equations and this will be done elsewhere.

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