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Research Article

Positive Fixed Points for Semipositone Operators in Ordered Banach Spaces and Applications

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The theory of semipositone integral equations and semipositone ordinary differential equations has been emerging as an important area of investigation in recent years, but the research on semipositone operators in abstract spaces is yet rare. By employing a well-known fixed point index theorem and combining it with a translation substitution, we study the existence of positive fixed points for a semipositone operator in ordered Banach space. Lastly, we apply the results to Hammerstein integral equations of polynomial type.

1. Introduction

Existence of fixed points for positive operators have been studied by many authors; see [1–9] and their references. The theory of semipositone integral equations and semipositone ordinary differential equations has been emerging as an important area of investigation in recent years (see [10–17]). But the research on semipositone operators in abstract spaces are yet rare up to now.

Inspired by a number of semipositone problems for integral equations and ordinary differential equations, we study the existence of positive fixed points for semipositone operators in ordered Banach spaces. Then the results are applied to Hammerstein integral equations of polynomial type.

Let E be a real Banach space with the norm $\|\cdot\|$, P a cone of E, and "$\leq$" the partial ordering defined by P, θ denoting the zero element of E, $P^+ = P\setminus\{\theta\}$, $[a,b]=\{x\in E \mid a\leq x\leq b\}$.

Recall that cone P is said to be normal if there exists a positive constant N such that $\theta\leq x\leq y$ implies $\|x\|\leq N\|y\|$, the smallest N is called the normal constant of P. An element $z\in E$ is called the least upper bound (i.e., supremum) of set $D\subset E$, if it satisfies two conditions: (i) $x\leq z$ for any $x\in D$; (ii) $x\leq y$, $x\in D$ implies $z\leq y$. We denote the least upper bound of D by $\sup D$, that is, $z=\sup D$.

Definition 1. Cone $P\subset E$ is said to be minihedral if $\sup\{x,y\}$ exists for each pair of elements $x, y\in E$. For any x in E we define $x^+=\sup\{x,\theta\}$.

Definition 2 (see [1, 3]). Let E_i be real Banach spaces, P_i cones of E_i, $i=1,2$, $T: P_1\to P_2$, and $\alpha\in R$. Then we say T is α-convex if and only if $T(tu)\leq t^\alpha Tu$ for all $(u,t)\in P_1\times(0,1)$.

Definition 3. Let E_i be real Banach spaces, P_i cones of E_i, and $i=1,2$. $P_1\subset D\subset E_1$, $T: D\to E_2$. T is said to be nondecreasing if $x_1\leq x_2 (x_1,x_2\in D)$ implies $Tx_1\leq Tx_2$; T is said to be positive if $Tx\in P_2$ for any $x\in P_1$; T is said to be semipositone if (i) there exists an element $x_0\in P_1$ such that $F(x_0)\notin P_2$ and (ii) there exists an element $q\in E_2$ such that $Tx+q\in P_2$ for any $x\in P_1$.

In order to prove the main results, we need the following lemma which is obtained in [18].

Lemma 4. *Let E be a real Banach space and Ω a bounded open subset of E, with $\theta\in\Omega$, and $A:\overline{\Omega}\cap Q\to Q$ is a completely continuous operator, where Q is a cone in E.*

(i) *Suppose that $Au\neq\mu u$, for all $u\in\partial\Omega\cap Q$, $\mu\geq 1$, then the fixed point index $i(A,\Omega\cap Q,Q)=1$.*

(ii) *Suppose that $Au\nleq u$, for all $u\in\partial\Omega\cap Q$, then $i(A,\Omega\cap Q,Q)=0$.*

The research on ordered Banach spaces, cones, fixed point index, and the above lemma can be seen in [18, 19].

2. Main Results and Their Proofs

Theorem 5. *Let E_i be Banach space, $P_i \subset E_i$ cones, and $i = 1, 2$. Suppose that operator $A : E_1 \rightarrow E_2$ can be expressed as $A = BF$, where the cone P_i and the operator F and B satisfy the following conditions:*

(H1) *when P_1 is normal and minihedral, P_2 is normal;*

(H2) *when $F : E_1 \rightarrow E_2$ is continuous, there exist $g \in P_2^+$, $q \in E_2$, a nondecreasing α-convex operator $G : P_1 \rightarrow P_2$, $(\alpha > 1)$, and a bounded functional $H : P_1 \rightarrow [0, +\infty)$ such that*

$$Gu \leq Fu + q \leq H(u)\, g, \quad \forall u \in P_1; \tag{1}$$

(H3) *when $B : E_2 \rightarrow E_1$ is linear completely continuous, there exists $e \in P_1^+$ such that*

$$Bx \geq \|Bx\|\, e \quad \forall x \in P_2; \quad Ge > \theta; \tag{2}$$

(H4) *when there exists a positive number r_0 such that*

$$\theta < Bq < r_0 e, \qquad h(r_0 N)\, \|Bg\| < \frac{r_0}{N}, \tag{3}$$

with $h(t) = \max_{u \in P_1, \|u\| \leq t} H(u)$, N is the normal constant of P_1. Then A has a fixed point $w \in P_1^+$.

Proof. For q in (H2) and e in (H3), we define that

$$x_0 = Bq, \qquad P_e = \{u \in P_1 \mid u \geq \|u\|\, e\}, \tag{4}$$

$$Ku = B\left(F\left([u - x_0]^+\right) + q\right), \quad \forall u \in P_1. \tag{5}$$

Clearly, $P_e \subset P_1$ is a normal cone of E_1. Since the cone P_1 is minihedral, $[u - x_0]^+$ makes sense. By (H4) and (4), we know that

$$x_0 < r_0 e \leq \frac{y}{\|y\|} r_0, \quad \forall y \in P_e^+. \tag{6}$$

From the condition (H3) and (4), we know that $x_0 \in P_e \subset P_1$, and hence $u - x_0 \leq u$ and

$$\theta \leq [u - x_0]^+ \leq u, \quad \forall u \in P_1. \tag{7}$$

By (7), we have $[u - x_0]^+ \in P_1$, using (H2) we know that

$$F\left([u - x_0]^+\right) + q \geq G\left([u - x_0]^+\right), \quad \forall u \in P_1^+. \tag{8}$$

That is, $F([u - x_0]^+) + q \in P_2$. This and (2) and (5) imply $Ku \in P_e$, for all $u \in P_1$. Hence,

$$K(P_e) \subset P_e. \tag{9}$$

Suppose that D is a bounded set of P_e, L is a positive number satisfying $\| u \| \leq L$, for all $u \in D$. By (7) and normality of P_1, we obtain that

$$\left\|[u - x_0]^+\right\| \leq N\|u\| \leq NL, \quad \forall u \in D. \tag{10}$$

Therefore, (H2) implies that $F([u - x_0]^+) \in [-q, h(NL)g]$, $u \in D$. Since P_2 is normal, the order interval $[-q, h(NL)g]$ is a bounded set of E_2; therefore, $\{F([u - x_0]^+) \mid u \in D\}$ is a bounded set of E_2. This together with (9), continuity of F, and the completely continuity of B, we obtain that K map P_e into P_e and is completely continuous.

For the r_0 in (H4), we let $\Omega_{r_0} = \{u \in E_1 \mid \| u \| < r_0\}$. By (7) we know that

$$\left\|[u - x_0]^+\right\| \leq N\|u\| \leq r_0 N, \quad \forall u \in \Omega_{r_0} \cap P_e. \tag{11}$$

Therefore, from (H2) we obtain that

$$F\left([u - x_0]^+\right) + q \leq H\left([u - x_0]^+\right) g \leq h(r_0 N)\, g, \quad \forall u \in \Omega_{r_0} \cap P_e, \tag{12}$$

where $h(t)$ is as in (H4).

We prove that

$$Ku \neq \mu u, \quad \forall u \in \partial\Omega_{r_0} \cap P_e, \ \mu \geq 1. \tag{13}$$

Assume there exist $\mu_0 \in (0, 1]$ and $z_0 \in \partial\Omega_{r_0} \cap P_e$, such that $z_0 = \mu_0 K z_0$. Using (12) we have

$$Kz_0 = B\left(F\left([z_0 - x_0]^+\right) + q\right) \leq h(r_0 N)\ Bg, \tag{14}$$

hence

$$r_0 = \|z_0\| = \|\mu_0 K z_0\| \leq \|Kz_0\| \leq N h(r_0 N)\, \|Bg\| \tag{15}$$

which contradicts the condition (3), thus (13) holds. By Lemma 4 we know

$$i\left(K, \Omega_{r_0} \cap P_e, P_e\right) = 1. \tag{16}$$

Take $m_0 > 0$ such that $m_0 < 1/r_0$, and set

$$R > \max\left\{2r_0, \left(m_0\|Bq\|\right)^{-1}, \frac{r_0}{1 - m_0 r_0}, N^{1/(\alpha-1)}\left(\left(m_0\|Bq\|\right)^{\alpha}\|BGe\|\right)^{-1/(\alpha-1)}\right\}, \tag{17}$$

where r_0 as in (3), N is the normal constant of P_1. In the following, we prove

$$u \ngeq Ku, \quad \forall u \in \partial\Omega_R \cap P_e. \tag{18}$$

Assume there exists $y_1 \in \partial\Omega_R \cap P_e$ such that $y_1 \geq K y_1$. Using (6), we have $x_0 < (y_1/\| y_1 \|) r_0 = (y_1/R) r_0$, thus it is obtained that

$$y_1 > \frac{R}{r_0} x_0, \qquad y_1 - x_0 \in P_1^+, \tag{19}$$

by (17). From (17) we know $R > r_0/(1 - m_0 r_0)$, thus $(R - r_0)/r_0 \geq m_0 R$. This and (H3), (4), and (19) imply

$$[y_1 - x_0]^+ = y_1 - x_0 > \left(\frac{R}{r_0} - 1\right) Bq \\ \geq m_0 RBq \geq m_0 R \|Bq\| e. \quad (20)$$

By α-convexity of G we know

$$G(su) \geq s^\alpha G(u), \quad \forall u \in P_1, \; s > 1. \quad (21)$$

By (17) we know $m_0 R \parallel Bq \parallel > 1$, hence (20) and (21) imply

$$G\left([y_1 - x_0]^+\right) \geq G(m_0 R \; \|Bq\| e) \geq (m_0 R \|Bq\|)^\alpha Ge. \quad (22)$$

This together with (5) and the condition (H2) imply

$$y_1 \geq Ky_1 = B\left(F\left([y_1 - x_0]^+\right) + q\right) \\ \geq B\left(G\left([y_1 - x_0]^+\right)\right) \geq (m_0 R \|Bq\|)^\alpha BGe. \quad (23)$$

This and (23) imply

$$NR = N\|y_1\| \geq (m_0 R \|Bq\|)^\alpha \|BGe\| \\ = R^\alpha (m_0 \|Bq\|)^\alpha \|BGe\|, \quad (24)$$

therefore,

$$N^{1/(\alpha-1)}\left((m_0 \|Bq\|)^\alpha \|BGe\|\right)^{-1/(\alpha-1)} \geq R, \quad (25)$$

which contradicts (17), thus (18) holds. Using Lemma 4 we have

$$i(K, \Omega_R \cap P_e, P_e) = 0. \quad (26)$$

By (16) and (26) and additivity of fixed point indexes we know that

$$i\left(K, \left(\Omega_R \setminus \overline{\Omega_{r_0}}\right) \bigcap P_e, P_e\right) = -1. \quad (27)$$

Thus, K has a fixed point z on $(\Omega_R \setminus \overline{\Omega_{r_0}}) \bigcap P_e$. Hence,

$$z = B\left(F\left([z - x_0]^+\right) + q\right), \quad z \in P_e, \; r_0 \leq \|z\| \leq R. \quad (28)$$

Let $w = z - x_0$. From (6) and $\parallel z \parallel \geq r_0$ we know $x_0 < (z/\parallel z \parallel) r_0 \leq z$, then $[z - x_0]^+ = w \in P_1^+$. This together with (4) and (28) imply $w = z - x_0 = BF(w) = A(w)$, so that w is a positive fixed point of A. □

3. Corollary and Applications

From Theorem 5 we obtain the following corollary.

Corollary 6. *Suppose that conditions (H1), (H2), and (H3) hold, and in addition assume the following.*

(H5) *For any $x \in P_2^+$, there exists a positive number L_x such that $Bx \leq L_x e$.*

Then there exists a small enough $\lambda^* > 0$ such that $u = \lambda Au$ has a positive solution for any $\lambda \in (0, \lambda^*)$.

Proof. For any fixed $r_0 > 0$, by (H5), we can all take $\overline{\lambda} = \overline{\lambda}(r_0)$, such that

$$\lambda Bq < r_0 e, \quad \lambda h(r_0 N) \|Bg\| < \frac{r_0}{N}, \quad \forall \lambda \in \left(0, \overline{\lambda}\right), \quad (29)$$

hence (H4) holds. We take that

$$F^*(t, u) = \lambda F(t, u), \qquad G^*(u) = \lambda G(u), \\ q^*(t) = \lambda q(t), \qquad g^*(t) = \lambda g(t). \quad (30)$$

Then for $\lambda A = B(\lambda F)$, the conditions in Theorem 5 are satisfied. Thus, λA has a positive fixed point, that is, $u = \lambda A$ has a positive solution, and the proof is complete. □

We consider the integral equation

$$u(x) = \int_G k(x, y)\left(\sum_{i=1}^m a_i(y) u(y)^{\alpha_i} + q(y) \\ \times \left(u(y)^\gamma - u(y)^\delta - w_0\right)\right) dy, \quad (31)$$

where G is a bounded closed domain in R^n and $\alpha_i \geq 0$, $a_i(x)$, $q(x) \in L(G, [0, \infty))$, $i = 1, 2, \ldots, m$, $k(x, y)$ is nonnegative continuous on $G \times G$.

Theorem 7. *Suppose that among α_i $(i = 1, 2, \ldots, m)$ there exists $\alpha_{i_0} > 1$ such that $\inf_{x \in G} a_{i_0}(x) > 0$, and there exist nontrivial nonnegative functions $a(x)$, $b(x) \in C(G)$, and a positive number c, γ, δ, w_0 such that*

$$ca(x) b(y) \leq k(x, y) \leq a(x), \\ k(x, y) \leq b(y), \quad \forall x, y \in G, \quad (32)$$

$$\gamma > \delta > 0, \qquad 0 < w_0 \leq 1 + \min_{t \in [0,1]} \left\{t^\gamma - t^\delta\right\}, \quad (33)$$

$$\int_G q(y) dy < c, \\ \int_G b(y) \cdot \max\left(\sum_{i=1}^m a_i(y), q(y)\right) dy < \frac{1}{2 - w_0}. \quad (34)$$

Then (31) *has a nontrivial nonnegative solution in $C(G)$.*

Proof. Let the Banach space $E_1 = C(G)$ with the sup norm $\|\cdot\|$,

$$P_1 = \{u \in E_1 \mid u(x) \geq 0, \forall x \in G\}, \quad (35)$$

$$E_2 = L(G), \qquad P_2 = \{u \in E_2 \mid u(x) \geq 0, \forall x \in G\}, \quad (36)$$

$$e = ca(x), \qquad q = q(x), \\ g(x) = \max\left\{q(x), \sum_{i=1}^m a_i(x)\right\}, \quad (37)$$

$$Gu = a_{i_0}(x)\, u(x)^{\alpha_{i_0}}, \quad \forall u(x) \in P_1, \tag{38}$$

$$Fu = \sum_{i=1}^{m} a_i(x)\, u(x)^{\alpha_i} + q(x)\left(u(x)^{\gamma} - u(x)^{\delta} - w_0\right), \quad \forall u(x) \in P_1, \tag{39}$$

$$Ju(x) = \begin{cases} u(x)^{\alpha}, & \text{if } u(x) \le 1, \\ u(x)^{\beta}, & \text{if } u(x) > 1, \end{cases} \quad \forall u(x) \in P_1, \tag{40}$$

with $\alpha = \min_{1\le i\le n}\{\alpha_i\}$, $\beta = \max_{1\le i\le n}\{\alpha_i\}$,

$$H(u) = \left\| Ju(x) + u(x)^{\gamma} - u(x)^{\delta} - w_0 + 1 \right\|_C, \quad \forall u(x) \in P_1, \tag{41}$$

$$Bu = \int_G k(x,y)\, u(y)\, dy, \qquad r_0 = 1. \tag{42}$$

Then $P_1 \subset E_1$ is normal minihedral, the normal constant $N = 1$, $e \in P_1^+$. P_2 is a cone of E_2, $q, g \in P_2^+$. $G : P_1 \to P_2$ is nondecreasing α_{i_0}-convex operator, and $Ge > \theta$. $F : P_1 \to E_2$ is continuous; $h : P_1 \to [0, +\infty)$.

It is known easily that

$$-1 < \min_{t\in[0,1]} \left\{t^{\gamma} - t^{\delta}\right\} \le t^{\gamma} - t^{\delta} < 0, \quad t \in (0,1), \tag{43}$$

thus w_0 exits in (33) and

$$t^{\gamma} - t^{\delta} - w_0 \le -w_0, \quad t \in [0,1]. \tag{44}$$

By (33), (43), and $\gamma > \delta$ we have

$$\begin{aligned} u(x)^{\gamma} - u(x)^{\delta} - w_0 &\ge u(x)^{\gamma} - u(x)^{\delta} - 1 - \min_{t\in[0,1]} \left\{t^{\gamma} - t^{\delta}\right\} \\ &\ge -1, \quad \forall u(x) \in P_1^+, \end{aligned} \tag{45}$$

therefore

$$u(x)^{\gamma} - u(x)^{\delta} - w_0 + 1 \ge 0, \quad \forall u(x) \in P_1^+. \tag{46}$$

From (33), (39), and (44) we know easily that there exists $u_0 \in P_1$ such that $Fu \notin P_2$. From (37)–(46), we obtain that

$$\begin{aligned} Gu \le Fu + q &= \sum_{i=1}^{m} a_i(x)\, u(x)^{\alpha_i} + q(x)\left(u(x)^{\gamma} - u(x)^{\delta} - w_0 + 1\right) \\ &\le \left((Ju)(x) + u(x)^{\gamma} - u(x)^{\delta} - w_0 + 1\right) g(x) \\ &\le H(u)\, g(x), \quad \forall x \in G,\ u \in P_1^+. \end{aligned} \tag{47}$$

Equations (32) and (42) imply that $\| Bu \| \le \int_G b(y) u(y) dy$, and hence

$$Bu \ge ca(x) \int_G b(y)\, u(y)\, dy \ge \|Bu\|\, e, \quad \forall u \in P_1. \tag{48}$$

By (42), (32), (34), and (37), we obtain that

$$Bq \le a(x) \int_G q(y)\, dy < ca(x) = r_0 e. \tag{49}$$

By (41) we have $h(r_0 N) = h(1) = \max_{\|u\|\le 1}\{H(u)\} = 2 - w_0$. This and (34) and (42) get that

$$\begin{aligned} Bg &= \int_G k(x,y)\, g(y)\, dy \\ &\le \int_G b(y)\, g(y)\, dy < \frac{1}{2 - w_0} = \frac{r_0}{h(r_0 N)}. \end{aligned} \tag{50}$$

From (35) and (36) we know that (H1) is satisfied. By (47) and (48) we obtain that (H2) and (H3) are satisfied. Equations (49) and (50) imply that (H4) is satisfied. Therefore, using Theorem 5, the integral equation (31) has a positive solution in $C(G)$. □

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