

Research Article

Fixed Points of Meromorphic Solutions for Some Difference Equations

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We investigate fixed points of meromorphic solutions $y(z)$ for the Pielou logistic equation and obtain some estimates of exponents of convergence of fixed points of $y(z)$ and its shifts $y(z+n)$, differences $\Delta y(z) = y(z+1) - y(z)$, and divided differences $\Delta y(z)/y(z)$.

1. Introduction and Results

In this paper, we assume the reader is familiar with basic notions of Nevanlinna's value distribution theory (see [1–3]). In addition, we use the notation $\sigma(f)$ to denote the order of growth of a meromorphic function f and $\lambda(f)$ and $\lambda(1/f)$ to denote, respectively, the exponents of convergence of zeros and poles of f . We also use the notation $\tau(f)$ to denote the exponent of convergence of fixed points of f that is defined as

$$\tau(f) = \overline{\lim}_{r \rightarrow \infty} \frac{\log N(r, 1/(f(z) - z))}{\log r}. \quad (1)$$

Recently, a number of papers (including [4–17]) focus on complex difference equations and difference analogues of Nevanlinna's theory.

The Pielou logistic equation

$$y(z+1) = \frac{R(z)y(z)}{Q(z) + P(z)y(z)}, \quad (2)$$

where $P(z)$, $Q(z)$, and $R(z)$ are nonzero polynomials, is an important difference equation, because it is obtained by transform from the well-known Verhulst Pearl equation (see [18, page 99])

$$x'(t) = x(t)[a - bx(t)] \quad (a, b > 0), \quad (3)$$

which is the most popular continuous model of growth of a population.

Chen [7] obtained the following theorem.

Theorem A. *Let $P(z)$, $Q(z)$, and $R(z)$ be polynomials with $P(z)Q(z)R(z) \neq 0$ and let $y(z)$ be a finite order transcendental meromorphic solution of (2). Then*

$$\lambda\left(\frac{1}{y}\right) = \sigma(y) \geq 1. \quad (4)$$

Example 1. The function $y(z) = z2^z/(2^z - 1)$ satisfies the Pielou logistic equation

$$y(z+1) = \frac{2(z+1)y(z)}{z+y(z)}, \quad (5)$$

where $y(z)$ satisfies

$$(y) = 0, \quad \lambda\left(\frac{1}{y}\right) = \sigma(y) = 1. \quad (6)$$

This example shows that the result of Theorem A is sharp.

One of the main purposes in this paper is to study fixed points of meromorphic solutions of the Pielou logistic equation (2).

The problem of fixed points of meromorphic functions is an important one in the theory of meromorphic functions.

Many papers and books (including [18–20]) investigate fixed points of meromorphic functions.

Now we consider fixed points of meromorphic functions and their shifts, differences, and divided differences. We see that there are many examples to show that either $f(z)$ may have no fixed point, for example, $f_1(z) = e^z + z$ or the shift $f(z+c)$ of $f(z)$, or the difference $\Delta_c f(z) = f(z+c) - f(z)$ of $f(z)$ may have only finitely many fixed points; for example, for the function $f_2(z) = e^z + z - 1$, its shift $f_2(z+1) = ee^z + z$, and its difference $\Delta_{2\pi i} f_2(z) = f_2(z+2\pi i) - f_2(z) = 2\pi i$ have only finitely many fixed points. Even if for a meromorphic function of small growth, Chen and Shon show that there exists a meromorphic function f_0 such that $\sigma(f_0) < 1$ and $\Delta_c f_0(z) = f_0(z+c) - f_0(z)$ has only finitely many fixed points (see Theorem 6 of [9]).

A divided difference $(f(z+c) - f(z))/f(z)$ may also have only finitely many fixed points; for example, the function $f(z) = ze^z$ satisfies that its divided difference $(f(z+1) - f(z))/f(z) = ((z+1)e - z)/z$ has only finitely many fixed points. Chen and Shon obtained Theorem B.

Theorem B (see [9]). *Let $c \in \mathbb{C} \setminus \{0\}$ be a constant and let f be a transcendental meromorphic function of order of growth $\sigma(f) = \sigma < 1$ or of the form $f(z) = h(z)e^{az}$, where $a \neq 0$ is a constant and $h(z)$ is a transcendental meromorphic function with $\sigma(h) < 1$. Suppose that $p(z)$ is a nonconstant polynomial. Then*

$$G(z) = \frac{f(z+c) - f(z)}{f(z)} - p(z) \tag{7}$$

has infinitely many zeros.

From Theorem B, we easily see that under conditions of Theorem B, the divided difference $G_1(z) = (f(z+c) - f(z))/f(z)$ has infinitely many fixed points. The previous example $f(z) = ze^z$ shows that result of Theorem B is sharp.

However, we discover that the properties on fixed points of meromorphic solutions of (2) are very good. We prove the following theorem.

Theorem 2. *Let $P(z)$, $Q(z)$, and $R(z)$ be nonzero polynomials such that*

$$\deg P(z) \geq \max \{ \deg R(z), \deg Q(z) \}, \quad \deg P(z) \geq 1. \tag{8}$$

Set $\Delta y(z) = y(z+1) - y(z)$. Then every finite order transcendental meromorphic solution $y(z)$ of (2) satisfies the following:

- (i) $\tau(y(z+n)) = \sigma(y(z)) \geq 1$ ($n = 0, 1, \dots$);
- (ii) if $R(z) - (z+1)Q(z) \neq 0$, then $\tau(\Delta y(z)/y(z)) = \sigma(y(z))$;
- (iii) if there is a polynomial $h(z)$ satisfying

$$(-R(z) + Q(z) + zP(z))^2 - 4zP(z)Q(z) = h(z)^2, \tag{9}$$

then $\tau(\Delta y(z)) = \sigma(y(z))$.

Remark 3. Generally, $\tau(f(z)) \neq \tau(f(z+c))$ for a meromorphic function $f(z)$ of finite order. For example, the function $f_1(z) = e^z + z$ satisfies

$$\tau(f_1(z)) = 0 \neq \tau(f_1(z+1)) = 1. \tag{10}$$

2. Proof of Theorem 2

We need the following lemmas for the proof of Theorem 2.

Lemma 4 (see [12, 17]). *Let $w(z)$ be a nonconstant finite order meromorphic solution of*

$$P(z, w) = 0, \tag{11}$$

where $P(z, w)$ is a difference polynomial in $w(z)$. If $P(z, a) \neq 0$ for a meromorphic function $a(z)$ satisfying $T(r, a) = S(r, w)$, then

$$m\left(r, \frac{1}{w-a}\right) = S(r, w) \tag{12}$$

holds for all r outside of a possible exceptional set with finite logarithmic measure.

Remark 5. Using the same method as in the proof of Lemma 4 (see [12]), we can prove that in Lemma 4, if all coefficients $b_\lambda(z)$ of $P(z, w)$ satisfy $\sigma(b_\lambda(z)) = \sigma_1 < \sigma(w(z)) = \sigma$ and if $P(z, a) \neq 0$ for a meromorphic function $a(z)$ satisfying $T(r, a) = S(r, w)$, then for a given ε ($0 < \varepsilon < \sigma - \sigma_1$),

$$m\left(r, \frac{1}{w(z) - a(z)}\right) = S(r, w(z)) + O(r^{\sigma_1 + \varepsilon}) \tag{13}$$

holds for all r outside of a possible exceptional set with finite logarithmic measure.

Lemma 6. *Suppose that $R(z)$, $Q(z)$, and $P(z)$ satisfy the condition (8) in Theorem 2 and that $y(z)$ is a nonconstant meromorphic function. Then*

$$\begin{aligned} f_1(z) &= (R(z) - zP(z))y(z) - zQ(z), \\ f_2(z) &= P(z)y(z) + Q(z) \end{aligned} \tag{14}$$

have at most finitely many common zeros.

Proof. Suppose that z_0 is a common zero of $f_1(z)$ and $f_2(z)$. Then $f_2(z_0) = P(z_0)y(z_0) + Q(z_0) = 0$. Thus, $y(z_0) = -Q(z_0)/P(z_0)$. Substituting $y(z_0) = -Q(z_0)/P(z_0)$ into $f_1(z)$, we obtain

$$\begin{aligned} f_1(z_0) &= -\frac{Q(z_0)}{P(z_0)}(R(z_0) - z_0P(z_0)) - z_0Q(z_0) \\ &= -\frac{R(z_0)Q(z_0)}{P(z_0)} \\ &= 0. \end{aligned} \tag{15}$$

Since $R(z)Q(z)/P(z)$ has only finitely many zeros, we see that $f_1(z)$ and $f_2(z)$ have at most finitely many common zeros. \square

Lemma 7 (see [14]). *Let $f(z)$ be a nonconstant finite order meromorphic function. Then*

$$T(r + 1, f(z)) = T(r, f(z)) + S(r, f(z)). \quad (16)$$

Goldberg and Ostrovskii [21, page 66] give that for any constant b ,

$$(1 + o(1))T(r - |b|, f(z)) \leq T(r, f(z + b)) \leq (1 + o(1))T(r + |b|, f(z)). \quad (17)$$

This and Lemma 7 give the following lemma.

Lemma 8. *Let $f(z)$ be a nonconstant finite order meromorphic function. Then*

$$T(r, f(z + 1)) = T(r, f(z)) + S(r, f(z)). \quad (18)$$

Using the same method as in the proof of Lemma 6, we can prove Lemmas 9 and 10.

Lemma 9. *Suppose that $R(z)$, $Q(z)$, and $P(z)$ satisfy the condition (8) in Theorem 2 and that $y(z)$ is a nonconstant meromorphic function. Then*

$$\begin{aligned} f_1(z) &= R(z) - (z + 1)Q(z) - (z + 1)P(z)y(z), \\ f_2(z) &= P(z)y(z) + Q(z) \end{aligned} \quad (19)$$

have at most finitely many common zeros.

Lemma 10. *Suppose that $R(z)$, $Q(z)$, and $P(z)$ satisfy the condition (8) in Theorem 2 and $y(z)$ is a nonconstant meromorphic function. Then*

$$\begin{aligned} f_1(z) &= P(z)y(z)^2 + [-R(z) + Q(z) + zP(z)]y(z) \\ &\quad + zQ(z), \\ f_2(z) &= P(z)y(z) + Q(z) \end{aligned} \quad (20)$$

have at most finitely many common zeros.

Proof of Theorem 2. (i) We prove that $\tau(y(z+n)) = \sigma(y(z)) \geq 1$ ($n = 0, 1, \dots$). Suppose that $n = 0$. Set $y(z) - z = g(z)$. So, $g(z)$ is transcendental, $T(r, g(z)) = T(r, y(z)) + O(\log r)$, and $S(r, g) = S(r, y)$. Substituting $y(z) = g(z) + z$ into (2), we obtain

$$\begin{aligned} K_0(z, g) &:= P(z)[g(z + 1) + z + 1][g(z) + z] \\ &\quad + Q(z)[g(z + 1) + z + 1] \\ &\quad - R(z)[g(z) + z] = 0. \end{aligned} \quad (21)$$

Thus,

$$K_0(z, 0) = z(z + 1)P(z) + (z + 1)Q(z) - zR(z). \quad (22)$$

By (8) and (22), we see that $K_0(z, 0) \neq 0$. Thus, by Lemma 4 and $K_0(z, 0) \neq 0$, we obtain

$$N\left(r, \frac{1}{g(z)}\right) = T(r, g(z)) + S(r, g(z)) \quad (23)$$

for all r outside of a possible exceptional set with finite logarithmic measure. Thus,

$$N\left(r, \frac{1}{y(z) - z}\right) = T(r, y(z)) + S(r, y(z)) \quad (24)$$

for all r outside of a possible exceptional set with finite logarithmic measure. So, by Theorem A and (24), we obtain $\tau(y(z)) = \sigma(y(z)) \geq 1$.

Now suppose that $n = 1$. By (2), we obtain

$$\begin{aligned} &y(z + 1) - z \\ &= \frac{(R(z) - zP(z))y(z) - zQ(z)}{Q(z) + P(z)y(z)} \\ &= \frac{(R(z) - zP(z))[y(z) - zQ(z)/(R(z) - zP(z))]}{Q(z) + P(z)y(z)}. \end{aligned} \quad (25)$$

By (8), we see that $R(z) - zP(z) \neq 0$. Since $P(z)$, $Q(z)$, and $R(z)$ are polynomials, by (25), we see that $y(z) - zQ(z)/(R(z) - zP(z))$ and $Q(z) + P(z)y(z)$ have the same poles, except possibly finitely many poles. By Lemma 6, we see that $(R(z) - zP(z))y(z) - zQ(z)$ and $Q(z) + P(z)y(z)$ have at most finitely many common zeros. Hence, by (25), we have that

$$\begin{aligned} \tau(y(z + 1)) &= \lambda(y(z + 1) - z) \\ &= \lambda\left(y(z) - \frac{zQ(z)}{R(z) - zP(z)}\right). \end{aligned} \quad (26)$$

Suppose that $\lambda(y(z) - zQ(z)/(R(z) - zP(z))) < \sigma(y(z))$. Thus, $y(z) - zQ(z)/(R(z) - zP(z))$ can be rewritten as the following form:

$$y(z) - \frac{zQ(z)}{R(z) - zP(z)} = z^s \frac{b_0(z)}{H_0(z)} e^{h(z)} = \frac{b(z)}{H(z)}, \quad (27)$$

where $h(z)$ is a polynomial with $\deg h(z) \leq \sigma(y(z))$, $b_0(z)$ and $H_0(z)$ are canonical products ($b_0(z)$ may be a polynomial) formed by nonzero zeros and poles of $y(z) - zQ(z)/(R(z) - zP(z))$, respectively, and s is an integer; if $s \geq 0$, then $b(z) = z^s b_0(z)$, $H(z) = H_0(z)e^{-h(z)}$; if $s < 0$, then $b(z) = b_0(z)$, $H(z) = z^{-s} H_0(z)e^{-h(z)}$. Combining Theorem A with properties of canonical product, we see that

$$\begin{aligned} \lambda(b(z)) &= \sigma(b(z)) \\ &= \lambda\left(y(z) - \frac{zQ(z)}{R(z) - zP(z)}\right) \\ &< \sigma(y(z)), \end{aligned} \quad (28)$$

$$\lambda(H(z)) = \sigma(H(z)) = \sigma(y(z)).$$

By (27), we obtain

$$\begin{aligned} y(z) &= \frac{zQ(z)}{R(z) - zP(z)} + b(z)f(z), \\ y(z + 1) &= \frac{(z + 1)Q(z + 1)}{R(z + 1) - (z + 1)P(z + 1)} \\ &\quad + b(z + 1)f(z + 1), \end{aligned} \quad (29)$$

where $f(z) = 1/H(z)$. Thus, by (28) and Lemma 8, we have that

$$\begin{aligned} \sigma(f(z)) &= \sigma(H(z)) = \sigma(y(z)), \\ \sigma(b(z+1)) &= \sigma(b(z)) < \sigma(f(z)). \end{aligned} \tag{30}$$

Substituting (29) into (2), we obtain

$$\begin{aligned} &K_1(z, f) \\ &:= \left\{ \frac{(z+1)Q(z+1)}{R(z+1) - (z+1)P(z+1)} + b(z+1)f(z+1) \right\} \\ &\cdot \left\{ Q(z) + P(z) \left[\frac{zQ(z)}{R(z) - zP(z)} + b(z)f(z) \right] \right\} \\ &- R(z) \left[\frac{zQ(z)}{R(z) - zP(z)} + b(z)f(z) \right] = 0. \end{aligned} \tag{31}$$

By (31), we obtain

$$\begin{aligned} &K_1(z, 0) \\ &= [(z+1)Q(z+1)Q(z)R(z) \\ &\quad - zQ(z)R(z)[R(z+1) - (z+1)P(z+1)]] \\ &\times ([R(z+1) - (z+1)P(z+1)][R(z) - zP(z)])^{-1}. \end{aligned} \tag{32}$$

By (8), we see that in the numerator of the right side of (32), there exists only one term $z(z+1)Q(z)R(z)P(z+1)$ being of the highest degree. So,

$$K_1(z, 0) \neq 0. \tag{33}$$

Thus, by (28), (33), Lemma 4, and its Remark 5, we obtain that for any given ε ($0 < \varepsilon < \sigma(y(z)) - \sigma(b(z))$)

$$\begin{aligned} N\left(r, \frac{1}{f(z)}\right) &= T(r, f(z)) + S(r, f(z)) \\ &\quad + O\left(r^{\sigma(b(z))+\varepsilon}\right) \end{aligned} \tag{34}$$

holds for all r outside of a possible exceptional set with finite logarithmic measure.

On the other hand, by $f(z) = 1/H(z)$ and the fact that $H(z)$ is an entire function, we see that

$$N\left(r, \frac{1}{f(z)}\right) = 0. \tag{35}$$

Thus, by this and (28), we see that (34) is a contradiction. Hence, $\lambda(y(z) - zQ(z)/(R(z) - zP(z))) = \sigma(y(z))$. By (26), we obtain

$$\tau(y(z+1)) = \sigma(y(z)). \tag{36}$$

Now suppose that $n = 2$. By (2), we obtain

$$g(z+1) = \frac{R(z+1)g(z)}{Q(z+1) + P(z+1)g(z)}, \tag{37}$$

where $g(z) = y(z+1)$. By Lemma 8, we have that $\sigma(g(z)) = \sigma(y(z))$. By (8), we have

$$\begin{aligned} \deg P(z+1) &\geq \max\{\deg R(z+1), \deg Q(z+1)\}, \\ \deg P(z+1) &\geq 1. \end{aligned} \tag{38}$$

Thus, for (37), applying the conclusion of $n = 1$ above, we obtain

$$\tau(y(z+2)) = \tau(g(z+1)) = \sigma(g(z)) = \sigma(y(z)). \tag{39}$$

Continuing to use the same method as above, we can obtain

$$\tau(y(z+n)) = \sigma(y(z)) \quad (n = 1, 2, \dots). \tag{40}$$

(ii) Suppose that $R(z) - (z+1)Q(z) \neq 0$. We prove that $\tau(\Delta y(z)/y(z)) = \sigma(y(z))$. By (2), we obtain

$$\begin{aligned} &\frac{\Delta y(z)}{y(z)} - z \\ &= \frac{y(z+1) - y(z)}{y(z)} - z \\ &= \frac{R(z) - (z+1)Q(z) - (z+1)P(z)y(z)}{Q(z) + P(z)y(z)} \\ &= -(z+1)P(z) \\ &\quad \times (y(z) - (R(z) - (z+1)Q(z))/(z+1)P(z)) \\ &\quad \times (Q(z) + P(z)y(z))^{-1}. \end{aligned} \tag{41}$$

Since $y(z) - (R(z) - (z+1)Q(z))/(z+1)P(z)$ and $Q(z) + P(z)y(z)$ have the same poles, except possibly finitely many poles, by Lemma 9 and (41), we only need to prove that

$$\lambda\left(y(z) - \frac{R(z) - (z+1)Q(z)}{(z+1)P(z)}\right) = \sigma(y(z)). \tag{42}$$

Set

$$h_1(z) = R(z) - (z+1)Q(z), \quad h_2(z) = (z+1)P(z). \tag{43}$$

Suppose that $\lambda(y(z) - h_1(z)/h_2(z)) < \sigma(y(z))$. Using the same method as in the proof of (i), $y(z) - h_1(z)/h_2(z)$ can be rewritten as the following form:

$$y(z) = \frac{h_1(z)}{h_2(z)} + b_2(z)f_2(z), \tag{44}$$

where $f_2(z) = 1/H_2(z)$ and $b_2(z)$ and $H_2(z)$ are nonzero entire functions, such that

$$\begin{aligned} \lambda(b_2(z)) &= \sigma(b_2(z)) < \sigma(y(z)), \\ \lambda(H_2(z)) &= \sigma(H_2(z)) = \sigma(y(z)). \end{aligned} \tag{45}$$

Substituting (44) into (2), we obtain

$$\begin{aligned}
 & D_2(z, f_2(z)) \\
 & := \left\{ \frac{h_1(z+1)}{h_2(z+1)} + b_2(z+1) f_2(z+1) \right\} \\
 & \quad \times \left\{ Q(z) + P(z) \frac{h_1(z)}{h_2(z)} + P(z) b_2(z) f_2(z) \right\} \\
 & \quad - R(z) \left\{ \frac{h_1(z)}{h_2(z)} + b_2(z) f_2(z) \right\} = 0, \\
 & D_2(z, 0) \\
 & = \frac{h_1(z+1)}{h_2(z+1)} \left\{ Q(z) + P(z) \frac{h_1(z)}{h_2(z)} \right\} - R(z) \frac{h_1(z)}{h_2(z)} \\
 & = \frac{h_1(z+1)}{h_2(z+1)} \left\{ Q(z) + P(z) \frac{R(z) - (z+1)Q(z)}{(z+1)P(z)} \right\} \\
 & \quad - R(z) \frac{h_1(z)}{h_2(z)} \\
 & = R(z) \frac{h_1(z+1)h_2(z) - (z+1)h_1(z)h_2(z+1)}{(z+1)h_2(z+1)h_2(z)}.
 \end{aligned} \tag{46}$$

Since $h_1(z)$ and $h_2(z)$ are polynomials, we see that

$$h_1(z+1)h_2(z) - (z+1)h_1(z)h_2(z+1) \neq 0, \tag{47}$$

that is,

$$D_2(z, 0) \neq 0. \tag{48}$$

Using the same method as in the proof of (i), we obtain a contradiction. Hence, (42) holds; that is, $\tau(\Delta y(z)/y(z)) = \sigma(y(z))$.

(iii) Suppose that there is a polynomial $h(z)$ satisfying

$$(-R(z) + Q(z) + zP(z))^2 - 4zP(z)Q(z) = h(z)^2. \tag{49}$$

Thus, by (8) and (49), we see that

$$\deg h(z) = \deg P(z) + 1. \tag{50}$$

Now we prove that $\tau(\Delta y(z)) = \sigma(y(z))$. By (2), we obtain

$$\begin{aligned}
 & y(z+1) - y(z) - z \\
 & = -\frac{P(z)y(z)^2 + [-R(z) + Q(z) + zP(z)]y(z) + zQ(z)}{Q(z) + P(z)y(z)}.
 \end{aligned} \tag{51}$$

By (49) and (51), we obtain

$$\begin{aligned}
 & y(z+1) - y(z) - z \\
 & = -P(z) \left\{ y(z) - \frac{-zP(z) + Q(z) - R(z) + h(z)}{2P(z)} \right\} \\
 & \quad \times \left\{ y(z) - \frac{-zP(z) + Q(z) - R(z) - h(z)}{2P(z)} \right\} \\
 & \quad \times (Q(z) + P(z)y(z))^{-1}.
 \end{aligned} \tag{52}$$

Since $P(z)$, $Q(z)$, and $R(z)$ are polynomials, we see that poles of $Q(z) + P(z)y(z)$ must be poles of $P(z)y(z)^2 + [-R(z) + Q(z) + zP(z)]y(z) + zQ(z)$. Thus, poles of $Q(z) + P(z)y(z)$ are not zeros of $y(z+1) - y(z) - z$. By Lemma 10, we see that the numerator and the denominator of the right side of (51) have at most finitely many common zeros. Thus, in order to prove $\tau(\Delta y(z)) = \sigma(y(z))$, by (52), we only need to prove that

$$\lambda \left(y(z) - \frac{-zP(z) + Q(z) - R(z) + h(z)}{2P(z)} \right) = \sigma(y(z)) \tag{53}$$

or

$$\lambda \left(y(z) - \frac{-zP(z) + Q(z) - R(z) - h(z)}{2P(z)} \right) = \sigma(y(z)). \tag{54}$$

By (50), we have $\deg(-zP(z)) = \deg h(z) = \deg P(z) + 1$. Combining this with (8), we see that there exists at least one of

$$\begin{aligned}
 & -zP(z) + Q(z) - R(z) + h(z), \\
 & -zP(z) + Q(z) - R(z) - h(z),
 \end{aligned} \tag{55}$$

such that its degree is equal to $\deg P(z) + 1$. Without loss of generality, we may suppose that

$$\deg(-zP(z) + Q(z) - R(z) + h(z)) = \deg P(z) + 1. \tag{56}$$

Now we prove that (53) holds. Suppose that

$$\lambda \left(y(z) - \frac{-zP(z) + Q(z) - R(z) + h(z)}{2P(z)} \right) < \sigma(y(z)). \tag{57}$$

Using a similar method as in the proof of (i), we see that $y(z) - (-zP(z) + Q(z) - R(z) + h(z))/2P(z)$ can be rewritten as the following form:

$$\begin{aligned}
 y(z) & = \frac{-zP(z) + Q(z) - R(z) + h(z)}{2P(z)} \\
 & \quad + b_3(z) f_3(z),
 \end{aligned} \tag{58}$$

where $f_3(z) = 1/H_3(z)$ and $b_3(z)$ and $H_3(z)$ are nonzero entire functions, such that

$$\begin{aligned}
 & \lambda(b_3(z)) = \sigma(b_3(z)) < \sigma(y(z)), \\
 & \lambda(H_3(z)) = \sigma(H_3(z)) = \sigma(y(z)).
 \end{aligned} \tag{59}$$

Substituting (58) into (2), we obtain

$$\begin{aligned}
 & D_3(z, f_3(z)) \\
 & := \left\{ \frac{-((z+1)P(z+1) + Q(z+1) - R(z+1)) + h(z+1)}{2P(z+1)} \right. \\
 & \quad \left. + b_3(z+1)f_3(z+1) \right\} \\
 & \cdot \left\{ Q(z) + P(z) \frac{-(zP(z) + Q(z) - R(z)) + h(z)}{2P(z)} \right. \\
 & \quad \left. + P(z)b_3(z)f_3(z) \right\} \\
 & - R(z) \frac{-(zP(z) + Q(z) - R(z)) + h(z)}{2P(z)} \\
 & - R(z)b_3(z)f_3(z) = 0, \\
 & D_3(z, 0) \\
 & := \left\{ \frac{-(z+1)P(z+1) - Q(z+1) + R(z+1) + h(z+1)}{2P(z+1)} \right\} \\
 & \cdot \left\{ \frac{-zP(z) + Q(z) + R(z) + h(z)}{2} \right\} \\
 & - R(z) \frac{-zP(z) - Q(z) + R(z) + h(z)}{2P(z)} \\
 & = \frac{W_1(z) - W_2(z)}{4P(z+1)P(z)}, \tag{60}
 \end{aligned}$$

where

$$\begin{aligned}
 W_1(z) &= [-(z+1)P(z+1) - Q(z+1) \\
 & \quad + R(z+1) + h(z+1)] \\
 & \quad \times [-zP(z) + Q(z) + R(z) + h(z)]P(z),
 \end{aligned}$$

$$W_2(z) = 2(-zP(z) - Q(z) + R(z) + h(z))R(z)P(z+1). \tag{61}$$

By (8), (50), and (56), we see that

$$\begin{aligned}
 & \deg(-(z+1)P(z+1) - Q(z+1) + R(z+1) + h(z+1)) \\
 & = \deg P(z) + 1, \\
 & \deg(-zP(z) + Q(z) + R(z) + h(z)) = \deg P(z) + 1. \tag{62}
 \end{aligned}$$

Thus, we obtain

$$\begin{aligned}
 & \deg W_1(z) = 3 \deg P(z) + 2, \\
 & \deg W_2(z) \leq 3 \deg P(z) + 1. \tag{63}
 \end{aligned}$$

So, by (60) and (63), we see that $D_3(z, 0) \neq 0$.

Using the same method as in the proof of (i), we see that (53) holds.

Thus, Theorem 2 is proved. \square

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