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Research Article

Periodic Solutions of Second-Order Difference Problem with Potential Indefinite in Sign

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We investigate the periodic solutions of second-order difference problem with potential indefinite in sign. We consider the compactness condition of variational functional and local linking at 0 by introducing new number λ_*. By using Morse theory, we obtain some new results concerning the existence of nontrivial periodic solution.

1. Introduction

We consider the second-order discrete Hamiltonian systems

$$\Delta^2 x_{n-1} + W'(n, x_n) = 0, \qquad x_{n+T} = x_n, \tag{1}$$

where $T \geq 2$ is a given integer, $n \in \mathbb{Z}$, $x_n \in \mathbb{R}^N$, $\Delta x_n = x_{n+1} - x_n$, $\Delta^2 x_n = \Delta(\Delta x_n)$, W' stands for the gradient of W with respect to the second variable. $W \in C^2(\mathbb{Z} \times \mathbb{R}^N, \mathbb{R})$ is T-periodic in the first variable and has the form $W(n,x) = (1/2)a|x|^2 + H(n,x)$, where $a = 4\sin^2(m\pi/T)$ for some $m \in Z[0,r]$, $r = [T/2]$, $[\cdot]$ stands for the greatest-integer function. For integers $a \leq b$, the discrete interval $\{a, a+1, \ldots, b\}$ is denoted by $Z[a,b]$.

In this paper we consider that H is sign changing, that is,

$$\begin{aligned} H(n,x) &= b(n)\left(\frac{1}{s}|x|^s + \overline{G}_s(n,x)\right) \\ &\triangleq \frac{1}{s} b(n)|x|^s + G_s(n,x), \end{aligned} \tag{2}$$

$\Omega_+ = \{n \in Z[1,T] | b(n) > 0)\}$, $\Omega_- = \{n \in Z[1,T] | b(n) < 0)\}$ are two nonempty subsets of $Z[1,T]$, where $s > 1$, $b(\cdot)$ is a T-periodic real function, $G_s \in C^1(\mathbb{Z} \times \mathbb{R}^N, \mathbb{R})$, and $G_s(n,0) = 0$.

Consider the second-order Hamiltonian system

$$\begin{aligned} \ddot{x}(t) + W'(t,x) = 0, \qquad x(0) = x(T), \\ \dot{x}(0) = \dot{x}(T), \end{aligned} \tag{3}$$

where $W \in C^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ is T-periodic in t, $W(t,x) = (1/2)(A(t)x, x) + H(t,x)$. Here $A(\cdot)$ is a continuous, T-periodic matrix-value function.

Systems (1) and (3) have been investigated by many authors using various methods, see [1–5]. The dynamical behavior of differential and difference equations was studied by using various methods, and many interesting results have obtained, see [6–10] and references therein. The critical point theory [11–14] is a useful tool to investigate differential equations. Morse theory [15–19] has also been used to solve the asymptotically linear problem. By minimax methods in critical point theory, Tang and Wu [4], Antonacci [20, 21] considered the problem (3) with potential indefinite in sign, where H is superquadratic at zero and infinity. By using Morse theory, Zou and Li [10] study the existence of T-periodic solution of (3), where H is asymptotically superquadratic and sign changing. Moroz [19] studies system (3) where H is asymptotically subquadratic and sign changing. Motivated by [5, 10, 19], we investigate periodic solutions for asymptotically superquadratic or subquadratic discrete system (1).

By expression of $H(n,x)$, system (1) possesses a trivial solution $x = 0$. Here we are interested in finding the nonzero T-periodic solution of (1), and we divide the problem into two cases: $s > 2$ and $1 < s < 2$. For $s = 2$, one can refer to [22].

Case 1 (asymptotically superquadratic case: $s > 2$). In this case, we replace p with s in (2). Letting $g_p(n,x) = G'_p(n,x)$, we rewrite (1) as

$$\Delta^2 x_{n-1} + ax_n + b(n)|x_n|^{p-2}x_n + g_p(n,x_n) = 0, \quad x_{n+T} = x_n. \tag{4}$$

Furthermore, for all $(n,x) \in \mathbb{Z}\times\mathbb{R}^N$, we assume that g_p satisfies

(A1) $g_p(n,x) = o(|x|)$ as $|x| \to \infty$ uniformly in n,

(A2) $g_p(n,x) = o(|x|^{p-1})$ as $|x| \to 0$ uniformly in n.

Case 2 (asymptotically subquadratic case: $1 < s < 2$). Here we replace q with s in (2). Letting $g_q(n,x) = G'_q(n,x)$, we rewrite (1) as

$$\Delta^2 x_{n-1} + ax_n + b(n)|x_n|^{q-2}x_n + g_q(n,x_n) = 0, \quad x_{n+T} = x_n. \tag{5}$$

For all $(n,x) \in \mathbb{Z}\times\mathbb{R}^N$, we assume that g_q satisfies

(B1) $g_q(n,x) = o(|x|^{q-1})$ as $|x| \to \infty$ uniformly in n,

(B2) $g_q(n,x) = o(|x|)$ as $|x| \to 0$ uniformly in n.

Before stating the main results, we introduce space $E_T = \{x = \{x_n\} \in S \mid x_{n+T} = x_n, n \in \mathbb{Z}\}$, where $S = \{x = \{x_n\} \mid x_n \in \mathbb{R}^N, n \in \mathbb{Z}\}$. For any $x, y \in S, a, b \in \mathbb{R}$, we define $ax + by = \{ax_n + by_n\}_{n\in\mathbb{Z}}$. Then S is a linear space. Let $\langle x,y\rangle_{E_T} = \sum_{n=1}^{T}(x_n, y_n)$, $\|x\|_{E_T} = (\sum_{n=1}^{T}|x_n|^2)^{1/2}$, for all $x, y \in E_T$, where $(\cdot,\cdot)$ and $|\cdot|$ are the usual inner product and norm in $\mathbb{R}^N$, respectively. Obviously, E_T is a Hilbert space with dimension NT and homeomorphism to $\mathbb{R}^{NT}$. For $r > 1$, let $\|x\|_r = (\sum_{n=1}^{T}|x_n|^r)^{1/r}$, $x \in E_T$. Moreover, for simplicity, we write $\langle x,y\rangle$ and $\|x\|$ instead of $\langle x,y\rangle_{E_T}$ and $\|x\|_{E_T}$, respectively.

Lemma 1. *There exist positive numbers a_1, a_2, such that $a_1 \| x\|_r \leq \| x \| \leq a_2 \| x\|_r$.*

Inspired by [10, 19], one introduces two numbers as follows:

$$\lambda_*(p) = \inf_{\|x\|=1}\left\{\|\Delta x\|^2 \mid \sum_{n=1}^{T} b(n)|x_n|^p = 0\right\}, \quad \lambda_*(q) = \inf_{\|x\|=1}\left\{\|\Delta x\|^2 \mid \sum_{n=1}^{T} b(n)|x_n|^q = 0\right\}. \tag{6}$$

Theorem 2. *If $a < \lambda_*(p)$, then* (4) *has a nonzero T-periodic solution.*

Theorem 3. *If $a < \lambda_*(q)$, then* (5) *has a nonzero T-periodic solution.*

This paper is divided into four sections. Section 2 contains some preliminaries, and the proofs of Theorems 2 and 3 are given in Sections 3 and 4, respectively.

2. Preliminaries

2.1. Variational Functional and (PS) Condition. For seeking T-periodic solution of (1), we consider variational functional J_p associated with (4) as $J_p(x) = (1/2)\sum_{n=1}^{T}|\Delta x_n|^2 - (1/2)a\sum_{n=1}^{T}|x_n|^2 - 1/p\sum_{n=1}^{T}b(n)|x_n|^p - \sum_{n=1}^{T}G_p(n,x_n)$, that is

$$J_p(x) = \frac{1}{2}\|\Delta x\|^2 - \frac{1}{2}a\|x\|^2 - \frac{1}{p}\sum_{n=1}^{T}b(n)|x_n|^p - \sum_{n=1}^{T}G_p(n,x_n), \quad x \in E_T. \tag{7}$$

Moreover, T-periodic solution of (5) is associated with the critical point of functional

$$J_q(x) = \frac{1}{2}\|\Delta x\|^2 - \frac{1}{2}a\|x\|^2 - \frac{1}{q}\sum_{n=1}^{T}b(n)|x_n|^q - \sum_{n=1}^{T}G_q(n,x_n), \quad x \in E_T. \tag{8}$$

We say that a C^1-functional φ on Hilbert space X satisfies the Palais-Smale (PS) condition if every sequence $\{x^{(j)}\}$ in X, such that $\{\varphi(x^{(j)})\}$, is bounded and $\varphi'(x^{(j)}) \to 0$ as $j \to \infty$ contains a convergent subsequence.

Lemma 4. *Functional J_p satisfies (PS) condition if $a < \lambda_*(p)$.*

Proof. Let $\{x^{(j)}\} \subset E_T$ be the (PS) sequence for functional J_p, such that $J_p(x^{(j)})$ is bounded, and $J'_p(x^{(j)}) \to 0$ as $j \to \infty$. Hence, for any $\varepsilon > 0$, there exist $N_\varepsilon > 0$ and constant $c_1 > 0$, such that

$$\left|\left\langle J'_p(x^{(j)}), x^{(j)}\right\rangle\right| \leq \varepsilon\left\|x^{(j)}\right\| \quad \text{for } j \geq N_\varepsilon, \qquad \left|J_p(x^{(j)})\right| \leq c_1. \tag{9}$$

To prove that J_p satisfies (PS) condition, it suffices to show that $\|x^{(j)}\|$ is bounded in E_T. Suppose not that there exists a subsequence $\{x^{(j_k)}\}$, $\| x^{(j_k)} \| \to \infty$ as $k \to \infty$. For simplicity, we write as $\{x^{(j)}\}$ instead of $\{x^{(j_k)}\}$. Without loss of generality, we assume that there exists $k \in Z[1,T]$, such that

$$\left|x_n^{(j)}\right| \longrightarrow \infty \quad \text{as} \quad j \longrightarrow \infty \quad \text{for } n \in Z[1,k], \qquad x_n^{(j)} \text{ are bounded for } n \in Z[k+1,T]. \tag{10}$$

Therefore for all $n \in [1,T]$, by assumption (A1), there exists $c_2 > 0$ such that

$$\left|G_p(n,x_n^{(j)})\right| \leq \varepsilon\left|x_n^{(j)}\right|^2 + c_2, \qquad \left|g_p(n,x_n^{(j)})\right| \leq \varepsilon\left|x_n^{(j)}\right| + c_2 \tag{11}$$

for large j. By the previous argument, it follows that

$$\left|\sum_{n=1}^{T}\left(g_p\left(n,x_n^{(j)}\right),x_n^{(j)}\right)\right| \le \sum_{n=1}^{T}\left|g_p\left(n,x_n^{(j)}\right)\right| \left|x_n^{(j)}\right| \le \varepsilon\left\|x^{(j)}\right\|^2 + c_2T\left\|x^{(j)}\right\|. \quad (12)$$

By (7), we have

$$pJ_p\left(x^{(j)}\right) - \left\langle J_p'\left(x^{(j)}\right),x^{(j)}\right\rangle = \left(\frac{p}{2}-1\right)\left(\left\|\Delta x^{(j)}\right\|^2 - a\left\|x^{(j)}\right\|^2\right) - p\sum_{n=1}^{T}G_p\left(n,x_n^{(j)}\right) + \sum_{n=1}^{T}\left(g_p\left(n,x_n^{(j)}\right),x_n^{(j)}\right). \quad (13)$$

In terms of (9) and (11), for large j, it follows that

$$\left(\frac{p}{2}-1\right)\left(\left\|\Delta x^{(j)}\right\|^2 - a\left\|x^{(j)}\right\|^2\right) \le pc_1 + \varepsilon\left\|x^{(j)}\right\| + (p+1)\,\varepsilon\left\|x^{(j)}\right\|^2 + pc_2T + c_2T\left\|x^{(j)}\right\|. \quad (14)$$

Set $y_n^{(j)} = x_n^{(j)}/\|x^{(j)}\|$. Dividing by $\|x^{(j)}\|^2$ in the previous formula, it follows that

$$\left\|\Delta y^{(j)}\right\|^2 \le a + \frac{2}{p-2}\left((p+1)\,\varepsilon + \frac{c_2T+\varepsilon}{\left\|x^{(j)}\right\|} + \frac{pc_2T+pc_1}{\left\|x^{(j)}\right\|^2}\right) \quad (15)$$

for large j. Therefore, by ε being chosen arbitrarily, there is a subsequence that converges to $y^0 \in E_T$ such that

$$\left\|\Delta y^0\right\|^2 \le a, \quad \left\|y^0\right\| = 1. \quad (16)$$

On the other hand, we have

$$J_p\left(x^{(j)}\right) - \frac{1}{2}\left\langle J_p'\left(x^{(j)}\right),x^{(j)}\right\rangle = \left(\frac{1}{2}-\frac{1}{p}\right)\sum_{n=1}^{T}b(n)\left|x_n^{(j)}\right|^p - \sum_{n=1}^{T}G_p\left(n,x_n^{(j)}\right) + \frac{1}{2}\sum_{n=1}^{T}\left(g_p\left(n,x_n^{(j)}\right),x_n^{(j)}\right). \quad (17)$$

Then, by (9) and (11), for large j, we get

$$\begin{aligned}\left|\left(\frac{1}{2}-\frac{1}{p}\right)\sum_{n=1}^{T}b(n)\left|x_n^{(j)}\right|^p\right| &= \left|J_p\left(x^{(j)}\right) - \frac{1}{2}\left\langle J_p'\left(x^{(j)}\right),x^{(j)}\right\rangle + \sum_{n=1}^{T}G_p\left(n,x_n^{(j)}\right) - \frac{1}{2}\sum_{n=1}^{T}\left(g_p\left(n,x_n^{(j)}\right),x_n^{(j)}\right)\right| \\ &\le c_1 + \frac{\varepsilon}{2}\left\|x^{(j)}\right\| + \varepsilon\left\|x^{(j)}\right\|^2 + c_2T + \frac{1}{2}\left(\varepsilon\left\|x^{(j)}\right\|^2 + c_2T\left\|x^{(j)}\right\|\right).\end{aligned} \quad (18)$$

By dividing by $\|x^{(j)}\|^p$ in the previous formula, then by $p > 2$, we have $\sum_{n=1}^{T} b(n)|y_n^{(j)}|^p \to 0$ as $j \to \infty$, that is, $\sum_{n=1}^{T} b(n)|y_n^0|^p = \lim_{j\to\infty}\sum_{n=1}^{T} b(n)|y_n^{(j)}|^p = 0$. By the definition of $\lambda_*(p)$, see (6), we have $\|\Delta y^0\|^2 \ge \lambda_*(p)$. This contradicts with (16) and assumption $a < \lambda_*(p)$. The proof is completed. □

Lemma 5. *Functional J_q satisfies (PS) condition if $a < \lambda_*(q)$.*

The proof is similar to that of Lemma 4 and is omitted.

2.2. Eigenvalue Problem. Consider eigenvalue problem:

$$-\Delta^2 x_{n-1} = \lambda x_n, \qquad x_{n+T} = x_n, \quad x_n \in \mathbb{R}^N, \quad (19)$$

that is, $x_{n+1}+(\lambda-2)x_n+x_{n-1} = 0$, $x_{n+T} = x_n$. By the periodicity, the difference system has complexity solution $x_n = e^{in\theta}c$ for $c \in \mathbb{C}^N$, where $\theta = 2k\pi/T$, $k \in \mathbb{Z}$. Moreover, $\lambda = 2-e^{-i\theta}-e^{i\theta} = 2(1-\cos\theta) = 4\sin^2(k\pi/T)$. Let η_k denote the real eigenvector corresponding to the eigenvalues $\lambda_k = 4\sin^2(k\pi/T)$, where $k \in Z[0,r]$ and $r = [T/2]$. Since $a = 4\sin^2(m\pi/T)$ for some $m \in Z[0,r]$, we can split space E_T as follows:

$$E_T = W^- \bigoplus W^0 \bigoplus W^+, \quad (20)$$

where

$$W^- = \text{span}\{\eta_k \mid k \in Z[0,m-1]\}, \qquad W^0 = \text{span}\{\eta_m\}, \qquad W^+ = \text{span}\{\eta_k \mid k \in Z[m+1,r]\}. \quad (21)$$

By means of eigenvalue problem, we have $|\Delta x_n|^2 - a|x_n|^2 = (\Delta x_n, \Delta x_n) - a(x_n,x_n) = (-\Delta^2 x_{n-1}, x_n) - a(x_n,x_n) = (\lambda - a)(x_n,x_n) = (\lambda-a)|x_n|^2$. Let

$$\delta = \begin{cases}\min\left\{4\sin^2\dfrac{(m+1)\pi}{T} - 4\sin^2\dfrac{m\pi}{T},\ 4\sin^2\dfrac{m\pi}{T} - 4\sin^2\dfrac{(m-1)\pi}{T}\right\}, & m \in Z[1,r], \\ 4\sin^2\dfrac{\pi}{T}, & m = 0.\end{cases} \quad (22)$$

Then $\pm(\|\Delta x\|^2 - a\|x\|^2) \ge \delta\|x\|^2$ for $x \in W^{\pm}$.

On the other hand, associating to numbers $\lambda_*(p)$ and $\lambda_*(q)$ (see (6)), we set

$$\Lambda_*(p)=\sum_{n=1}^{T}b(n)\left|e_n\right|^p,\qquad \Lambda_*(q)=\sum_{n=1}^{T}b(n)\left|e_n\right|^q, \tag{23}$$

where $e_n = u \in \mathbb{R}^N$ $(n \in [1,T])$ is the real eigenvector corresponding to eigenvalue $\lambda_0 = 0$. $e = (e_1^T, e_2^T, \dots, e_N^T)^T = (u^T, u^T, \dots, u^T)^T \in E_T$, where $\bullet^T$ denotes the transpose of a vector or a matrix. Moreover, letting $|u| = T^{-1/2}$, we have $\|e\| = 1$, $\|\Delta e\| = 0$. Therefore, by definition of $\lambda_*(p)$, if $\Lambda_*(p) = 0$ then $\lambda_*(p) = 0$.

However, by assumption $\lambda_*(p) > a = 4\sin^2(m\pi/T)$ for some $m \in Z[0,r]$, thus $\lambda_*(p) > 0$. That is to say the equality $\Lambda_*(p) = 0$ cannot hold. Therefore our discussion will be distinguished in two cases: $\Lambda_*(p) > 0$ and $\Lambda_*(p) < 0$.

2.3. Preliminaries. Let X be a Hilbert space, and let $\varphi \in C^1(X,\mathbb{R})$ be a functional satisfying the (PS) condition. Write $\operatorname{crit}(\varphi) = \{x \in X \mid \varphi'(x) = 0\}$ for the set of critical points of functional φ and $\varphi^c = \{x \in X \mid \varphi(x) \le c\}$ for the level set. Denote by $H_k(A,B)$ the kth singular relative homology group with integer coefficients. Let $x_0 \in \operatorname{crit}(\varphi)$ be an isolated critical point with value $c = \varphi(x_0)$, $c \in \mathbb{R}$, the group $C_k(\varphi, x_0) = H_k(\varphi^c \cap U, (\varphi^c \cap U) \setminus \{x_0\})$, and $k \in \mathbb{Z}$ is called the kth critical group of φ at x_0, where U is a closed neighbourhood of u. Due to the excision of homology [13], $C_k(\varphi, x_0)$ is dependent on U.

Suppose that $\varphi(\operatorname{crit}(\varphi))$ is strictly bounded from below by $a \in \mathbb{R}$, then the critical groups of φ at infinity are formally defined [11] as $C_k(\varphi,\infty) = H_k(X, \varphi^a)$, $k \in \mathbb{Z}$.

Proposition 6 (Proposition 2.3, [11]). *Assume that C^2-functional φ satisfying (PS) condition has a local linking at* 0 *with respect to $X = X_0^+ \bigoplus X_0^-$; that is, there exists $\rho > 0$ such that*

$$\varphi(x)\le\varphi(0)\quad \text{for } x\in X_0^- \text{ and } \|x\|\le\rho,\qquad \varphi(x)>\varphi(0)\quad \text{for } x\in X_0^+ \text{ and } 0<\|x\|\le\rho. \tag{24}$$

Then $C_k(\varphi,0) \neq 0$, $k = \dim X_0^-$.

By Propostion 6, one proves the following lemmas with respect to $E_T = X^+ \bigoplus X^-$.

Lemma 7. *If $a < \lambda_*(p)$, then $C_k(J_p, 0) \neq 0$, $k = \dim X^-$, where $X^- = W^- \bigoplus W^0$ as $\Lambda_*(p) > 0$, $X^- = W^-$ as $\Lambda_*(p) < 0$. $\Lambda_*(p)$ is defined by* (23).

Proof. We first consider the following.

Case 1 ($\Lambda_*(p) > 0$ and $X^+ = W^+$, $X^- = W^- \bigoplus W^0$). By $p > 2$, $|x|^p = o(|x|^2)$ as $|x| \to 0$, then there exists $\theta \in (0,1)$ suitably small, such that $|x|^p \le \delta/3(b/p+\varepsilon)|x|^2$ as $|x| < \theta$, where $\delta > 0$ see (22) and $b = \max\{|b(1)|, \dots, |b(T)|\} > 0$. By assumption (A2) and $G_p(n,0) = 0$, for any given $\varepsilon > 0$, there exists $\rho_n \in (0,\theta)$, such that $|G_p(n,x_n)| \le \varepsilon |x_n|^p$ as $|x_n| \le \rho_n$, $n \in Z[1,T]$. Thus

$$\begin{aligned}\frac{1}{p}\sum_{n=1}^{T}b(n)\left|x_n\right|^p+\sum_{n=1}^{T}G_p(n,x_n)\\ \le\left(\frac{b}{p}+\varepsilon\right)\sum_{n=1}^{T}\left|x_n\right|^p\le\frac{1}{3}\delta\|x\|^2.\end{aligned} \tag{25}$$

Let $\rho = \min\{\rho_1, \dots, \rho_T\}$. For $0 < \| x \| \le \rho < 1$, it follows that

$$J_p(x)\ge\frac{1}{2}\delta\|x\|^2-\frac{1}{3}\delta\|x\|^2>0,\quad x\in W^+=X^+. \tag{26}$$

We need to prove that $J_p(x) \le 0$ for $x \in X^- = W^- \bigoplus W^0$, $\|x\| \le \rho$. We first claim that

$$\sum_{n=1}^{T}b(n)\left|x_n\right|^p>0,\quad \forall x\in W^-\bigoplus W^0,\ x\neq 0. \tag{27}$$

Indeed, by contradiction, assume that $\sum_{n=1}^{T} b(n)|x_n|^p \le 0$, for some $x \in W^- \bigoplus W^0$, $x \neq 0$. Since $\Lambda_*(p) = \sum_{n=1}^{T} b(n)|e_n|^p > 0$, where $e = (e_1^T, e_2^T, \dots, e_N^T)^T = (u^T, u^T, \dots, u^T)^T \in W^- \bigoplus W^0$, and $(W^- \bigoplus W^0) \setminus \{0\}$ is arcwise connected, then there exists a $x^0 \in (W^- \bigoplus W^0) \setminus \{0\}$, such that $\sum_{n=1}^{T} b(n)|x_n^0|^p = 0$. Thus $\|\Delta x^0\|^2 \ge \lambda_*(p)\|x^0\|^2$ by the definition of $\lambda_*(p)$. On the other hand, by the definition of $W^- \bigoplus W^0$, we have $\|\Delta x^0\|^2 \le a\,\|x^0\|^2$. This is a contradiction with assumption $a < \lambda_*(p)$. So the claim (27) holds.

There exists $c_4 > 0$ by (27), such that $\sum_{n=1}^{T} b(n)|x_n|^p \ge c_4\|x\|_p^p$ for all $x \in W^- \bigoplus W^0 \setminus \{0\}$, where $\|x\|_p = (\sum_{n=1}^{T} |x_n|^p)^{1/p}$. For $x \in W^- \bigoplus W^0$, $\|x\| \le \rho$, ε sufficiently small, we have

$$\begin{aligned}J_p(x)&\le-\frac{1}{p}\sum_{n=1}^{T}b(n)\left|x_n\right|^p-\sum_{n=1}^{T}G_p(n,x_n)\\ &\le-\frac{c_4}{p}\|x\|_p^p+\varepsilon\|x\|_p^p\le 0.\end{aligned} \tag{28}$$

Since $J_p(0) = 0$ and J_p satisfies (PS) condition by Lemma 4, so by Proposition 6, we obtain that $C_k(J_p, 0) \neq 0$ for $k = \dim(W^- \bigoplus W^0)$.

Case 2 ($\Lambda_*(p) < 0$, $X^+ = W^+ \bigoplus W^0$, $X^- = W^-$). It is easy to see that $J_p(x) \le 0$ by $\|\Delta x\|^2 - a\,\|x\|^2 \le -\delta\,\|x\|^2$ and $p > 2$, where $x \in W^-$ and $\|x\| \le \rho$. We need to claim that $J_p(x) > 0$, for $x \in W^+ \bigoplus W^0$, $0 < \|x\| \le \rho$.

Suppose not that there exists a sequence $\{x^{(j)}\} \subset E_T$ such that

$$\left\{x^{(j)}\right\}\subset W^+\bigoplus W^0\setminus\{0\},\quad 0<\left\|x^{(j)}\right\|\le\rho,\qquad J_p\left(x^{(j)}\right)\le 0, \tag{29}$$

for large j. For $\| x^{(j)} \| \le \rho$, by Lemma 1, we get

$$\left|\sum_{n=1}^{T}\left[\frac{1}{p}b(n)\left|x_n^{(j)}\right|^p + G_p\left(n, x_n^{(j)}\right)\right]\right| \le \sum_{n=1}^{T}\left[\frac{b}{p}\left|x_n^{(j)}\right|^p + \varepsilon\left|x_n^{(j)}\right|^p\right] \le \left(\frac{b}{p}+\varepsilon\right)\left(\frac{1}{a_1}\right)^p\left\|x^{(j)}\right\|^p. \tag{30}$$

Set $y_n^{(j)} = x_n^{(j)}/\|x^{(j)}\|$. Then by (29) and the previous formula, we have

$$0 \ge \frac{J_p\left(x^{(j)}\right)}{\left\|x^{(j)}\right\|^2} \ge \frac{1}{2}\left(\left\|\Delta y^{(j)}\right\|^2 - a\right) - \left(\frac{b}{p}+\varepsilon\right)\left(\frac{1}{a_1}\right)^p\left\|x^{(j)}\right\|^{p-2}. \tag{31}$$

On the other hand, $\|\Delta y^{(j)}\|^2 \ge a$ by the definition of $W^+ \bigoplus W^0$. Hence by $p > 2$, there exists a subsequence converges to $y^0 \in E_T$, such that $\|\Delta y^0\|^2 = a$, that is $y^0 \in W^0$ and $\|y^0\| = 1$. Since $\|\Delta x^{(j)}\|^2 \ge a\|x^{(j)}\|^2$ for $\{x^{(j)}\} \subset W^+ \bigoplus W^0$, it follows from $J_p(x^{(j)}) \le 0$ that

$$0 \le \frac{1}{p}\sum_{n=1}^{T}b(n)\left|x_n^{(j)}\right|^p + \sum_{n=1}^{T}G_p\left(n, x_n^{(j)}\right) \le \frac{1}{p}\sum_{n=1}^{T}b(n)\left|x_n^{(j)}\right|^p + \varepsilon\left(\frac{1}{a_1}\right)^p\left\|x^{(j)}\right\|^p. \tag{32}$$

Dividing by $\|x^{(j)}\|^p$ in the previous inequality, then $\sum_{n=1}^{T} b(n)|y_n^0|^p = \lim_{j\to\infty}\sum_{n=1}^{T} b(n)|y_n^{(j)}|^p \ge 0$.

Since e, $y^0 \in W^- \bigoplus W^0$, $\Lambda_*(p) = \sum_{n=1}^{T} b(n)|e_n|^p < 0$ and $(W^- \bigoplus W^0)\setminus\{0\}$ is arcwise connected, then there exists a $\overline{y} \in (W^- \bigoplus W^0)\setminus\{0\}$ such that $\sum_{n=1}^{T} b(n)|\overline{y}_n|^p = 0$. Thus $\|\Delta\overline{x}\|^2 \ge \lambda_*(p)\|\overline{x}\|^2$ by the definition of $\lambda_*(p)$. On the other hand, $\|\Delta\overline{x}\|^2 \le a\|\overline{x}\|^2$ by the definition of $W^- \bigoplus W^0$. This is a contradiction with assumption $a < \lambda_*(p)$. That is to say, the claim is valid.

By Proposition 6, we obtain $C_k(J_p, 0) \ne 0$, $k = \dim W^-$. The proof is completed. □

Lemma 8. *If $a < \lambda_*(q)$, then $C_k(J_q, \infty) \ne 0$ for $k = \dim X^-$, where $X^- = W^- \bigoplus W^0$ as $\Lambda_*(q) > 0$, $X^- = W^-$ as $\Lambda_*(q) < 0$.*

The proof is similar to that of Lemma 7 and is omitted.

3. Proof of Theorem 2

Lemma 9. *Let $a < \lambda_*(p)$. If there exists $K_1 > 0$ such that for any $K > K_1$, $J_p(x) \le -K$, then one has $\sum_{n=1}^{T} b(n)|x_n|^p > 0$, and $(d/dt)J_p(tx)|_{t=1} < 0$.*

Proof. We first claim that $\| x \|$ is sufficiently large, if x satisfies condition of Lemma 9. Suppose not there exists $M > 0$ such that $\| x \| \le M$. So there exists $\{x^{(j)}\} \subset E_T$, $x^0 \in E_T$, such that $x^{(j)} \to x^0$ as $j \to \infty$. Since for any $j > K_1$, we have $J_p(x^{(j)}) \le -j$, thus $J_p(x^0) = \lim_{j\to\infty}J_p(x^{(j)}) = -\infty$. It is a contradiction with $J_p(x^0) = c$.

If $\|x\|$ is large enough, then we can assume that $|x_n|$ is large enough for $n \in Z[1,k]$ and $|x_n|$ are bounded for $n \in Z[k+1,T]$. Therefore, by assumption (A1), for any given $\varepsilon > 0$, there exists $M_1 > 0$ such that

$$\left|g_p(n,x_n)\right| \le \varepsilon|x_n| + \frac{M_1}{T}, \qquad \left|G_p(n,x_n)\right| \le \varepsilon|x_n|^2 + \frac{M_1}{T}, \qquad \forall (n,x_n) \in Z[1,T]\times\mathbb{R}^N. \tag{33}$$

We claim that $\sum_{n=1}^{T} b(n)|x_n|^p > 0$. Suppose not that, for $j > K_1$, there exists $\{x^{(j)}\} \subset E_T$ such that

$$\sum_{n=1}^{T}b(n)\left|x_n^{(j)}\right|^p \le 0. \tag{34}$$

By $J_p(x^{(j)}) \le -j \le 0$, (33) and (34), we have

$$\frac{1}{2}\left\|\Delta x^{(j)}\right\|^2 \le \frac{a}{2}\left\|x^{(j)}\right\|^2 + \sum_{n=1}^{T}G_p\left(n, x_n^{(j)}\right) \le \frac{a}{2}\left\|x^{(j)}\right\|^2 + \varepsilon\left\|x^{(j)}\right\|^2 + M_1. \tag{35}$$

Set $y_n^{(j)} = x_n^{(j)}/\|x^{(j)}\|$ and divided by $\|x^{(j)}\|^2$ in the previous inequality. Since ε can be small enough, then there exists a subsequence that converges to $y^0 \in E_T$, such that $\|\Delta y^0\|^2 \le a$, $\|y^0\| = 1$. Moreover, by (33) and (34), we get

$$0 \ge \frac{1}{p}\sum_{n=1}^{T}b(n)\left|x_n^{(j)}\right|^p \ge j + \frac{1}{2}\left\|\Delta x^{(j)}\right\|^2 - \frac{a}{2}\left\|x^{(j)}\right\|^2 - \sum_{n=1}^{T}G_p\left(n, x_n^{(j)}\right) \ge -\left(\frac{a}{2}+\varepsilon\right)\left\|x^{(j)}\right\|^2 - M_1. \tag{36}$$

Since $p > 2$ and $\lim_{j\to\infty}\|x^{(j)}\| = \infty$, divided by $\|x^{(j)}\|^p$ in the previous inequality, we have $\sum_{n=1}^{T} b(n)|y_n^0|^p = \lim_{j\to\infty}\sum_{n=1}^{T} b(n)|y_n^{(j)}|^p = 0$, that is, $\|\Delta y^0\| \ge \lambda_*(q)$, which deduce a contradiction. So the claim $\sum_{n=1}^{T} b(n)|x_n|^p > 0$ holds.

Next we prove that $(d/dt)J_p(tx)|_{t=1} < 0$ holds. By contradiction, there exists a sequence $\{x^{(j)}\} \subset E_T$ such that, for $j > K_1$,

$$\left.\frac{d}{dt}J_p\left(tx^{(j)}\right)\right|_{t=1} \ge 0. \tag{37}$$

Then, by (7), we get

$$\left.\frac{d}{dt}J_p\left(tx^{(j)}\right)\right|_{t=1} = \left\|\Delta x^{(j)}\right\|^2 - a\left\|x^{(j)}\right\|^2 - \sum_{n=1}^{T}b(n)\left|x_n^{(j)}\right|^p - \sum_{n=1}^{T}\left(g_p\left(n, x_n^{(j)}\right), x_n^{(j)}\right), \tag{38}$$

and by (37) and $J_p(x^{(j)}) \le -j < 0$, it follows that

$$\left(1-\frac{p}{2}\right)\left(\left\|\Delta x^{(j)}\right\|^2 - a\left\|x^{(j)}\right\|^2\right) - \sum_{n=1}^{T}\left(g_p\left(n, x_n^{(j)}\right), x_n^{(j)}\right) + p\sum_{n=1}^{T} G_p\left(n, x_n^{(j)}\right) = \frac{d}{dt}J_p\left(tx^{(j)}\right)\Big|_{t=1} - pJ_p\left(x^{(j)}\right) \ge 0. \quad (39)$$

Set $y_n^{(j)} = x_n^{(j)}/\|x^{(j)}\|$ and divided by $\|x^{(j)}\|^2$ in the previous formula; since $p > 2$ and ε can be small enough, then there exists a subsequence converges to $y^0 \in E_T$ such that $\| \Delta y^0\|^2 \le a$, $\|y^0\| = 1$. Moreover, by (37) and the first claim, we get

$$0 < \sum_{n=1}^{T} b(n)\left|x_n^{(j)}\right|^p \le \left\|\Delta x^{(j)}\right\|^2 - a\left\|x^{(j)}\right\|^2 - \sum_{n=1}^{T}\left(g_p\left(n, x_n^{(j)}\right), x_n^{(j)}\right). \quad (40)$$

Divided by $\| x^{(j)}\|^p$ in the previous formula, and by $p > 2$, it follows that $\sum_{n=1}^{T} b(n)|y_n^0|^p = 0$. This is a contradiction with the definition of $\lambda_*(p)$ and condition $a < \lambda_*(p)$. So the second claim holds. The proof is completed. □

Based on Lemma 9, we introduce the following notations:

$$J_p^{-K} = \left\{x \in E_T : J_p(x) \le -K\right\},$$
$$E_p^+ = \left\{x \in E_T : \sum_{n=1}^{T} b(n)\left|x_n\right|^p > 0\right\},$$
$$E(\Omega_+) = \left\{x \in E_T : x_n = 0 \text{ for } n \in Z[1,T] \setminus \Omega_+\right\} \setminus \{0\}. \quad (41)$$

Clearly, $E(\Omega_+) \subset E_p^+$. And by Lemma 9, we have $J_p^{-K} \subset E_p^+$. In order to describe the $H_q(E_T, J_p^{-K})$, we need to show the following lemma.

Lemma 10. *If $a < \lambda_*(p)$, then there exists $K_1 > 0$, such that for any $K > K_1$, J_p^{-K} is a strong deformation retraction of E_p^+. Moreover, $E(\Omega_+)$ and E_p^+ are homotopy equivalent.*

Proof. Now we prove that J_p^{-K} is a strong deformation retraction of E_p^+.

By Lemma 9, we have $J_p^{-K} \subset E_p^+$. Let $x \in E_p^+$. By Lemma 9, there exists a unique $t_p = t_p(x) > 0$ such that $J_p(t_p x) = -K$. By applying Implicit Function Theorem, $t_p(x)$ is a continuous function in E_p^+. Let $T_p(x) = \max\{t_p(x), 1\}$ and define $f_p(s,x) = (1-s)x + sT_p(x)x$, then $f_p : [0,1] \times E_p^+ \to J_p^{-K}$ is a strong deformation retraction. Thus J_p^{-K} is a strong deformation retraction of E_p^+.

We next claim that $E(\Omega_+)$ is a strong deformation retraction of E_p^+. Clearly, in terms of the notations, we have $E(\Omega_+) \subset E_p^+$. Let $\xi_p : Z[1,T] \to \mathbb{R}$ be a function such that

$$\xi_p(n) = 1 \quad \text{if } n \in \Omega_+, \qquad \xi_p(n) = 0 \quad \text{if } n \in \Omega_-,$$
$$\xi_p(n) \in [0,1] \quad \text{if } n \in Z[1,T] \setminus \left(\Omega_+ \cup \Omega_-\right). \quad (42)$$

Define

$$\zeta_p(s, x_n) = \begin{cases} (1-2s)x_n + 2s\xi_p(n)x_n & \text{if } 0 \le s \le \dfrac{1}{2}, \\ 2(1-s)\xi_p(n)x_n + 2\left(s-\dfrac{1}{2}\right)P\left(\xi_p(n)x_n\right) & \text{if } \dfrac{1}{2} \le s \le 1, \end{cases} \quad (43)$$

where $P : E_T \to E(\Omega_+)$ is a projection operator. Then $\zeta_p : [0,1] \times E_p^+ \to E(\Omega_+)$ is a deformation retraction. Indeed,

$$\zeta_p(0,x) = x, \quad \zeta_p(1,x) \in E(\Omega_+), \quad \text{for } x \in E_p^+,$$
$$\zeta_p(s,x) = x, \quad \text{for } x \in E(\Omega_+) \text{ and } s \in [0,1]. \quad (44)$$

For $x \in E_p^+$, if $s \in [0, 1/2]$, then

$$\sum_{n=1}^{T} b(n)\left|\zeta_p(s,x_n)\right|^p = \sum_{n\in\Omega_+} b(n)\left|x_n\right|^p + \sum_{n\in\Omega_-} b(n)(1-2s)^p\left|x_n\right|^p \ge \sum_{n=1}^{T} b(n)\left|x_n\right|^p > 0, \quad (45)$$

where $0 \le (1-2s)^p \le 1$, that is, $\zeta_p(s,x) \in E_p^+$. If $s \in (1/2, 1]$, it follows that

$$\sum_{n=1}^{T} b(n)\left|\zeta_p(s,x_n)\right|^p = \sum_{n\in\Omega_+} b(n)\left|2(1-s)\xi_p(n)x_n + 2\left(s-\frac{1}{2}\right)P\left(\xi_p(n)x_n\right)\right|^p \ge 0. \quad (46)$$

We claim that the equality of the previous formula cannot hold. Otherwise, $Px_n = -((1-s)/(s-(1/2)))x_n$, for $n \in \Omega_+$, which implies that $Px_n = 0$. Hence $x_n = 0$ in Ω_+, which contradicts with the fact $x \in E_p^+$. So $\sum_{n=1}^{T} b(n)|\zeta_p(s,x_n)|^p > 0$, that is, $\zeta_p(s,x) \in E_p^+$ as $s \in (1/2, 1]$. Therefore, ζ_p is a deformation retraction from E_p^+ onto $E(\Omega_+)$, and this completes the proof. □

Proof of Theorem 2. Since $E(\Omega_+)$ is well known to be contractile in itself, and by Lemma 10, it follows that J_p^{-K} is homotopically equivalent to $E(\Omega_+)$ for K large enough, then the Betti numbers (cf. [11, 13]) are

$$\beta_k = \dim C_k(J_p, \infty) = \dim H_k(E_T, J_p^{-K}) = \dim H_k(E_T, E(\Omega_+)) = 0, \quad k \in Z[0, NT]. \tag{47}$$

Now we suppose that system (4) has only trivial solution; that is, J_p has only critical point $x = 0$, then we have the Morse-type numbers $M_k = \dim C_k(J_p, 0)$ for $k \in Z[0, NT]$ (cf. [13]). Moreover, by Lemma 7, $C_k(J_p, 0) \neq 0$ for $k = \dim W^-$ or $k = \dim(W^- \bigoplus W^0)$. Since J_p satisfies (PS) condition by Lemma 4, then using Morse Relation, we have the following.

$$0 = \sum_{k=0}^{NT} (-1)^k \beta_k = \sum_{k=0}^{NT} (-1)^k M_k \neq 0, \tag{48}$$

which is a contradiction. Therefore, J_p has at least one critical point $x^* \neq 0$ and system (4) has at least a nonzero T-periodic solution. □

4. Proof of Theorem 3

For convenience, we introduce the following notations:

$$J_q^c = \{x \in E_T : J_q(x) \le c\}, \quad c \in \mathbb{R},$$
$$E_q^+ = \left\{x \in E_T : \sum_{n=1}^{T} b(n) |x_n|^q > 0\right\}. \tag{49}$$

Clearly, $E_q^+ \cup \{0\}$ is star-shaped with respect to the origin and $E(\Omega_+) \subset E_q^+$, where $E(\Omega_+)$ is given in Section 3. Similarly with the proof of Lemmas 9 and 10, we have the following.

Lemma 11. *Let $a < \lambda_*(q)$. Then there exists $\rho > 0$ such that $(d/dt)J_q(tx)|_{t=1} > 0$ for any $x \in M_\rho = \{x \in B_\rho \cap E_q^+ : J_q(x) \ge 0\}$, where B_ρ stands for the closed ball in E_T of radius $\rho > 0$ with the center at zero.*

Lemma 12. *Let $a < \lambda_*(q)$. Then there exists $\rho > 0$ such that $(J_q^0 \cap B_\rho) \setminus \{0\}$ is a retract of $E_q^+ \cap B_\rho$, and $E(\Omega^+)$ is a strong deformation retraction of E_q^+.*

Proof of Theorem 3. We first prove that $J_q^0 \cap B_\rho$ is contractible in itself. In fact, it is sufficient to show that $J_q^0 \cap B_\rho$ is starshaped with respect to the origin; that is, $x \in J_q^0 \cap B_\rho$ implies that $tx \in J_q^0 \cap B_\rho$ for all $t \in [0, 1]$.

Assume, by a contradiction, that there exists $x_0 \in J_q^0 \cap B_\rho$ and $t_0 \in (0, 1)$, such that $J_q(t_0 x_0) > 0$. It follows from Lemma 11 that $(d/dt)J_q(t_0 x_0) > 0$. By the monotonicity arguments, this implies that

$$J_q(tx_0) > 0 \quad \forall t \in [t_0, 1]. \tag{50}$$

This contradicts the assumption $x_0 \in J_q^0$, which implies $J_q(x_0) \le 0$.

On the other hand, since $E(\Omega_+)$ is contractible in itself, and $E_q^+ \cup \{0\}$ is starshaped with respect to the origin, then $E_q^+ \cap B_\rho$ is contractible in itself. The retract of the set which is contractible in itself is also contractible (cf. [19]); it follows that the set $(J_q^0 \cap B_\rho) \setminus \{0\}$ is contractible by Lemma 12.

Combining the previous argument, $J_q^0 \cap B_\rho$ and $(J_q^0 \cap B_\rho) \setminus \{0\}$ are contractible in themselves.

$$\dim C_k(J_q, 0) = \dim H_k(J_q^0 \cap B_\rho, (J_q^0 \cap B_\rho) \setminus \{0\}) = 0, \quad k \in Z[0, NT]. \tag{51}$$

By Lemma 8, $C_k(J_q, \infty) \neq 0$ for $k = \dim(W^- \bigoplus W^0)$ or $k = \dim W^-$. Therefore, by Morse Relation and the same methods in proof of Theorem 2, it follows that J_q has at least one critical point $x^* \neq 0$ and system (5) has at least a nonzero T-periodic solution. □

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References

[1] Z. Guo and J. Yu, “Existence of periodic and subharmonic solutions for second-order superlinear difference equations,” *Science in China A*, vol. 46, no. 4, pp. 506–515, 2003.

[2] Z. Guo and J. Yu, “The existence of periodic and subharmonic solutions of subquadratic second order difference equations,” *Journal of the London Mathematical Society*, vol. 68, no. 2, pp. 419–430, 2003.

[3] W. G. Kelley and A. C. Peterson, *Difference Equations: An Introduction with Applications*, Academic Press, Boston, Mass, USA, 1991.

[4] C.-L. Tang and X.-P. Wu, “Periodic solutions for second order Hamiltonian systems with a change sign potential,” *Journal of Mathematical Analysis and Applications*, vol. 292, no. 2, pp. 506–516, 2004.

[5] Y.-T. Xu and Z.-M. Guo, “Existence of periodic solutions to second-order Hamiltonian systems with potential indefinite in sign,” *Nonlinear Analysis. Theory, Methods & Applications A*, vol. 51, no. 7, pp. 1273–1283, 2002.

[6] C. D. Ahlbrandt and A. C. Peterson, *Discrete Hamiltonian Systems: Difference Equations, Continued Fractions, and Riccati Equations*, vol. 16, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.

[7] H. Amann and E. Zehnder, “Nontrivial solutions for a class of nonresonance problems and applications to nonlinear differential equations,” *Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie IV*, vol. 7, no. 4, pp. 539–603, 1980.

[8] Z. Huang, C. Feng, and S. Mohamad, “Multistability analysis for a general class of delayed Cohen-Grossberg neural networks,” *Information Sciences*, vol. 187, pp. 233–244, 2012.

[9] Z. Huang and Y. N. Raffoul, "Biperiodicity in neutral-type delayed difference neural networks," *Advances in Difference Equations*, vol. 2012, article 5, 2012.

[10] W. Zou and S. Li, "Infinitely many solutions for Hamiltonian systems," *Journal of Differential Equations*, vol. 186, no. 1, pp. 141–164, 2002.

[11] T. Bartsch and S. Li, "Critical point theory for asymptotically quadratic functionals and applications to problems with resonance," *Nonlinear Analysis. Theory, Methods & Applications A*, vol. 28, no. 3, pp. 419–441, 1997.

[12] K. C. Chang, *Critical Point Theory and Its Applications*, Science and Technical Press, Shanghai, China, 1980.

[13] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, vol. 74, Springer, New York, NY, USA, 1989.

[14] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, vol. 65 of *CBMS Regional Conference Series in Mathematics*, American Mathematical Society, Providence, RI, USA, 1986.

[15] V. Benci, "A new approach to the Morse-Conley theory and some applications," *Annali di Matematica Pura ed Applicata*, vol. 158, pp. 231–305, 1991.

[16] C. Conley, *Isolated Invariant Sets and the Morse Index*, vol. 38 of *CBMS Regional Conference Series in Mathematics*, American Mathematical Society, Providence, RI, USA, 1978.

[17] K. C. Chang, "Solutions of asymptotically linear operator equations via Morse theory," *Communications on Pure and Applied Mathematics*, vol. 34, no. 5, pp. 693–712, 1981.

[18] K.-C. Chang, *Infinite-Dimensional Morse Theory and Multiple Solution Problems*, Birkhäuser, Boston, Mass, USA, 1993.

[19] V. Moroz, "On the Morse critical groups for indefinite sublinear elliptic problems," *Nonlinear Analysis. Theory, Methods & Applications A*, vol. 52, no. 5, pp. 1441–1453, 2003.

[20] F. Antonacci, "Periodic and homoclinic solutions to a class of Hamiltonian systems with indefinite potential in sign," *Unione Matematica Italiana. Bollettino B*, vol. 10, no. 2, pp. 303–324, 1996.

[21] F. Antonacci, "Existence of periodic solutions of Hamiltonian systems with potential indefinite in sign," *Nonlinear Analysis. Theory, Methods & Applications A*, vol. 29, no. 12, pp. 1353–1364, 1997.

[22] H.-H. Bin, J.-S. Yu, and Z.-M. Guo, "Nontrivial periodic solutions for asymptotically linear resonant difference problem," *Journal of Mathematical Analysis and Applications*, vol. 322, no. 1, pp. 477–488, 2006.