Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 708281, 13 pages
http://dx.doi.org/10.1155/2013/708281

Research Article

Computation of Positive Solutions for Nonlinear Impulsive Integral Boundary Value Problems with p-Laplacian on Infinite Intervals

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Received 14 October 2012; Accepted 11 January 2013

Academic Editor: Patricia J. Y. Wong

This paper deals with the existence and iteration of positive solutions for nonlinear second-order impulsive integral boundary value problems with p-Laplacian on infinite intervals. Our approach is based on the monotone iterative technique.

1. Introduction

The theory of impulsive differential equations has been emerging as an important area of investigation in recent years. It has been extensively applied to biology, biologic medicine, optimum control in economics, chemical technology, population dynamics, and so on. It is much richer because all the structure of its emergence has deep physical background and realistic mathematical model and coincides with many phenomena in nature. For an introduction of the basic theory of impulsive differential equations in R^n, the reader is referred to see Lakshmikantham et al. [1, 2], Samoĭlenko and Perestyuk [3], and the references therein.

Boundary value problems on infinite intervals arise quite naturally in the study of radially symmetric solutions of nonlinear elliptic equations and models of gas pressure in a semi-infinite porous medium; see [4–7], for example. In a recent paper [8], by means of a fixed-point theorem due to Avery and Peterson, Li and Nieto obtained some new results on the existence of multiple positive solutions for the following multipoint boundary value problem with a finite number of impulsive times on an infinite interval:

$$\begin{gathered} u''(t) + q(t) f(t, u(t)) = 0, \\ \forall 0 < t < \infty, \quad t \neq t_k, \quad k = 1, 2, \ldots, p, \\ \Delta u(t_k) = I_k(u(t_k)), \quad k = 1, 2, \ldots, p, \\ u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad u'(\infty) = 0, \end{gathered} \tag{1}$$

where $f \in C([0, +\infty) \times [0, +\infty), [0, +\infty))$, $I_k \in C([0, +\infty), [0, +\infty))$, $u'(\infty) = \lim_{t \to \infty} u'(t)$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < +\infty$, $0 < t_1 < t_2 < \cdots < t_p < +\infty$, and $q \in C([0, +\infty), [0, +\infty))$.

Boundary value problems with integral boundary conditions for ordinary differential equations arise in different fields of applied mathematics and physics such as heat conduction, chemical engineering, underground water flow, thermoelasticity, and plasma physics. Moreover, boundary value problems with Riemann-Stieltjes integral conditions constitute a very interesting and important class of problems. They include two-point, three-point, and multipoint boundary value problems as special cases; see [9–14]. For boundary value problems with other integral boundary conditions and comments on their importance, we refer the reader to the papers [11–20] and the references therein.

There are relatively few papers available for integral boundary value problems for impulsive differential equations on an infinite interval with an infinite number of impulsive times up to now. In [21], Zhang et al. investigated the existence of minimal nonnegative solution for the following second-order impulsive differential equation

$$\begin{gathered} -x''(t) = f\left(t, x(t), x'(t)\right) = 0, \quad t \in J, t \neq t_k, \\ \Delta x|_{t=t_k} = I_k\left(x\left(t_k\right)\right), \quad k = 1, 2, \ldots, \\ \Delta x'\Big|_{t=t_k} = \bar{I}_k\left(x\left(t_k\right)\right), \quad k = 1, 2, \ldots, \\ x(0) = \int_0^{+\infty} g(t) x(t)\, dt, \quad x'(\infty) = 0, \end{gathered} \tag{2}$$

where $f \in C(J\times J\times J, J)$, $I_k \in C(J, J)$, $\bar{I}_k \in C(J, J)$, $J = [0, +\infty)$, $0 < t_1 < t_2 < \cdots < t_k < \cdots$, $t_k \to \infty$, and $g(t) \in L[J, J]$ with $\int_0^{+\infty} g(t)dt < 1$. $\Delta x|_{t=t_k}$ denotes the jump of $x(t)$ at $t = t_k$, that is,

$$\Delta x|_{t=t_k} = x\left(t_k^+\right) - x\left(t_k^-\right), \tag{3}$$

where $x(t_k^+)$ and $x(t_k^-)$ represent the right-hand limit and left-hand limit of $x(t)$ at $t = t_k$, respectively. $\Delta x'|_{t=t_k}$ has a similar meaning to $x'(t)$.

In the past few years, the existence and the multiplicity of bounded or unbounded positive solutions to nonlinear differential equations on infinite intervals have been studied by several different techniques; we refer the reader to [5–8, 21–29] and the references therein. However, most of these papers only considered the existence of positive solutions under various boundary value conditions. Seeing such a fact, a natural question which arises is "how can we find the solutions when they are known to exist?" More recently, Ma et al. [30] and Sun et al. [31, 32] established iterative schemes for approximating the solutions for some boundary value problems defined on finite intervals by virtue of the iterative technique.

However, to the author's knowledge, the corresponding theory for impulsive integral boundary value problems with p-Laplacian operator and infinite impulsive times on infinite intervals has not been considered till now. Motivated by previous papers, the purpose of this paper is to obtain the existence of positive solutions and establish a corresponding iterative scheme for the following impulsive integral boundary value problem of second-order differential equation with p-Laplacian on an infinite interval

$$\begin{gathered} \left(\varphi_p\left(x'(t)\right)\right)' + q(t) f\left(t, x(t), x'(t)\right) = 0, \quad t \in J'_+, \\ \Delta x|_{t=t_k} = I_k\left(x\left(t_k\right)\right), \quad k = 1, 2, \ldots, \\ x(0) = \int_0^{+\infty} g(t) x(t)\, dt, \quad x'(\infty) = x_\infty, \end{gathered} \tag{4}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $J = [0, +\infty)$, $J_+ = (0, +\infty)$, $J'_+ = J_+ \setminus \{t_1, t_2, \ldots, t_k, \ldots\}$, $0 < t_1 < t_2 < \cdots < t_k < \cdots$, $t_k \to \infty$, and $g(t) \in L[J, J]$ with $\int_0^{+\infty} g(t)dt < 1$, $\int_0^{+\infty} tg(t)dt < +\infty$, and $0 \leq x'(\infty) = \lim_{t\to+\infty} x'(t)$.

It is clear that

$$\varphi_p(s+t) \leq \begin{cases} 2^{p-1}\left(\varphi_p(s) + \varphi_p(t)\right), & p \geq 2,\ s, t > 0, \\ \varphi_p(s) + \varphi_p(t), & 1 < p < 2,\ s, t > 0, \end{cases} \tag{5}$$

$$\varphi_p^{-1}(s+t) \leq \begin{cases} 2^{1/(p-1)}\left(\varphi_p^{-1}(s) + \varphi_p^{-1}(t)\right), & p \geq 2,\ s, t > 0, \\ \varphi_p^{-1}(s) + \varphi_p^{-1}(t), & 1 < p < 2,\ s, t > 0. \end{cases} \tag{6}$$

Throughout this paper, we adopt the following assumptions.

(H_1) $f(t, u, v) \in C(J \times J \times J, J)$, $f(t, 0, 0) \not\equiv 0$ on any subinterval of J, and when u, v are bounded, $f(t, (1+t)u, v)$ is bounded on J.

(H_2) $q(t)$ is a nonnegative measurable function defined in J_+ and $q(t)$ does not identically vanish on any subinterval of J_+, and

$$\begin{gathered} 0 < \int_0^{+\infty} q(t)\, dt < +\infty, \\ 0 < \int_0^{+\infty} \varphi_p^{-1}\left(\int_s^{+\infty} q(\tau)\, d\tau\right) ds < +\infty. \end{gathered} \tag{7}$$

(H_3) $I_k \in C(J, J)$, and there exist $a_k \geq 0$, $b_k \geq 0$ such that

$$\begin{gathered} 0 \leq I_k(x) \leq a_k + b_k x, \quad \text{for } x \in J \ (k = 1, 2, 3, \ldots), \\ a^* = \sum_{k=1}^{\infty} a_k < +\infty, \qquad b^* = \sum_{k=1}^{\infty} b_k\left(1 + t_k\right) < +\infty, \end{gathered} \tag{8}$$

with $b^* < (1/3)(1 - \int_0^{+\infty} g(t)dt)$.

If $p = 2$, $I_k = 0$ $(k = 1, 2, \ldots)$, $g(t) \equiv 0$, $x'(\infty) = 0$, then BVP (4) reduces to the following two-point boundary value problem:

$$\begin{gathered} -x''(t) = f\left(t, x(t), x'(t)\right) = 0, \quad t \in J, \\ x(0) = 0, \quad x'(\infty) = 0, \end{gathered} \tag{9}$$

which has been studied in [23].

Compared with [8, 21], the main features of the present paper are as follows. Firstly, second-order differential operator is replaced by a more general p-Laplacian operator. Secondly, in this paper, x_∞ in boundary value conditions may not be zero which will bring about computational difficulties. Thirdly, by applying monotone iterative techniques, we construct successive iterative schemes starting off with simple known functions. It is worth pointing out that the first terms of our iterative schemes are simple functions. Therefore, the iterative schemes are significant and feasible.

The rest of this paper is organized as follows. In Section 2, we give some preliminaries and establish several lemmas. The main theorems are formulated and proved in Section 3. Then, in Section 4, an example is presented to illustrate the main results.

2. Preliminaries and Several Lemmas

Definition 1. Let E be a real Banach space. A nonempty closed set $P \subset E$ is said to be a cone provided that

(1) $au + bv \in P$ for all $u, v \in P$ and all $a \geq 0$, $b \geq 0$,

(2) $u, -u \in P$ implies that $u = 0$.

Definition 2. A map $\alpha : P \to [0, +\infty)$ is said to be concave on P, if $\alpha(tu + (1-t)v) \geq t\alpha(u) + (1-t)\alpha(v)$ for all $u, v \in P$ and $t \in [0, 1]$.

Let $PC[J, R] = \{x : x$ is a map from J into R such that $x(t)$ is continuous at $t \neq t_k$, left continous at $t = t_k$ and $x(t_k^+)$ exists for $k = 1, 2, \ldots, \}$, $PC^1[J, R] = \{x \in PC[J, R] : x'(t)$ exists and is continuous at $t \neq t_k$, left continous at $t = t_k$ and $x'(t_k^+)$ exists for $k = 1, 2, \ldots, \}$

$$FPC[J, R] = \left\{x \in PC[J, R] : \sup_{t\in J} \frac{|x(t)|}{1+t} < \infty\right\},$$
$$E = DPC^1[J, R] = \left\{x \in PC^1[J, R] : \sup_{t\in J} \frac{|x(t)|}{1+t} < \infty,\ \sup_{t\in J} \left|x'(t)\right| < \infty\right\}. \tag{10}$$

Obviously, $DPC^1[J, R] \subset FPC[J, R]$. It is clear that $FPC[J, R]$ is a Banach space with the norm

$$\|x\|_F = \sup_{t\in J} \frac{|x(t)|}{1+t}, \tag{11}$$

and $DPC^1[J, R]$ is also a Banach space with the norm

$$\|x\|_D = \max\left\{\|x\|_F, \left\|x'\right\|_B\right\}, \tag{12}$$

where $\|x'\|_B = \sup_{t\in J}|x'(t)|$. Let $J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}]$ $(k = 1, 2, 3, \ldots)$. Define a cone $P \subset E$ by

$$P = \{x \in E : x \text{ is concave and nondecreasing on } J,\ x(t) \geq 0,\ x'(t) \geq 0,\ t \in J\}. \tag{13}$$

Remark 3. If x satisfies (4), then $(\varphi_p(x'(t)))' = -q(t)f(t, x(t), x'(t)) \leq 0$, and $t \in [0, +\infty)$ which implies that $\varphi_p(x'(t))$ is nonincreasing on J; that is, $x'(t)$ is also nonincreasing on J. Thus, x is concave on $[0, +\infty)$. Moreover, if $x'(\infty) = x_\infty \geq 0$, then $x'(t) \geq 0$, $t \in [0, +\infty)$, and so x is monotone increasing on $[0, +\infty)$.

Lemma 4. *Let conditions (H_1)–(H_3) hold. Then, $x \in P$ with $(\varphi_p(x'(t)))' \in C(0, +\infty)$ is a solution of BVP (4) if and only if $x \in C[0, +\infty)$ is a fixed point of the following operator equation:*

$$\begin{aligned}(Ax)(t) &= \frac{1}{1 - \int_0^{+\infty} g(t)\, dt} \times \int_0^{+\infty} g(t) \times \Bigg[\int_0^t \varphi_p^{-1} \times \left(\int_s^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right) d\tau + \varphi_p(x_\infty)\right) ds \\ &\quad + \sum_{t_k<t} I_k(x(t_k))\Bigg] dt \\ &\quad + \int_0^t \varphi_p^{-1}\left(\int_s^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right) d\tau + \varphi_p(x_\infty)\right) ds \\ &\quad + \sum_{t_k<t} I_k(x(t_k)).\end{aligned} \tag{14}$$

Proof. Suppose that $x \in P$ with $(\varphi_p(x'(t)))' \in C(0, +\infty)$ is a solution of BVP (4). For $t \in J$, integrating (4) from t to $+\infty$, we have

$$\int_t^{+\infty} \varphi_p\left(x'(\tau)\right)' d\tau = -\int_t^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right) d\tau. \tag{15}$$

That is

$$\varphi_p\left(x'(\infty)\right) - \varphi_p\left(x'(t)\right) = -\int_t^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right) d\tau, \tag{16}$$

which implies that

$$x'(t) = \varphi_p^{-1}\left(\int_t^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right) d\tau + \varphi_p(x_\infty)\right). \tag{17}$$

If $t_1 < t \leq t_2$, integrating (17) from 0 to t_1, we get that

$$x(t_1) - x(0) = \int_0^{t_1} \varphi_p^{-1}\left(\int_s^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right) d\tau + \varphi_p(x_\infty)\right) ds. \tag{18}$$

Integrating (17) from t_1 to t, we obtain

$$x(t) - x(t_1^+) = \int_{t_1}^{t} \varphi_p^{-1}\left(\int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\, \mathrm{d}\tau + \varphi_p(x_\infty)\right) \mathrm{d}s. \tag{19}$$

Adding (18) and (19) together, we have

$$\begin{aligned} x(t) &= x(0) + \int_0^{t} \varphi_p^{-1}\left(\int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\, \mathrm{d}\tau + \varphi_p(x_\infty)\right) \mathrm{d}s \\ &\quad + I_1(x(t_1)), \quad t_1 < t \le t_2. \end{aligned} \tag{20}$$

Repeating previous process, we get that

$$\begin{aligned} x(t) &= x(0) + \int_0^{t} \varphi_p^{-1}\left(\int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\, \mathrm{d}\tau + \varphi_p(x_\infty)\right) \mathrm{d}s + \sum_{t_k<t} I_k(x(t_k)) \\ &= \int_0^{+\infty} g(t) x(t)\, \mathrm{d}t + \int_0^{t} \varphi_p^{-1}\left(\int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\, \mathrm{d}\tau + \varphi_p(x_\infty)\right) \mathrm{d}s + \sum_{t_k<t} I_k(x(t_k)). \end{aligned} \tag{21}$$

It follows that

$$\begin{aligned} \int_0^{+\infty} g(t) x(t)\, \mathrm{d}t &= \frac{1}{1 - \int_0^{+\infty} g(t)\, \mathrm{d}t} \times \int_0^{+\infty} g(t) \times \left[\int_0^t \varphi_p^{-1} \times \left(\int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\, \mathrm{d}\tau + \varphi_p(x_\infty)\right) \mathrm{d}s + \sum_{t_k<t} I_k(x(t_k))\right] \mathrm{d}t. \end{aligned} \tag{22}$$

Substituting (22) into (21), we get that

$$\begin{aligned} x(t) &= \frac{1}{1 - \int_0^{+\infty} g(t)\, \mathrm{d}t} \times \int_0^{+\infty} g(t) \times \left[\int_0^t \varphi_p^{-1} \times \left(\int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\, \mathrm{d}\tau + \varphi_p(x_\infty)\right) \mathrm{d}s + \sum_{t_k<t} I_k(x(t_k))\right] \mathrm{d}t \\ &\quad + \int_0^{t} \varphi_p^{-1}\left(\int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\, \mathrm{d}\tau + \varphi_p(x_\infty)\right) \mathrm{d}s + \sum_{t_k<t} I_k(x(t_k)). \end{aligned} \tag{23}$$

For $x \in P$, there exists r_0 such that $\|x\|_D < r_0$. Set $B_{r_0} = \sup\{f(t, (1+t)u, v) \mid (t, u, v) \in J \times [0, r_0] \times [0, r_0]\}$, and we have by ($\mathrm{H}_1$) and ($\mathrm{H}_3$) that

$$\begin{gathered} \int_0^{+\infty} q(s) f(s, x(s), x'(s))\, \mathrm{d}s \le \int_0^{+\infty} q(s)\, \mathrm{d}s \cdot B_{r_0}, \\ \sum_{t_k<t} I_k(x(t_k)) \le \sum_{k=1}^{\infty} I_k(x(t_k)) \le a^* + b^* r_0 < +\infty. \end{gathered} \tag{24}$$

By (6), (24), we have

$$\begin{aligned} x(t) &= \frac{1}{1 - \int_0^{+\infty} g(t)\, \mathrm{d}t} \times \int_0^{+\infty} g(t) \times \left[\int_0^t \varphi_p^{-1} \times \left(\int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\, \mathrm{d}\tau + \varphi_p(x_\infty)\right) \mathrm{d}s + \sum_{t_k<t} I_k(x(t_k))\right] \mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \varphi_p^{-1} \\
&\qquad \times \left(\int_s^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right) \mathrm{d}\tau + \varphi_p(x_\infty)\right) \mathrm{d}s \\
&+\sum_{t_k<t} I_k\left(x\left(t_k\right)\right) \\
&\leq \frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t} \\
&\quad \times \int_0^{+\infty} t g(t)\,\mathrm{d}t \cdot \varphi_p^{-1} \\
&\qquad \times \left(\int_0^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right) \mathrm{d}\tau + \varphi_p(x_\infty)\right) \\
&\quad + t\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right) \mathrm{d}\tau + \varphi_p(x_\infty)\right) \\
&\quad + \frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}\left(a^* + b^* r_0\right) \\
&\leq 2^{1/(p-1)}\left(\frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t} \cdot \int_0^{+\infty} t g(t)\,\mathrm{d}t + t\right) \\
&\quad \times \left[\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right) \mathrm{d}\tau\right) + x_\infty\right] \\
&\quad + \frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}\left(a^* + b^* r_0\right).
\end{aligned} \tag{25}
$$

It follows from (24) and (25) that the right term in (23) is well defined. Thus, we have proved that x is a fixed point of the operator A defined by (14).

Conversely, suppose that $x \in C[0,+\infty)$ is a fixed point of the operator equation (14). Evidently,

$$
\Delta x|_{t=t_k} = I_k\left(x\left(t_k\right)\right) \quad (k = 1, 2, \ldots). \tag{26}
$$

Direct differentiation of (14) implies that, for $t \neq t_k$,

$$
\begin{gathered}
x'(t) = \varphi_p^{-1}\left(\int_t^{+\infty} q(s) f\left(s, x(s), x'(s)\right) \mathrm{d}s + \varphi_p(x_\infty)\right), \\
\Delta x'\big|_{t=t_k} = 0 \quad (k = 1, 2, \ldots), \\
\left(\varphi_p\left(x'(t)\right)\right)' = -q(t) f\left(t, x(t), x'(t)\right),
\end{gathered} \tag{27}
$$

which means that $(\varphi_p(x'(t)))' \in C(J')$. It is easy to verify that $x(0) = \int_0^{+\infty} g(t)x(t)\mathrm{d}t$, $x'(\infty) = x_\infty$. The proof of Lemma 4 is complete. □

To obtain the complete continuity of A, the following lemma is still needed.

Lemma 5 (see [33, 34]). *Let W be a bounded subset of P. Then, P is relatively compact in E if $\{W(t)/(1+t)\}$ and $\{W'(t)\}$ are both equicontinuous on any finite subinterval $J_k \cap [0,T]$ $(k = 1, 2, \ldots)$ for any $T > 0$, and for any $\varepsilon > 0$, there exists $N > 0$ such that*

$$
\left|\frac{x(t')}{1+t'} - \frac{x(t'')}{1+t''}\right| < \varepsilon, \quad \left|x'(t') - x'(t'')\right| < \varepsilon, \quad \forall t', t'' \geq N, \tag{28}
$$

uniformly with respect to $x \in W$ as $t', t'' \geq N$, where $W(t) = \{x(t) \mid x \in W\}$, $W'(t) = \{x'(t) \mid x \in W\}$, $t \in [0,+\infty)$.

This lemma is a simple improvement of the Corduneanu theorem in [33, 34].

Lemma 6. *Let (H_1)–(H_3) hold. Then $A : P \to P$ is completely continuous.*

Proof. For any $x \in P$, by (14), we have

$$
\begin{gathered}
\varphi_p\left((Ax)'\right)(t) = \int_t^{+\infty} q(s) f\left(s, x(s), x'(s)\right) \mathrm{d}s + \varphi_p(x_\infty), \\
\left(\varphi_p(Ax)'(t)\right)' = -q(t) f\left(t, x(t), x'(t)\right).
\end{gathered} \tag{29}
$$

It follows from (14), (29), and (H_1) that $(Ax)(t) \geq 0$, $(Ax)'(t) \geq x_\infty \geq 0$, $(Ax)''(t) \leq 0$, that is, $A(P) \subset P$. Now, we prove that A is continuous and compact respectively. Let $x_n \in P$, $x_n \to x$ as $n \to \infty$, then there exists r_0 such that $\sup_{n \in N\setminus\{0\}} \|x_n\| < r_0$. Let $B_{r_0} = \sup\{f(t,(1+t)u,v) \mid (t,u,v) \in J \times [0,r_0] \times [0,r_0]\}$. By (H_1) and (H_2), we have

$$
\begin{aligned}
&\int_0^{+\infty} q(\tau)\left|f\left(\tau, x_n(\tau), x_n'(\tau)\right) - f\left(\tau, x(\tau), x'(\tau)\right)\right| \mathrm{d}\tau \\
&\qquad \leq 2B_{r_0} \cdot \int_0^{+\infty} q(s)\,\mathrm{d}s < +\infty.
\end{aligned} \tag{30}
$$

It follows from (30) and dominated convergence theorem that

$$
\begin{aligned}
&\int_0^{+\infty} q(\tau)\left|f\left(\tau, x_n(\tau), x_n'(\tau)\right) - f\left(\tau, x(\tau), x'(\tau)\right)\right| \mathrm{d}\tau \\
&\longrightarrow \int_0^{+\infty} q(\tau)\left|f\left(\tau, x(\tau), x'(\tau)\right) - f\left(\tau, x(\tau), x'(\tau)\right)\right| \mathrm{d}\tau,
\end{aligned} \tag{31}
$$

which implies that

$$
\begin{aligned}
&\Bigg|\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau) f\left(\tau, x_n(\tau), x_n'(\tau)\right) \mathrm{d}\tau + \varphi_p(x_\infty)\right) \mathrm{d}s \\
&\quad -\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right) \mathrm{d}\tau + \varphi_p(x_\infty)\right) \mathrm{d}s\Bigg| \\
&\longrightarrow 0, \quad n \longrightarrow \infty.
\end{aligned} \tag{32}
$$

By (30)–(32), (H_3) and dominated convergence theorem, we get that

$$\begin{aligned}
&\|Ax_n - Ax\|_F \\
&= \sup_{t\in J}\left\{\frac{1}{1+t}\right. \\
&\quad\times\left|\frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}\right. \\
&\quad\times\left[\int_0^{+\infty} g(t)\right. \\
&\quad\times\int_0^t \varphi_p^{-1} \\
&\quad\times\left(\int_s^{+\infty} q(\tau) f\left(\tau, x_n(\tau), x_n'(\tau)\right)\mathrm{d}\tau\right. \\
&\quad\left.+\varphi_p(x_\infty)\right)\mathrm{d}s\,\mathrm{d}t \\
&\quad-\int_0^{+\infty} g(t) \\
&\quad\times\int_0^t \varphi_p^{-1}\left(\int_s^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right)\mathrm{d}\tau\right. \\
&\quad\left.+\varphi_p(x_\infty)\right)\mathrm{d}s\,\mathrm{d}t \\
&\quad+\int_0^{+\infty} g(t) \\
&\quad\left.\cdot\sum_{t_k<t}\left(I_k(x_n(t_k)) - I_k(x(t_k))\right)\right]\mathrm{d}t \\
&\quad+\int_0^t\left[\varphi_p^{-1}\left(\int_s^{+\infty} q(\tau) f\left(\tau, x_n(\tau), x_n'(\tau)\right)\mathrm{d}\tau\right.\right. \\
&\quad\left.+\varphi_p(x_\infty)\right) \\
&\quad-\varphi_p^{-1}\left(\int_s^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right)\mathrm{d}\tau\right. \\
&\quad\left.\left.+\varphi_p(x_\infty)\right)\right]\mathrm{d}s \\
&\quad\left.\left.+\sum_{t_k<t}\left(I_k(x_n(t_k)) - I_k(x(t_k))\right)\right|\right\} \\
&\le \frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t} \\
&\quad\times\int_0^{+\infty} t g(t)\,\mathrm{d}t \\
&\quad\cdot\left|\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau) f\left(\tau, x_n(\tau), x_n'(\tau)\right)\mathrm{d}\tau + \varphi_p(x_\infty)\right)\right. \\
&\quad\left.-\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right)\mathrm{d}\tau + \varphi_p(x_\infty)\right)\right| \\
&\quad+\left|\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau) f\left(\tau, x_n(\tau), x_n'(\tau)\right)\mathrm{d}\tau + \varphi_p(x_\infty)\right)\right. \\
&\quad\left.-\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right)\mathrm{d}\tau + \varphi_p(x_\infty)\right)\right| \\
&\quad+\frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t} \\
&\quad\times\int_0^{+\infty} g(t)\cdot\sum_{t_k<t}\left|I_k(x_n(t_k)) - I_k(x(t_k))\right|\mathrm{d}t \\
&\quad+\sum_{t_k<t}\left|I_k(x_n(t_k)) - I_k(x(t_k))\right| \longrightarrow 0 \quad (n\longrightarrow\infty), \\
&\left\|(Ax_n)' - (Ax)'\right\|_B \\
&= \sup_{t\in J}\left\{\left|\varphi_p^{-1}\left(\int_t^{+\infty} q(s) f\left(s, x_n(s), x_n'(s)\right)\mathrm{d}s + \varphi_p(x_\infty)\right)\right.\right. \\
&\quad\left.\left.-\varphi_p^{-1}\left(\int_t^{+\infty} q(s) f\left(s, x(s), x'(s)\right)\mathrm{d}s + \varphi_p(x_\infty)\right)\right|\right\} \\
&\longrightarrow 0 \quad (n\longrightarrow\infty).
\end{aligned} \tag{33}$$

It follows from (33) that A is continuous.

Let $\Omega \subset P$ be any bounded subset. Then, there exists $r > 0$ such that $\|x\|_D \le r$ for any $x \in \Omega$. Obviously,

$$\begin{aligned}
&\|Ax\|_F \\
&= \sup_{t\in J}\left\{\frac{1}{1+t}\right. \\
&\quad\times\left|\frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}\right. \\
&\quad\times\int_0^{+\infty} g(t) \\
&\quad\times\left[\int_0^t \varphi_p^{-1}\right. \\
&\quad\times\left(\int_s^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right)\mathrm{d}\tau\right. \\
&\quad\left.+\varphi_p(x_\infty)\right)\mathrm{d}s \\
&\quad\left.+\sum_{t_k<t} I_k(x(t_k))\right]\mathrm{d}t \\
&\quad+\int_0^t \varphi_p^{-1} \\
&\quad\times\left(\int_s^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right)\mathrm{d}\tau\right. \\
&\quad\left.+\varphi_p(x_\infty)\right)\mathrm{d}s \\
&\quad\left.\left.+\sum_{t_k<t} I_k(x(t_k))\right|\right\}
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t} \\
&\quad \times \int_0^{+\infty} t g(t)\,\mathrm{d}t \cdot \varphi_p^{-1} \\
&\qquad \times \left(\int_0^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right)\mathrm{d}\tau + \varphi_p\left(x_\infty\right)\right) \\
&\quad + \varphi_p^{-1}\left(\int_0^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right)\mathrm{d}\tau + \varphi_p\left(x_\infty\right)\right) \\
&\quad + \frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}\left(a^* + b^* r\right) \\
&\le 2^{1/(p-1)}\left[\frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}\int_0^{+\infty} t g(t)\,\mathrm{d}t + 1\right] \\
&\quad \times \left[\varphi_p^{-1}\left(B_r\right)\cdot \varphi_p^{-1}\left(\int_0^{+\infty} q(\tau)\,\mathrm{d}\tau\right) + x_\infty\right] \\
&\quad + \frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}\left(a^* + b^* r\right), \\
&\left\|(Ax)'\right\|_B \\
&= \sup_{t\in J}\left\{\left|\varphi_p^{-1}\left(\int_t^{+\infty} q(s) f\left(s, x(s), x'(s)\right)\mathrm{d}s + \varphi_p\left(x_\infty\right)\right)\right|\right\} \\
&\le 2^{1/(p-1)}\left[\varphi_p^{-1}\left(\int_0^{+\infty} q(s)\,\mathrm{d}s\right)\cdot \varphi_p^{-1}\left(B_r\right) + x_\infty\right].
\end{aligned}
\tag{34}
$$

From (34), (H_2), and (H_3), we know that $A\Omega$ is bounded.

For any $T > 0, x \in \Omega, t', t'' \in J_k \cap [0,T]$ with $t' < t''$, by the absolute continuity of the integral, we have

$$
\begin{aligned}
&\left|\frac{(Ax)\left(t'\right)}{1+t'} - \frac{(Ax)\left(t''\right)}{1+t''}\right| \\
&\le \frac{1}{\left(1+t''\right)\left(1-\int_0^{+\infty} g(t)\,\mathrm{d}t\right)} \\
&\quad \cdot \int_0^{+\infty} g(t)\,\mathrm{d}t \\
&\quad \cdot \int_{t'}^{t''} \varphi_p^{-1} \\
&\qquad \times \left(\int_s^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right)\mathrm{d}\tau + \varphi_p\left(x_\infty\right)\right)\mathrm{d}s \\
&\quad + \frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t} \\
&\quad \times \int_0^{+\infty} g(t)\,\mathrm{d}t \\
&\quad \cdot \int_0^{t'} \varphi_p^{-1} \\
&\qquad \times \left(\int_s^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right)\mathrm{d}\tau + \varphi_p\left(x_\infty\right)\right)\mathrm{d}s \\
&\quad \cdot \left|\frac{1}{1+t''} - \frac{1}{1+t'}\right| \\
&\quad + \frac{1}{1+t''} \\
&\quad \times \int_{t'}^{t''} \varphi_p^{-1} \\
&\qquad \times \left(\int_s^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right)\mathrm{d}\tau + \varphi_p\left(x_\infty\right)\right)\mathrm{d}s \\
&\quad + \int_0^{t'} \varphi_p^{-1} \\
&\qquad \times \left(\int_s^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right)\mathrm{d}\tau + \varphi_p\left(x_\infty\right)\right)\mathrm{d}s \\
&\quad \cdot \left|\frac{1}{1+t''} - \frac{1}{1+t'}\right| \\
&\quad + \frac{a^* + b^* r}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}\cdot\left|\frac{1}{1+t''} - \frac{1}{1+t'}\right| \\
&\le \frac{2^{1/(p-1)}}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t} \\
&\quad \times \left[\int_{t'}^{t''}\left(\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau)\,\mathrm{d}\tau\right)\cdot\varphi_p^{-1}\left(B_r\right) + x_\infty\right)\mathrm{d}s\right. \\
&\qquad + \int_0^{t'}\left(\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau)\,\mathrm{d}\tau\right)\cdot\varphi_p^{-1}\left(B_r\right) + x_\infty\right)\mathrm{d}s \\
&\qquad \left.\cdot\left|\frac{1}{1+t''} - \frac{1}{1+t'}\right|\right] \\
&\quad + \frac{a^* + b^* r}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}\cdot\left|\frac{1}{1+t''} - \frac{1}{1+t'}\right| \\
&\quad \longrightarrow 0 \quad \text{uniformly as} \quad t' \longrightarrow t'', \\
&\left|\varphi_p\left((Ax)'\left(t'\right)\right) - \varphi_p\left((Ax)'\left(t''\right)\right)\right| \\
&= \left|\int_{t'}^{t''} q(s) f\left(s, x(s), x'(s)\right)\mathrm{d}s\right| \\
&\le B_r \cdot \left|\int_{t'}^{t''} q(s)\,\mathrm{d}s\right| \\
&\longrightarrow 0 \quad \text{uniformly as} \quad t' \longrightarrow t''.
\end{aligned}
\tag{35}
$$

Thus, we have proved that $A\Omega$ is equicontinuous on any $J_k \cap [0,T]$.

Next, we prove that for any $\varepsilon > 0, x \in \Omega$, there exits sufficiently large $N > 0$ such that

$$\left|\frac{(Ax)(t')}{1+t'} - \frac{(Ax)(t'')}{1+t''}\right| < \varepsilon,$$
$$\left|(Ax)'(t') - (Ax)'(t'')\right| < \varepsilon, \quad \forall t', t'' \geq N. \tag{36}$$

For any $x \in \Omega$, we have

$$\begin{aligned}
&\lim_{t\to+\infty}\frac{1}{1+t} \cdot \left[\frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t} \times \int_0^{+\infty} g(t)\sum_{t_k<t} I_k(x(t_k))\,\mathrm{d}t + \sum_{t_k<t} I_k(x(t_k))\right] \\
&\leq \lim_{t\to+\infty}\frac{1}{1+t}\cdot\frac{a^*+b^*r}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t} = 0, \\
&\lim_{t\to+\infty}\frac{1}{1+t} \cdot \int_0^{+\infty} g(t) \cdot \left[\int_0^t \varphi_p^{-1}\left(\int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\,\mathrm{d}\tau + \varphi_p(x_\infty)\right)\mathrm{d}s\right]\mathrm{d}t \\
&\leq \lim_{t\to+\infty}\frac{1}{1+t}2^{1/(p-1)} \times \left(\frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}\int_0^{+\infty} tg(t)\,\mathrm{d}t\right) \times \left[\varphi_p^{-1}(B_r)\cdot\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau)\,\mathrm{d}\tau\right) + x_\infty\right] = 0, \\
&\lim_{t\to+\infty}\frac{1}{1+t} \cdot \int_0^t \varphi_p^{-1}\left(\int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\,\mathrm{d}\tau + \varphi_p(x_\infty)\right)\mathrm{d}s \\
&= \lim_{t\to+\infty}\varphi_p^{-1}\left(\int_t^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\,\mathrm{d}\tau + \varphi_p(x_\infty)\right) \\
&= x_\infty.
\end{aligned} \tag{37}$$

It follows from (37) that

$$\begin{aligned}
&\lim_{t\to\infty}\left|\frac{(Ax)(t)}{1+t}\right| \\
&= \lim_{t\to\infty}\frac{1}{1+t} \times \left\{\frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t} \times \int_0^{+\infty} g(t) \times \left[\int_0^t \varphi_p^{-1} \times \left(\int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\,\mathrm{d}\tau + \varphi_p(x_\infty)\right)\mathrm{d}s + \sum_{t_k<t} I_k(x(t_k))\right]\mathrm{d}t \right.\\
&\quad \left. + \int_0^t \varphi_p^{-1} \times \left(\int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\,\mathrm{d}\tau + \varphi_p(x_\infty)\right)\mathrm{d}s + \sum_{t_k<t} I_k(x(t_k))\right\} \\
&= x_\infty.
\end{aligned} \tag{38}$$

On the other hand, we arrive at

$$\begin{aligned}
&\lim_{t\to\infty}\left|(Ax)'(t)\right| \\
&= \lim_{t\to\infty}\varphi_p^{-1}\left(\int_t^{+\infty} q(s) f(s, x(s), x'(s))\,\mathrm{d}s + \varphi_p(x_\infty)\right) \\
&= x_\infty.
\end{aligned} \tag{39}$$

Thus, (36) can be easily derived from (38) and (39). So, by Lemma 5, we know that $A\Omega$ is relatively compact. Thus, we have proved that $A : P \to P$ is completely continuous. □

3. Main Results

For notational convenience, we denote that

$$m = 2^{1/(p-1)}\left(\frac{1}{1-\int_0^{+\infty} g(t)\,dt}\int_0^{+\infty} tg(t)\,dt + 1\right) \cdot \varphi_p^{-1}\left(\int_0^{+\infty} q(\tau)\,d\tau\right), \tag{40}$$

$$m' = \left(\frac{1}{1-\int_0^{+\infty} g(t)\,dt}\int_0^{+\infty} tg(t)\,dt + 1\right) \cdot \varphi_p^{-1}\left(\int_0^{+\infty} q(\tau)\,d\tau\right), \tag{41}$$

$$n = 2^{1/(p-1)}\left(\frac{1}{1-\int_0^{+\infty} g(t)\,dt}\int_0^{+\infty} tg(t)\,dt + 1\right) x_\infty, \tag{42}$$

$$n' = \left(\frac{1}{1-\int_0^{+\infty} g(t)\,dt}\int_0^{+\infty} tg(t)\,dt + 1\right) x_\infty, \tag{43}$$

$$\Lambda = \max\left\{\frac{a^*}{1-\int_0^{+\infty} g(t)\,dt - 3b^*}, n\right\}, \quad \Lambda' = \max\left\{\frac{a^*}{1-\int_0^{+\infty} g(t)\,dt - 3b^*}, n'\right\}. \tag{44}$$

Theorem 7. *Assume that* (H_1)–(H_3) *hold, and there exists*

$$d > \begin{cases} 3\Lambda, & \text{as } p \geq 2, \\ 3\Lambda', & \text{as } 1 < p < 2 \end{cases} \tag{45}$$

such that

(A_1) $f(t, x_1, y_1) \leq f(t, x_2, y_2)$ *for any* $0 \leq t < +\infty, 0 \leq x_1 \leq x_2 \leq d, 0 \leq y_1 \leq y_2 \leq d$,

(A_2)

$$f(t, (1+t)u, v) \leq \begin{cases} \varphi_p\left(\dfrac{d}{3m}\right), & \text{as } p \geq 2, \\ \varphi_p\left(\dfrac{d}{3m'}\right), & \text{as } 1 < p < 2, \end{cases} \quad (t, u, v) \in [0, +\infty) \times [0, d] \times [0, d], \tag{46}$$

(A_3) $I_k(x_1) \leq I_k(x_2)$ $(k = 1, 2, \ldots,)$, *for any* $0 \leq x_1 \leq x_2$.

Then, the boundary value problem (4) *admits positive, nondecreasing on* $[0, +\infty)$ *and concave solutions* w^* *and* v^* *such that* $0 < \|w^*\|_D \leq d$, *and* $\lim_{n\to\infty} w_n = \lim_{n\to\infty} A^n w_0 = w^*$, *where*

$$w_0(t) = d + dt, \quad t \in J, \tag{47}$$

and $0 < \|v^*\|_D \leq d$, $\lim_{n\to\infty} v_n = \lim_{n\to\infty} A^n v_0 = v^*$, *where* $v_0(t) = 0, t \in J$.

Proof. We only prove the case that $p \geq 2$. Another case can be proved in a similar way. By Lemma 6, we know that $A : P \to P$ is completely continuous. From the definition of A, (A_1), and (A_3), we can easily get that $Ax_1 \leq Ax_2$ for any $x_1, x_2 \in P$ with $x_1 \leq x_2, x_1' \leq x_2'$. Denote that

$$\overline{P}_d = \{x \in P \mid \|x\|_D \leq d\}. \tag{48}$$

In what follows, we first prove that $A : \overline{P}_d \to \overline{P}_d$. If $x \in \overline{P}_d$, then $\|x\|_D \leq d$. By (6), (40), (42), (44), (H_3), (A_2), and (A_3), we get that

$$\begin{aligned}
\|Ax\|_F &= \sup_{t\in J}\left\{\frac{1}{1+t}\left|\frac{1}{1-\int_0^{+\infty} g(t)\,dt}\int_0^{+\infty} g(t)\left[\int_0^t \varphi_p^{-1}\left(\int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\,d\tau + \varphi_p(x_\infty)\right) ds + \sum_{t_k<t} I_k(x(t_k))\right] dt \right.\right. \\
&\quad \left.\left. + \int_0^t \varphi_p^{-1}\left(\int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\,d\tau + \varphi_p(x_\infty)\right) ds + \sum_{t_k<t} I_k(x(t_k))\right|\right\} \\
&\leq \frac{1}{1-\int_0^{+\infty} g(t)\,dt}\int_0^{+\infty} tg(t)\,dt \cdot \varphi_p^{-1}\left(\int_0^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\,d\tau + \varphi_p(x_\infty)\right)
\end{aligned}$$

$$
\begin{aligned}
&+\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right) \mathrm{d}\tau + \varphi_p(x_\infty)\right)\\
&+\frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}(a^* + b^* d)\\
&\le 2^{1/(p-1)}\left[\frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}\int_0^{+\infty} t g(t)\,\mathrm{d}t + 1\right]\\
&\times\left[\varphi_p^{-1}\left(\varphi_p\left(\frac{d}{3m}\right)\right)\cdot\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau)\,\mathrm{d}\tau\right) + x_\infty\right]\\
&+\frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}(a^* + b^* d)\\
&\le \frac{d}{3}+\frac{d}{3}+\frac{d}{3} = d,
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
&\left\|(Ax)'\right\|_B\\
&= \sup_{t\in J}\left\{\left|\varphi_p^{-1}\left(\int_t^{+\infty} q(s) f\left(s, x(s), x'(s)\right)\mathrm{d}s + \varphi_p(x_\infty)\right)\right|\right\}\\
&\le 2^{1/(p-1)}\left[\varphi_p^{-1}\left(\int_0^{+\infty} q(s)\,\mathrm{d}s\right)\varphi_p^{-1}\left(\varphi_p\left(\frac{d}{3m}\right)\right) + x_\infty\right]\\
&\le d.
\end{aligned}
\tag{50}
$$

Thus, we get that $\|Ax\|_D \le d$. Hence, we have proved that $A : \overline{P}_d \to \overline{P}_d$.

Let $w_0(t) = d + dt$, $0 \le t < +\infty$, then $w_0(t) \in \overline{P}_d$. Let $w_1 = Aw_0$, $w_2 = A^2 w_0$, then by Lemma 6, we have that $w_1 \in \overline{P}_d$ and $w_2 \in \overline{P}_d$. Denote that

$$
w_{n+1} = Aw_n = A^{n+1} w_0, \quad n = 0, 1, 2, \ldots. \tag{51}
$$

Since $A : \overline{P}_d \to \overline{P}_d$, we have that

$$
w_n \in A\left(\overline{P}_d\right) \subset \overline{P}_d, \; n = 1, 2, 3, \ldots. \tag{52}
$$

It follows from the complete continuity of A that $\{w_n\}_{n=1}^{\infty}$ is a sequentially compact set. We assert that $\{w_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{w_{n_k}\}_{k=1}^{\infty}$, and there exists $w^* \in \overline{P}_d$ such that $w_{n_k} \to w^*$.

By (51), (A_1)–(A_3), we get that

$$
\begin{aligned}
&w_1(t)\\
&= \frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}\\
&\times\int_0^{+\infty} g(t)\\
&\times\left[\int_0^t \varphi_p^{-1}\right.\\
&\times\left(\int_s^{+\infty} q(\tau) f\left(\tau, w_0(\tau), w_0'(\tau)\right)\mathrm{d}\tau\right.\\
&\left.+\varphi_p(x_\infty)\right)\mathrm{d}s\\
&\left.+\sum_{t_k<t} I_k\left(w_0(t_k)\right)\right]\mathrm{d}t\\
&+\int_0^t \varphi_p^{-1}\\
&\times\left(\int_s^{+\infty} q(\tau) f\left(\tau, w_0(\tau), w_0'(\tau)\right)\mathrm{d}\tau + \varphi_p(x_\infty)\right)\mathrm{d}s\\
&+\sum_{t_k<t} I_k\left(w_0(t_k)\right)\\
&\le \frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}\\
&\times\int_0^{+\infty} t g(t)\,\mathrm{d}t\cdot\varphi_p^{-1}\\
&\times\left(\int_0^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right)\mathrm{d}\tau + \varphi_p(x_\infty)\right)\\
&+t\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right)\mathrm{d}\tau + \varphi_p(x_\infty)\right)\\
&+\frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}(a^* + b^* d)\\
&\le 2^{1/(p-1)}\left(\frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}\cdot\int_0^{+\infty} t g(t)\,\mathrm{d}t\right)\\
&\times\left[\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau)\,\mathrm{d}\tau\right)\varphi_p^{-1}\left(\varphi_p\left(\frac{d}{3m}\right)\right) + x_\infty\right]\\
&+2^{1/(p-1)} t\left[\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau)\,\mathrm{d}\tau\right)\varphi_p^{-1}\left(\varphi_p\left(\frac{d}{3m}\right)\right) + x_\infty\right]\\
&+\frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}(a^* + b^* d)\\
&\le d + dt = w_0(t),\\
&w_1'(t)\\
&= (Aw_0)'(t)\\
&= \varphi_p^{-1}\left(\int_t^{+\infty} q(s) f\left(s, w_0(s), w_0'(s)\right)\mathrm{d}s + \varphi_p(x_\infty)\right)\\
&\le 2^{1/(p-1)}\left[\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau)\,\mathrm{d}\tau\right)\varphi_p^{-1}\left(\varphi_p\left(\frac{d}{3m}\right)\right) + x_\infty\right]\\
&\le d\\
&= w_0'(t),\quad 0 \le t < +\infty.
\end{aligned}
\tag{53}
$$

So, by (53) (A_1)–(A_3) we have

$$
\begin{gathered}
w_2(t) = \left(Aw_1\right)(t) \le \left(Aw_0\right)(t) = w_1(t), \quad 0 \le t < +\infty,\\
w_2'(t) = \left(Aw_1\right)'(t) \le \left(Aw_0\right)'(t) = \left(w_1\right)'(t), \quad 0 \le t < +\infty.
\end{gathered}
\tag{54}
$$

By induction, we get that

$$w_{n+1}(t) \le w_n(t),$$
$$w'_{n+1}(t) \le w'_n(t), \tag{55}$$
$$0 \le t < +\infty, \quad n = 0, 1, 2, \ldots.$$

Hence, we claim that $w_n \to w^*$ as $n \to \infty$. Applying the continuity of A and $w_{n+1} = Aw_n$, we get that $Aw^* = w^*$.

Let $v_0(t) = 0$, $0 \le t < +\infty$, then $v_0(t) \in \overline{P}_d$. Let $v_1 = Av_0, v_2 = A^2 v_0$. By Lemma 6, we have that $v_1 \in \overline{P}_d$ and $v_2 \in \overline{P}_d$. Denote

$$v_{n+1} = Av_n = A^{n+1} v_0, \quad n = 0, 1, 2, \ldots. \tag{56}$$

Since $A : \overline{P}_d \to \overline{P}_d$, we have that $v_n \in A(\overline{P}_d) \subset \overline{P}_d, n = 1, 2, 3, \ldots$. It follows from the complete continuity of A that $\{v_n\}_{n=1}^{\infty}$ is a sequentially compact set. And, we assert that $\{v_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ and there exists $v^* \in \overline{P}_d$ such that $v_{n_k} \to v^*$.

Since $v_1 = Av_0 \in \overline{P}_d$, we have

$$v_1(t) = (Av_0)(t) = (A0)(t) \ge 0, \quad 0 \le t < +\infty,$$
$$v'_1(t) = (Av_0)'(t) = (A0)'(t) \ge 0 = v'_0(t), \quad 0 \le t < +\infty. \tag{57}$$

By (A_1)–(A_3), we have

$$v_2(t) = (Av_1)(t) \ge (A0)(t) = v_1(t), \quad 0 \le t < +\infty,$$
$$v'_2(t) = (Av_1)'(t) \ge (A0)'(t) = v'_1(t), \quad 0 \le t < +\infty. \tag{58}$$

By induction, we get that

$$v_{n+1}(t) \ge v_n(t),$$
$$v'_{n+1}(t) \ge v'_n(t), \tag{59}$$
$$0 \le t < +\infty, \quad n = 0, 1, 2, \ldots.$$

Hence, we claim that $v_n \to v^*$ as $n \to \infty$. Applying the continuity of A and $v_{n+1} = Av_n$, we get that $Av^* = v^*$.

Since $f(t, 0, 0) \not\equiv 0$, $0 \le t < \infty$, then the zero function is not the solution of BVP (4). Thus, v^* is a positive solution of BVP (4). By Lemma 4 we know that w^* and v^* are positive, nondecreasing on $[0, +\infty)$ and concave solutions of the BVP (4).

We can easily get that Theorem 7 holds for $1 < p < 2$ in a similar manner. □

Remark 8. The iterative schemes in Theorem 7 are $w_0(t) = d + dt, w_{n+1} = Aw_n = A^{n+1} w_0, n = 0, 1, 2, \ldots$ and $v_0(t) = 0, v_{n+1} = Av_n = A^{n+1} v_0$, $n = 0, 1, 2, \ldots$. They start off with a known simple linear function and the zero function respectively. This is convenient in application.

Theorem 9. *Assume that (H_1)–(H_3) hold, and there exist*

$$d_n > d_{n-1} > \cdots > d_1 > \begin{cases} 3\Lambda, & \text{as } p \ge 2, \\ 3\Lambda', & \text{as } 1 < p < 2 \end{cases} \tag{60}$$

such that

(A'_1) *$f(t, x_1, y_1) \le f(t, x_2, y_2)$ for any $0 \le t < +\infty$, $0 \le x_1 \le x_2$, $0 \le y_1 \le y_2$.*

(A'_2)

$$f(t, (1+t)u, v) \le \begin{cases} \varphi_p\left(\dfrac{d_k}{3m}\right), & \text{as } p \ge 2, \\ \varphi_p\left(\dfrac{d_k}{3m'}\right), & \text{as } 1 < p < 2 \end{cases}$$
$$(t, u, v) \in [0, +\infty) \times [0, d_k] \times [0, d_k], k = 1, 2, \ldots, n. \tag{61}$$

(A'_3) *$I_k(x_1) \le I_k(x_2)$ $(k = 1, 2, \ldots,)$, for any $0 \le x_1 \le x_2$.*

Then, the boundary value problem (4) *admits positive nondecreasing on* $[0, +\infty)$ *and concave solutions w_k^* and v_k^*, such that $0 < \|w_k^*\|_D \le d_k$, and $\lim_{n\to\infty} w_{kn} = \lim_{n\to\infty} A^n w_{k0} = w_k^*$, where*

$$w_0(t) = d_k + d_k t, \quad t \in J, \tag{62}$$

and $0 < \|v_k^*\|_D \le d_k$, $\lim_{n\to\infty} v_{kn} = \lim_{n\to\infty} A^n v_{k0} = v_k^*$, where $v_0(t) = 0$, $t \in J$.

Remark 10. It is easy to see that w^* and v^* in Theorem 7 may coincide, and then the boundary value problem (4) has only one solution in P. Similarly, positive solutions w_k^* and v_k^* may also coincide.

4. An Example

Example 11. Consider the following impulsive integral boundary value problem:

$$\left(|x'|\, x'\right)' + e^{-6t} f\left(t, x(t), x'(t)\right) = 0, \quad t \in J_+,$$
$$\Delta x|_{t=k} = \frac{1}{9}\left[\frac{1}{2^{k+2}} x(k) + \frac{1}{2^{k+1}} (1 + x(k))^{1/6}\right], \tag{63}$$
$$x(0) = \int_0^{+\infty} \frac{1}{(1+t)^3} x(t)\, dt, \quad x'(\infty) = \frac{\sqrt{2}}{3},$$

where

$$f(t, u, v) = \begin{cases} \dfrac{1}{64}|\sin(101t + 20)| + \dfrac{1}{72}\left(\dfrac{u}{1+t}\right)^3 + \dfrac{1}{10}\left(\dfrac{v}{20}\right), & u \le 2, \\ \dfrac{1}{64}|\sin(101t + 20)| + \dfrac{1}{72}\left(\dfrac{2}{1+t}\right)^3 + \dfrac{1}{10}\left(\dfrac{v}{20}\right), & u \ge 2. \end{cases} \tag{64}$$

It is clear that conditions (H_1), (A_1), and (A_3) hold for $p = 3$, $q(t) = e^{-6t}$, $g(t) = 1/(1+t)^3$. By direct computation, we obtain that

$$\int_0^{+\infty} q(t)\, dt = \frac{1}{6}, \qquad \int_0^{+\infty} \varphi_p^{-1}\left(\int_s^{+\infty} q(\tau)\, d\tau\right) ds = \frac{\sqrt{3}}{18}, \tag{65}$$

which implies that (H_2) holds.

Obviously, $I_k \in C(J, J)$. Using a simple inequality

$$(1+u)^{\alpha} \le 1 + \alpha u, \quad \forall u \ge 0,\ 0 < \alpha < 1, \tag{66}$$

we get that

$$\begin{aligned} I_k(x(k)) &\le \frac{1}{9}\left[\frac{1}{2^{k+2}}x(k) + \frac{1}{2^{k+1}}\left(1 + \frac{1}{6}x(k)\right)\right] \\ &\le \frac{1}{9}\cdot\frac{1}{2^{k+1}} + \frac{2}{27}\cdot\frac{1}{2^{k+1}}x(k). \end{aligned} \tag{67}$$

Thus, (H_3) holds for $a_k = (1/9)\cdot(1/2^{k+1})$, $b_k = (2/27)\cdot(1/2^{k+1})$. Considering that

$$\begin{aligned} &\int_0^{+\infty} g(t)\,\mathrm{d}t = \int_0^{+\infty}\frac{1}{(1+t)^3}\mathrm{d}t = \frac{1}{2}, \\ &\int_0^{+\infty} tg(t)\,\mathrm{d}t = \int_0^{+\infty}\frac{t}{(1+t)^3}\mathrm{d}t = \frac{1}{2}, \\ &\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau)\,\mathrm{d}\tau\right) = \varphi_p^{-1}\left(\frac{1}{6}\right) = \frac{\sqrt{6}}{6}, \end{aligned} \tag{68}$$

we can obtain that

$$\begin{gathered} a^* = \sum_{k=1}^{\infty} a_k = \frac{1}{18}, \\ b^* = \frac{2}{27}\sum_{k=1}^{\infty}\frac{1+k}{2^{k+1}} = \frac{2}{27}\left(\sum_{k=1}^{\infty}\frac{1}{2^{k+1}} + \sum_{k=1}^{\infty}\frac{k}{2^{k+1}}\right) = \frac{1}{9}, \\ m = \sqrt{3}, \quad n = 2, \quad \Lambda = 2. \end{gathered} \tag{69}$$

Take $d = 8$. In this case, we have

$$\varphi_p\left(\frac{d}{3m}\right) = \varphi_p\left(\frac{8}{3\sqrt{3}}\right) = \frac{64}{27}. \tag{70}$$

On the other hand, nonlinear term f satisfies

$$\begin{aligned} &f(t, (1+t)u, v) \\ &\le \frac{1}{64} + \frac{1}{9} + \frac{1}{25} = \frac{2401}{14400}, \quad t \in [0, +\infty),\ u, v \in [0, 8], \end{aligned} \tag{71}$$

which means that (A_2) holds. Thus, we have checked that all the conditions of Theorem 7 are satisfied. Therefore, the conclusion of Theorem 7 holds.

Acknowledgments

This paper is supported financially by the Foundation for Outstanding Middle-Aged and Young Scientists of Shandong Province (BS2010SF004), the Project of Shandong Province Higher Educational Science and Technology Program (no. J10LA53), and the National Natural Science Foundation of China (10971179).

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